Prospective elementary teachers learning to reason flexibly with sums and differences: Number sense development viewed through the lens of collective activity

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Abstract

I present a viable learning trajectory for prospective elementary teachers’ number sense development with a focus on whole-number place value, addition, and subtraction. I document a chronology of classroom mathematical practices in a Number and Operations course. The findings provide insights into prospective elementary teachers’ number sense development. These include the role of standard algorithms and their relationship to the evolution of classroom mathematical practices that involve reasoning flexibly about number composition, sums, and differences.

Keywords: prospective elementary teachers, number sense, collective activity, classroom mathematical practice, initial mathematical practice.
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This study addresses the important problem of how to improve the mathematics content knowledge of prospective elementary teachers (PTs). K–12 students in the United States are now expected to engage in meaningful mathematical activity, including problem solving, argumentation, and recognizing and making use of the structure of the number system (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). For elementary teachers to facilitate such activity, they must have a deep understanding of, and flexibility with, the mathematics that they teach (Ball, Thames, & Phelps, 2008). However, such understanding and orientation is rare among both prospective and practicing elementary teachers in the United States, as it is in many (but not all) countries (Ma, 1999; Thanheiser et al., 2014). The research literature documents a trend in which PTs approach mathematical tasks by relying on standard procedures, rather than reasoning flexibly or meaningfully about operations and quantitative relationships (e.g., Newton, 2008; Simon, 1993; Thanheiser et al., 2014; Vest, 1978; Yang, Reys, & Reys, 2009).

Researchers agree that PTs need improved mathematical knowledge to better support children’s learning of mathematics; however, little is known about how this goal can be accomplished (Thanheiser et al., 2014). As Mewborn (2001, p. 33) noted, much of the literature provides “snapshots” of teachers’ knowledge at points in time, as opposed to longitudinal “videotape” studies that document PTs’ knowledge development. Thanheiser et al. (2014) echoed this critique in their review of the research literature through 2012, noting that many studies have identified deficiencies in PTs’ mathematical knowledge, whereas very few studies have provided analyses of the development of their knowledge. For mathematics teacher
educators to be better equipped to support PTs’ number sense development, the field is in need of analyses that illuminate viable learning trajectories.

PTs in the United States tend to have experienced mathematics primarily as a domain of facts and procedures to be memorized. This is unsurprising because US mathematics classes have rarely invited students to reason flexibly, to invent their own methods of solution, or to engage in mathematical argumentation (Stigler & Hiebert, 1999). The goal of improving PTs’ number sense is not merely a matter of enhancing the content knowledge of individuals; rather, it involves a change of culture. Mathematics teacher educators seek to support PTs in engaging in practices that run counter to traditional school-math experiences. Therefore, the development of PTs’ number sense may be conceptualized in terms of participation in an evolving classroom community in which ideas and activities that are indicative of number sense gradually become normative.

Given this view of number sense development as increasing participation in a certain set of practices, this article presents the results of an analysis of collective activity (Rasmussen & Stephan, 2008) in a mathematics content course for PTs. As reported elsewhere, the PTs in this class improved their number sense over the course of the semester (Whitacre, 2015; Whitacre & Nickerson, 2016a). To illuminate the process by which that improvement occurred, this study focuses on the evolution of collective activity that took place in the class. Rather than focusing on “snapshots” of number sense pre- and post-instruction, this article presents a “videotape” study of PTs’ number sense development—in other words, it documents the learning process (Mewborn, 2001, p. 33). It does so through a social lens by focusing on the evolution of collective classroom activity over time.
Background

Number sense and prospective elementary teachers

In this section, I describe number sense, its relationship to mental computation, and the relevance of these to PTs’ mathematical preparation. I then describe classroom mathematical practices (CMPs) as a construct for documenting learning in terms of participation in an evolving classroom community. According to McIntosh, Reys, and Reys (1992),

Number sense refers to a person’s general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful strategies for handling numbers and operations. (p. 3, emphasis added)

Number sense is an important goal of mathematics instruction (National Council of Teachers of Mathematics [NCTM], 2000; National Research Council [NRC], 2001). Even when not named explicitly, number sense resonates with contemporary recommendations for students’ mathematical learning. For example, the Common Core State Standards for Mathematics call for second graders to “fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction,” as well as for all K-12 students to “look for and make use of structure” (NGA & CCSSO, 2010, (2.NBT.5) and (SMP.8)). These emphases align closely with the goal of promoting students’ number sense development.

In contrast to the previous recommendations, however, students’ experiences of school mathematics often emphasize the rote application of algorithms and do not support number sense development (Hiebert, Stigler, Jacobs, & Givvin, 2005; Schoenfeld, 1987; Stigler & Hiebert, 1999). This problem is not unique to the United States. For example, Reys et al. (1999) found
that students aged 8–14 in Australia, Sweden, Taiwan, and the United States tended to rely on rote procedures, rather than reasoning meaningfully or flexibly about mathematical tasks. To address this problem, mathematics teacher educators need to effectively support PTs’ number sense development so that PTs can, in turn, support children’s number sense development. A deep and flexible understanding of elementary mathematics is one of the necessary components of the knowledge base that teachers need to teach mathematics effectively (Ball et al., 2008).

There is considerable research literature concerning number sense, as well as recommendations regarding instruction to promote number sense development. A consistent recommendation in the mathematics education literature is that number sense cannot be taught directly; instead, number sense development requires engagement in student-driven, goal-oriented, and meaningful mathematical activity (Greeno, 1991; Howden, 1989; McIntosh, 1998; Sowder, 1992; Yang, 2002). Such approaches have been effective in promoting the development of middle-grades students’ number sense (e.g., Markovits & Sowder, 1994).

Unfortunately, studies of PTs’ number sense have reported disappointing results (e.g., Tsao, 2005; Yang, 2007; Yang et al., 2009). For example, Newton’s (2008) intervention study reported a lack of improvement in PTs’ flexibility, which is a hallmark of number sense. Thus, there is a need for studies that illuminate cases of PTs’ number sense development, particularly in classroom settings and under normal institutional constraints (Mewborn, 2001).

Number sense and flexible mental computation

Mental computation is strongly associated with number sense (Heirdsfield & Cooper, 2004; Markovits & Sowder, 1994; Sowder, 1992; Yang et al., 2009). Mental computation is indicative of number sense when it is characterized by flexibility, including the use of nonstandard strategies, and is supported by understanding of number composition and properties.
of operations (Heirdsfield & Cooper, 2004; Markovits & Sowder, 1994). This article focuses on the activity of whole-number mental computation of sums and differences as a microcosm of number sense in action.

Heirdsfield and Cooper (2004) described the processes of individuals who perform mental computation flexibly or inflexibly.1 For those who reason inflexibly in mental computation, operations map to particular algorithms. Their way of performing an operation mentally is simply to use the mental analogue of the standard algorithm for that operation. Individuals who perform mental computation flexibly, by contrast, make a choice of strategy that is sensitive to the numbers involved in the computation and informed by a web of knowledge and beliefs—knowledge of numeration and number facts, understanding of the effect of operations on numbers, knowledge of strategies, and beliefs about strategies (Heirdsfield & Cooper, 2004). Thus, when choices are made, the person’s number sense is exercised. Engagement in mental computation can lead to improved number sense (Sowder, 1992; Whitacre & Nickerson, 2006). Furthermore, a person’s performance of mental computation provides a window into her number sense (Markovits & Sowder, 1994).

Markovits and Sowder (1994) related whole-number mental computation strategies to number sense according to the extent to which strategies depart from the standard written algorithms. Mental computation strategies are grouped into four categories: standard, transition, nonstandard with no reformulation, and nonstandard with reformulation. Strategies that are less similar to the mental analogue of the standard algorithm are regarded as indicative of better number sense. This connection hinges on the idea that individuals who use nonstandard strategies make a choice of strategy depending on the particular numbers involved in the computation; furthermore, when a person chooses to use a nonstandard strategy, this tends to be
a strategy that makes sense to that person (Heirdsfield & Cooper, 2004; Sowder, 1992). To be more precise, it is not the strategy itself but the flexible use of such strategies that reflects a person’s number sense.

In this article, mental computation strategies are categorized and related to number sense in a way that incorporates the work of several authors (Heirdsfield & Cooper, 2004; Markovits & Sowder, 1994; Whitacre, 2007; 2015). I use the four broad categories of strategies described by Markovits and Sowder, with revisions to the details of these based on (a) the focus in this article on addition and subtraction strategies and (b) nuances of PTs’ mathematical thinking. The following modified definitions of the four categories of mental computation strategies are used in this paper:

Standard: Using the mental analogue of the standard addition or subtraction algorithm,

Transition: Using a right-to-left or left-to-right process,

Nonstandard with no reformulation: Beginning with one of the given numbers and increasing or decreasing according to the other. and

Nonstandard with reformulation: Rounding one or both numbers, computing, and then compensating if necessary.

Table 1 presents specific mental addition and subtraction strategies belonging to each category. The categorization of strategies is informed by the scheme of Heirdsfield and Cooper (2004) but, again, modified with sensitivity to PTs’ mathematical thinking, based on my previous and related research (Whitacre, 2007, 2015).

Standard strategies for mental computation of sums and differences are simply the mental analogues of standard paper-and-pencil algorithms. Transition strategies for mental computation of sums and differences correspond to Heirdsfield and Cooper’s (2004) category of separation
strategies. Numbers are decomposed into tens and ones, the person computes with these separately, and then combines them (e.g., 95 + 27 is computed as 5 + 7 = 12, 90 + 20 = 110, 110 + 12 = 122). Nonstandard strategies with no reformulation correspond to aggregation strategies, wherein one or the other of the given numbers is taken as the starting amount and then the person adds or subtracts incrementally, while keeping a running subtotal (e.g., 95 + 27 computed by starting with 95 and adding a total of 27 to it, working in convenient chunks, such as 95 + 5 = 100, 100 + 22 = 122). Nonstandard strategies with reformulation correspond to Heirdsfield and Cooper’s Wholistic category. All of these strategies involve some form of compensation. One or both of the given numbers is rounded prior to computing, the person computes the sum or difference of the rounded numbers, and then compensates if necessary (e.g., 95 + 27 may be computed by rounding 27 to 30; 95 + 30 = 125, and then 125 – 3 = 122).

Table 1

*Prospective elementary teachers’ mental addition and subtraction strategies.*

<table>
<thead>
<tr>
<th></th>
<th>Standard</th>
<th>Transition</th>
<th>Nonstandard with no reform.</th>
<th>Nonstandard with reformulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition</td>
<td>Mental Analogue of</td>
<td>Right to Left</td>
<td>Aggregation</td>
<td>Levelling</td>
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<tr>
<td></td>
<td>Standard Algorithm</td>
<td>Left to Right</td>
<td></td>
<td>Single Compensation</td>
</tr>
<tr>
<td>Subtraction</td>
<td>Mental Analogue of</td>
<td>Right to Left</td>
<td>Aggregation (Up or Down)</td>
<td>Minuend Compensation</td>
</tr>
<tr>
<td></td>
<td>Standard Algorithm</td>
<td>Left to Right</td>
<td></td>
<td>Subtrahend Compensation</td>
</tr>
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<td></td>
<td>Shifting the Difference</td>
</tr>
</tbody>
</table>

Framing these strategies in terms of the spectrum from standard to nonstandard is helpful for conceptualizing their relationship to number sense. As PTs develop improved number sense, they move from dependence on the mental analogues of the standard algorithms to using various nonstandard strategies. That is, the range of strategies that they use widens, and the distribution
of these shifts toward the nonstandard end of the spectrum (Whitacre, 2007; Whitacre, 2015). The framework of Markovits and Sowder provides a useful theoretical link between number sense and mental computation. Because its categories span operations, it provides an overarching framework to conceptualize the process of number sense development more broadly. I have found this framework useful in previous analyses of the mental computation strategies used by individual PTs in response to interview tasks (Whitacre, 2007; Whitacre, 2015).

Theoretical Framework

The perspective on learning that informs this study can be broadly described as situated (Cobb & Bowers, 1999). Learning occurs in the doing of activities within a culture. The nature of those activities and the culture in which they are situated profoundly shape what is learned. Knowledge becomes meaningful and useful in the practice of authentic activities, which “are most simply defined as the ordinary practices of the culture” (Brown, Collins, & Duguid, 1989, p. 34).

More specifically, this study views learning as occurring through mathematical activity in a classroom setting. The characteristics of this activity are described in terms of classroom social norms, sociomathematical norms, and classroom mathematical practices (Cobb, Stephan, McClain, & Gravemeijer, 2001). Social norms for whole-class discussion include explaining and justifying solutions and indicating agreement or disagreement. Social norms are nonmathematical in the sense that the social norm that solutions should be justified does not entail specific criteria for mathematical justifications. Socio-mathematical norms, by contrast, refer to normative aspects of a classroom culture that are specific to mathematical activity. Examples include “what counts as a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical
explanation and justification” (Cobb, 2000, p. 323). Classroom mathematical practices refer to the mathematically specific, accepted means of justification and explanation that develop within a classroom community (Cobb et al., 2001).

Analyzing learning through this social lens entails characterizing the nature of collective activity and documenting changes in such activity over time. Through this lens, learning takes the form of evolution in the normative aspects of mathematical activity. In a class in which students engage in accountable argumentation (Horn, 2008), they are socially accountable for making sense of each other’s thinking, remembering and using previous arguments and established ideas, and constructing viable arguments. In such a classroom community, certain mathematical ideas that are used in students’ arguments become normative. Progress in collective activity can be described in terms of the normative sets of ideas and related activities that become established and that evolve over time.

Classroom mathematical practices

CMPs describe collective mathematical activity within a classroom community (Cobb et al., 2001; Rasmussen & Stephan, 2008). Whereas individual students have their own particular beliefs, conceptions, and preferred strategies, CMPs describe the normative aspects of a classroom community that are specific to mathematical explanations and justifications (Cobb & Yackel, 1996). It is important to understand that CMPs characterize a specific classroom community. These are not to be confused with the mathematical practices described in the Standards for Mathematical Practice, which specify ways of reasoning about and engaging in mathematical activity that are considered desirable goals for students in general (NGA & CCSSO, 2010). When I refer to CMPs in this article, I am describing collective activity that took place in a particular classroom.
Rigorous analysis of collective activity presents challenges. While in informal settings, people might freely attribute adjectives to couples or communities—for example, “Oh, the Smiths, they’re a fun couple” or “the Robinsons are argumentative” (Rasmussen & Stephan, 2008, p. 195)—a researcher is not making a claim about any one particular individual nor about all individuals that make up that community in attributing a normative way of reasoning to the classroom community. Neither is the researcher making a claim about a certain proportion of the individuals that make up the community. Bowers and Nickerson (2001) addressed this issue in their discussion of a modal view of classroom research. A researcher taking a modal view might attribute ways of reasoning to a community based on a quantitative criterion (e.g., if at least 80% of students behave in a particular way, then that way of reasoning is attributed to the class as a whole). The authors contrast this modal view with their own:

We can distinguish between a process that identifies how the mode of the class is thinking or acting at any given time with our approach that involves creating a chronology of the interrelations among students’ views as they progress over time. In fact, those students who do not follow the developmental trajectory of the mode often add the spice and initiative needed to propel the evolution of new practices. (Bowers & Nickerson, 2001, p. 3).

Thus, these authors make the point that a chronological characterization of the mode may not enable researchers to account for shifts in CMPs. To second this point, Stephan, Cobb, and Gravemeijer (2003) highlighted the distinct ways in which particular students participated in emerging mathematical practices. Ultimately, one student’s interpretation, or way of participating in a practice, may become accepted, while another student’s interpretation is
rejected. Nonetheless, each contributes to the negotiation of the particular mathematical ideas that constitute the classroom mathematical practice.

In their analysis of CMPs in a first-grade classroom, Stephan et al. (2003) used Toulmin’s (1958/2003) model of argumentation. This model frames the anatomy of an argument as consisting of claim, evidence, warrant, and backing.4 In a given argument, the claim is the assertion being made or the conclusion that the person is drawing. The person provides evidence or data in support of that claim. The warrant of the argument explains how the evidence supports the claim. Backing may also be provided to justify the warrant. Not all of these elements need to be present in an argument. Claim, evidence, and warrant constitute the core of an argument. Stephan et al. (2003) used Toulmin’s model to analyze students’ arguments and to provide operational criteria for the establishment of mathematical practices. By analyzing collective mathematical argumentation, Stephan et al. (2003) described the process of negotiation by which each of several mathematical practices developed over the course of a teaching experiment.

A key construct in the analysis of mathematical practices is the notion of taken-as-shared. According to Bowers, Cobb, and McClain (1999), “taken-as-shared understandings refer to the collective knowing of the classroom community” (p. 44). These as-if-shared understandings undergird CMPs. Stephan and Rasmussen (2002) analyzed CMPs in a differential equations class. They articulated their operational definition for taken-as-shared as follows:

We contend that mathematical ideas become taken-as-shared when either (1) the backings and/or warrants for an argumentation no longer appear in students’ explanations and therefore the mathematical idea expressed in the core of the argument stands as self-evident, or (2) any of the four parts of an argument (data, warrant, claim, backing) shift position (i.e., function) within subsequent arguments and are unchallenged. For example,
when students used a previously justified claim as unchallenged justification (data, warrant or backing) for future arguments, we concluded that the mathematical idea expressed in the claim had become taken-as-shared. When either of these instances occurred, no member of the community rejected the argumentation, and/or if the argumentation was rejected and the student’s rejection was rejected, we documented that the mathematical idea had become established. (p. 462).

Stephan and Rasmussen (2002) documented the emergence and establishment of CMPs in relation to particular mathematical ideas. They characterized CMPs in terms of sets of related ideas expressed and used in the context of certain mathematical activities.

To clarify, for Stephan and Rasmussen (2002), a particular CMP may involve several mathematical ideas. For example, in their analysis of a differential equations class, the authors identified the CMP of creating and structuring a slope field as it relates to predicting. This practice involved three distinct mathematical ideas concerning slopes and slope fields. One of these was invariance of slopes across time. This idea was introduced at one point during the semester and, as the authors documented, later became taken-as-shared. Such ideas emerged as students participated in the activity of creating and structuring slope fields, and the ideas became normative as the CMP became established.

In the analysis of CMPs described by Stephan et al. (2003), the authors found that “each practice grew out of practices previously established by the classroom community” (p. 100). That is, CMPs developed sequentially, each building upon the previous. By contrast, Stephan and Rasmussen (2002) found that “the emergence of classroom mathematical practices can be nonsequential in both time and structure” (p. 486). Nonsequential in time means that CMPs can emerge simultaneously. One CMP need not strictly succeed the previous practice. The
emergence of CMPs being nonsequential in structure means that taken-as-shared mathematical ideas can contribute to more than one CMP. As the authors put it,

taken-as-shared ideas do not have to always be viewed as an element of only one classroom mathematical practice—they may contribute to the emergence of other practices and form a network of practices instead of a sequential chain of practices with distinct taken-as-shared ideas. (Stephan & Rasmussen, 2002, p. 487–488).

A final note concerning argumentation in analyses of CMPs: Although the reports cited have focused on the construct of CMPs, social norms are nonetheless relevant. The inferences that researchers make in documenting the development of CMPs are dependent on characteristics of the classroom culture. In a class in which it is the norm for students to justify their statements mathematically, an inference can be made when a particular statement ceases to require justification; as in the examples cited previously, the researchers could claim that a mathematical idea had become taken as shared. However, in a class in which mathematical justification occurs rarely or sporadically, such an inference would be unwarranted. Stephan and Rasmussen (2002) noted that in classrooms in which students engage in mathematical argumentation, “the construct of a classroom mathematical practice is a way to document and characterize the learning of the classroom community” (p. 489). In classrooms with different norms, this may not be the case.

Previous analyses of CMPs and their evolution have focused on the learning of K-12 or undergraduate students who were encountering new mathematical topics. Examples include primary-grades students learning linear measurement (Gravemeijer, Bowers, & Stephan, 2003), middle-school students learning integer arithmetic (Stephan & Akyuz, 2012), and undergraduate students learning differential equations (Stephan & Rasmussen, 2002). Mathematics content courses for PTs contrast with such settings. Rather than encountering mathematical topics that
are new to them, PTs in a Number and Operations course are asked to revisit and reconceptualize familiar topics (Sowder, Sowder, & Nickerson, 2014). The goal of number sense development concerns making sense of these familiar topics in new ways. As PTs develop improved number sense, they come to reason flexibly with numbers and operations, relying less on standard algorithms and more on their own understanding of number composition and properties of operations. Furthermore, mathematics instruction for PTs is intended to contribute to their preparation to teach mathematics. In other words, it is concerned with their development of specialized content knowledge (Ball et al., 2008).

Given these differences in setting, there are limitations to the relevance of previous analyses of CMPs to research involving PTs. In particular, PTs’ prior knowledge of standard algorithms and long history of applying these in school mathematics distinguishes the challenge of supporting PTs’ number sense development from the challenge of supporting, for example, the learning of students in a differential equations course. These differences are reflected in the evolution of CMPs, especially in terms of the influence of PTs’ prior knowledge on the mathematical activities that take place. I use the notion of an initial mathematical practice (Whitacre & Nickerson, 2016b) to highlight the relationship between PTs’ prior mathematical experiences and their activity at the beginning of the course. I further illustrate how PTs initially used standard algorithms that were warranted only by appeals to authority; however, such activity helped to facilitate progress in the direction of number sense development. Over time, PTs became more flexible in their reasoning about number composition, sums, and differences, and they became able to justify the standard algorithms that they had been using previously. In a different setting that is unlike a course for PTs, such an evolution in collective activity might seem perverse. Instead, students might be expected to use informal methods initially and to
eventually reinvent the standard algorithms as generalizations of their less formal activity (e.g., Carpenter, Fennema, Franke, Levi, & Empson, 1999; Stephan & Akyuz, 2012). However, PTs represent a different student population. They already know the standard algorithms. Thus, the evolution of collective activity in a course for PTs may look different than in other classroom settings. This study builds upon established methods and perspectives to document learning in terms of a progression of CMPs. It also turns the literature on its head in some respects, due to differences arising from the setting of a mathematics content course for PTs and the goal of number sense development.

Methods

Context for the study

Data collection took place during fall semester 2010 in a mathematics content course taught at a large, urban university in the southwestern United States. There were 39 students enrolled in the course, 38 of whom were female. The course was designed specifically for PTs. It is the first of four mathematics content courses that the university requires for students who have indicated by their choice of major that they plan to teach elementary school. The majority of the students were freshmen and were recent high school graduates. The instructor of the course was a mathematics educator and an experienced teacher of mathematics courses for PTs.

In general, the course is intended to engage PTs in collaborative problem solving and meaningful mathematical discussions in which they make their reasoning explicit and provide justifications for their claims (Sowder et al., 2014). In addition to attempting to foster a classroom culture characterized by problem solving and sense making, the instructor privileged mental computation with the goal that it would be an authentic activity in the class (Brown et al., 1989). It was the instructor’s intention that performing computations mentally when the need to
compute arose would be one of “the ordinary practices of the culture” (p. 34). Mental
computation of sums and differences was not confined to a particular les-son. Rather, it was part
of the problem-solving activity across multiple curricular units. During group work, students
regularly performed computations mentally—often in service of solving larger problems. During
whole-class discussions, attention was given to both why and how students had performed their
computations.

In a related study, I found that PTs who participated in this teaching experiment
improved their number sense. Scores on an established measure of number sense improved by
more than one standard deviation (Whitacre & Nickerson, 2016a). Furthermore, mental
computation interviews with a subset of the PTs in the class revealed increased flexibility, as
well as shifts toward favoring nonstandard strategies (Whitacre, 2015). These results provide
compelling evidence of improved number sense in general and flexibility in particular, and they
motivate further investigation into how those changes happened.

The present study asked the following research question: In a Number and Operations
course for PTs, which is designed to promote the development of number sense, and in which
PTs’ number sense has been found to improve, how does collective activity related to place
value, sums, and differences evolve? In this article, I report the succession of CMPs related to
place value and addition and subtraction. These results represent a viable learning trajectory
toward number sense development, viewed in terms of collective activity.

Design-based research

The study reported here is part of a larger design-based research effort (Cobb & Bowers,
1999; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), which focuses on PTs’ number sense
development. This research takes the form of classroom teaching experiments (Cobb, 2000) and
proceeds through cycles of data collection, analysis, and theory building. The focus of this study is to document a viable learning trajectory for the development of PTs’ flexibility. In particular, the learning goal was for PTs to move from dependence on standard algorithms to reasoning flexibly about numbers and operations. I conceptualize this process in terms of movement along the spectrum from standard to nonstandard. This is not to say that PTs necessarily leave behind standard or Transition strategies as they make sense of and use nonstandard strategies. Rather, development involves PTs broadening their repertoires of strategies, and thus becoming more able to choose a suitable strategy for a given computation. For the purposes of this study, I conceive of the learning process in terms of the evolution of collective activity. From this perspective, learning involves a succession of CMPs. Thus, the expectation was that standard and Transition strategies and related mathematical ideas would function as accepted means of justification and explanation prior to nonstandard strategies functioning in this way.

Data collection and analysis

The class met twice weekly for 75 min per meeting. Three cameras were used to record classroom activity throughout the semester. The analysis reported here is focused on activity involving place value, addition, and subtraction in the whole-number domain. The data source for this analysis consisted of videotapes of Days 3, 6, 7, 8, 9, 11, and 12 of the semester-long course. Thus, the data span the second to the sixth week of class. From video and transcripts of the whole-class discussions that took place during that period, I identified 118 arguments related to whole-number place value, sums, and differences.

The methodology of Rasmussen and Stephan (2008) was used to analyze collective activity. This methodology involves coding arguments using Toulmin’s (1958/2003) model, which describes the anatomy of an argument in terms of claim, data, warrant, and backing. The
claim is the assertion being made. The data is evidence offered in support of the claim. The warrant explains how the data supports the claim. Backing serves to justify the validity of the warrant (Toulmin, 1958/2003). Arguments consist of ideas, and these ideas may be expressed verbally, as well as through gesturing and inscriptions (e.g., Rasmussen, Stephan, & Allen, 2004). For example, in an activity that involved reasoning flexibly about differences (see CMP4 in the Findings section), Valerie made an argument that 364 – 79 equaled 285, based on a strategy that the class called shifting the difference (364 – 79 = 365 – 80 = 385 – 100 = 285). Valerie used her hands to represent the minuend (364) and subtrahend (79) as points on a number line. She demonstrated that adding 21 to both numbers did not change the distance between them, because both moved the same distance in the same direction. In Valerie’s argument, the claim was that 364 – 79 = 285, the data were the computational steps that were per-formed, the warrant was that adding the same amount to the minuend and subtrahend did not change the difference, and the backing was that subtraction can be thought of in terms of the distance between number-locations.

The methodology of Rasmussen and Stephan (2008) consists of a three-phase process. In the first phase, whole-class discussions are transcribed. Then the transcripts are coded for all instances of claims.7 For each claim, an argumentation scheme is constructed, which explicitly identifies each of the components of the argument.8 This analysis yields a chronological argumentation log.

In the second phase, criteria are applied to identify ideas that functioned as if shared in whole-class discussions. A chronological record of all instances of a given mathematical idea is produced. An idea is considered to function as if shared if one or more of the following criteria are satisfied: (a) warrants or backings dropping off, (b) an element of an argument shifting roles
(e.g., when a previously established result is used as a warrant in support of a new claim), and (c) repeated use of data or warrants in support of different claims (Cole et al., 2011). As an example of the first criterion, early on in the course, the use of compensation strategies was unfamiliar and required justification. However, as time went on, particular compensation strategies became established ways of computing sums and differences. Rather than provide an elaborate justification for a previously justified and now-established strategy, students simply referenced the name that the class had given to the strategy. At that point, the underlying justification functioned as if shared. In layman’s terms, it went without saying.

In the third phase of the methodology, the as-if shared ideas are organized according to related mathematical activities to describe CMPs. Rasmussen and Stephan (2008) defined a CMP as a “collection of as-if shared ideas that are integral to the development of a more general mathematical activity” (p. 201). The way in which as-if shared ideas are organized into CMPs depends on the research focus and the theoretical framing of the phenomena of interest. In the case of the analysis reported here, a focus on number sense development and the use of the standard-to-nonstandard framework informed the categorization of CMPs.

Note that the methodology of Rasmussen and Stephan (2008) makes assumptions regarding class-room social norms. The criteria applied in the methodology are appropriate for the analysis of collective activity in classes in which students engage in problem solving and argumentation and the instructor does not function as the primary locus of authority. This methodology was appropriate to the analysis of collective activity in the class that was studied. Evidence for its appropriateness includes the fact that a large number of arguments were made in the class, most of the arguments were made by students, the majority of arguments included warrants, and many arguments also included backing (Whitacre, 2012).
I analyzed all of the data as part of a larger study (Whitacre, 2012). The evidence and interpretations were discussed with two other experienced researchers, one of them being an expert in the methodology of Rasmussen and Stephan (2008). They provided feedback regarding the coding of arguments, and they vetted the application of the criteria described previously. They also contributed to the organization and description of the CMPs.

The description of the CMPs was reflexively related to further analyses focused on the nature of the strategies and discourse belonging to each CMP. I coded the strategies under discussion in each subset of arguments according to the standard-to-nonstandard framework. I also identified discursive themes that distinguished the CMPs through a process of constant comparative analysis (Corbin & Strauss, 2008). Themes were generated through open coding of sets of arguments. These themes were tested against additional arguments belonging to the same CMP and then were compared across CMPs. The themes are descriptive in nature and help to characterize the collective activity that constituted each CMP.

Findings: A viable route to flexible mental computation in a content course for prospective elementary teachers

This section presents the five CMPs that were identified—the first of which is described as an initial mathematical practice. Table 2 provides an overview of the succession of CMPs. The strategies row describes the categories of strategies that were central to the mathematical activity that characterized each practice. The discourse row describes the discursive themes that characterized each practice. The name of the CMP highlights the central mathematical activity that characterizes that CMP. In other words, it answers the question, “What were the PTs doing with place value, sums, and differences?” For each CMP, I present one or more classroom vignettes. These were selected to illustrate the characteristic activities of that CMP. The vignettes
should be understood as snapshots in time. These snapshots necessarily provide descriptions of
the activities of particular class members during portions of that day’s lesson. As described in the
Methods, collective activity is conceptualized and analyzed in terms of trends over time in those
snapshots of activity. Thus, the vignettes, serve the purpose of illustrating the activities that took
place, whereas the criteria for as-if-shared ideas provide evidence of the collective nature of
those activities. (See Appendix A.)

The CMPs are presented chronologically to convey the learning trajectory traversed by
the class. Note that there was some temporal overlap between CMPs—thus, collective activity
did not proceed strictly from the end of one CMP to the beginning of the next—but there was a
clear chronological order to their emergence. Furthermore, the initial mathematical practice did
cease decisively with the emergence of CMP1. It is also important to clarify that I do not claim
that all of the PTs in the course were reasoning in the same ways as they participated in the class
activities. Rather, the learning trajectory that I present provides a description of how collective
activity evolved. Viewed as a social phenomenon, this evolution in collective activity is what
learning looks like. On the level of a whole class—in this case, consisting of 39 PTs—this viable
learning trajectory provides valuable insights into how PTs’ number sense development can be
fostered.

The collective activity to be described spanned the first 6 weeks of the semester, as the
class moved through three distinct curricular units. The five practices that follow belong to a
strand of activity that spanned curricular units. In the pages that follow, I elaborate on each
practice, describing the nature of the collective activity that took place, and I illustrate this
activity with one or two vignettes that exemplify the relevant activities and themes.
Table 2.

Strategies and discourse characteristic of each classroom mathematical practice.

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Initial mathematical practice: Using standard algorithms by default

Strategies

During problem-solving activity early in the course, PTs used the standard algorithms by default. The standard algorithm tended to be the first way PTs suggested for performing a calculation, and the use of algorithms went unquestioned and without substantive justification. A
few nonstandard strategies for various operations were introduced sporadically, and the
instructor pressed for justification of these.

Discourse

Two discursive themes characterized the initial mathematical practice: (1) distinguishing
operations from ways of performing those operations, and (2) offering unprecedented
justifications—arguments involving warrants for which there was no precedent in previous
discussions. These tended to involve either commonsense mathematical ideas, such as reasoning
about subtraction as a take-away process, or appeals to the authority of the standard algorithms.
The arguments that students made to justify their nonstandard strategies eventually led to the
establishment of important (even if familiar) mathematical ideas that contributed to the
advancement of mathematical activity.

Vignette: Nonstandard strategies require justification

The following vignette illustrates activity belonging to the initial mathematical practice.
On Day 3 of the semester, students were given a story problem that involved comparing the
heights of two pairs of siblings. While solving the problem, the need for computations arose and
the instructor asked students to attempt these mentally. Students recorded their mental
computations on small whiteboards. In the subsequent whole-class discussion, the quantitative
relationships involved in the height problem and the need to perform certain computations were
discussed first. Then the details of those computations were discussed. This emphasis on
quantitative reasoning (Smith & Thompson, 2007; Thompson, 1993) was typical of
mathematical activity in the course.

Aaron shared his group’s methods for performing two subtraction computations, $64 - 43$
and $70 - 19$. For the first computation, he used the standard subtraction algorithm (Figure 1a).
This computation was discussed briefly, and the validity of the approach went unquestioned.

Aaron then presented his group’s method for computing 70 – 19. He reported that they subtracted 10 from 70, obtaining 60. Then they subtracted 9 from 60, for an answer of 51 (Figure 1b). In the following exchange, the instructor challenged the legitimacy of this strategy, and Aaron argued for its validity:

Aaron: Well, we, um, trying to find the difference between Olivia and Oscar [referring to their heights], we took 70 … and subtracted ten first, and got 60 [writing on screen].

Instructor: Could you write the problem up here and remind us what that computation was? 70 minus 19 [Aaron writes 70 – 19 on screen]

Aaron: And then we subtracted nine from that, and we got 51.

Instructor: So, one question is, how come—how do you know it works like this? You took 70 minus 19. That’s not the same thing as you would do if you were, you know [Instructor enacts standard subtraction algorithm on screen] taking this, and borrowing like this, and—this isn’t exactly the same algorithm. How come this one gives the same answer, and why are you confident that that works? Can you …

Aaron: Just because you’re still subtracting the whole 19. It’s just you’re taking the ten out first.
In this exchange, Aaron presented an argument that $70 - 19$ equaled 51. As data, he reported two computations ($70 - 10 = 60$ and $60 - 9 = 51$). Aaron’s warrant was left implicit until the instructor pressed for further justification. He then explained that by subtracting 10 and then 9 more, he had subtracted a total of 19 (“Just because you’re still subtracting the whole 19.”). In other words, he was reasoning about subtraction as a cumulative process. Aaron further explicated his reasoning when he said, “It’s just you’re taking the ten out first.” In his previous statements, Aaron had used the words difference and subtracted. He provided the backing of his argument by conveying a meaning for subtraction as a take-away process. If subtraction is a process of removal (“taking … out”), it follows that subtraction is cumulative. By taking out 10 and then taking out 9 more, he had taken out a total of 19. In this way, the take-away meaning was essential to Aaron’s justification for his group’s strategy.

It is worth noting that subtraction did not appear as a take-away process in the problem context. Rather, it was being used to compare heights. The class had agreed that subtraction was the appropriate operation to use to solve this comparison problem. Once it was determined that they would use subtraction, Aaron described the strategy that had been used and discussed in his group, and that strategy (at its core) involved reasoning in terms of take-away. This way of reasoning about subtraction would later come to function as if shared and would play an important role in the evolution of collective activity related to subtraction.

Summary of the initial mathematical practice

As was typical of PTs early on in the course, Aaron’s group initially used the standard subtraction algorithm. Subsequently, they entertained an alternative strategy. In the discussion, the use of the familiar, standard algorithm went unquestioned. By contrast, the instructor pressed for justification of the non-standard strategy. This vignette is representative of the initial
mathematical practice in which the PTs drew on their prior mathematical knowledge as they
were thrown into a classroom environment in which they were expected to solve novel problems
and explain their reasoning.

Vignette 0 further illustrates two themes. First, whereas many PTs tend to rely on
standard algorithms and use these by default—thus, they may tend to equate the operation with
the algorithm—the whole-class discussions clearly distinguished the operations themselves from
how those operations were performed. The instructor played an important role in influencing this
trend by separating discussions of quantitative relationships from discussions of computations
and then by pressing students to describe and justify their nonstandard strategies. For instance, in
her exchange with Aaron, the instructor explicitly justified her request for backing by contrasting
Aaron’s strategy with the standard algorithm. Second, these requests led to unprecedented
justifications, which provided opportunities for PTs to voice common sense mathematical ideas,
such as reasoning about subtraction as a take-away process. In subsequent CMPs, these ideas
came to function as if shared and proved important to advancing the mathematical activity
(Rasmussen, Zandieh, King, & Teppo, 2005).

Classroom math practice 1: Making sense of place value in terms of grouping actions Strategies
and activity

Classroom math practice 1 (CMP1) involved attending to, and making sense of, place
value, which established foundations for Transition strategies. During days 6–8, students
participated in a variety of activities involving counting, grouping, and regrouping in various
bases. On days 6 and 7, students solved a variety of tasks in the Andrew’s Apple Farm context,
which implicitly involved base eight. Students converted between base-ten notation and
Andrew’s special way of packing apples and recording numbers of apples. In the latter part of
day 7, the class began to talk explicitly about different bases and to count and represent numbers in base three. Activity in base three was not related to a story-problem context but instead involved multilink cubes. Students acted out counting in base three and converting between bases three and ten by using the cubes, with one person playing the role of each place value.

Discourse

The discursive theme related to CMP1 was distinguishing numbers from numerals. Prospective elementary teachers tend to exhibit room for growth in their understanding of multidigit numbers, especially in relation to the standard addition and subtraction algorithms (Thanheiser, 2009). Before the PTs in the class were asked to make sense of the standard algorithms, ideas were established concerning the meanings of digits in numerals as groups (and groups of groups) of a given size. In the discourse belonging to CMP1, students reasoned about the values of digits in numerals in two important ways: in terms of relationships between neighboring place values and in terms of groups of ones.

Vignette: Notating groups

On day 7, students counted and represented numbers in base three. After students had worked on counting up to 27 in their small groups, they counted together as a class and discussed some of the reasoning involved. The following exchange occurred at the end of the counting discussion and regarded the transition from 26 (written as 222three) to 27 (written as 1000three):

Instructor: First, two two two base three. What number does that represent?

Molly: Two ones, two groups of three, and two groups of nine.

Instructor: Two ones, two groups of three, two groups of nine. Okay, so that’s twenty-six in base ten. And what happened here? [Instructor points to 1000 written on board.]
Molly: You add a one to the two ones, and that makes a group of three. And then you have three groups of three.

Instructor: Then you have three groups of three.

Molly: Yeah, and that’s nine. So, then you take that and the other two groups of nine, and that makes three groups of nine. So, that equals one twenty-seven.

In the previous exchange, Molly justified the transition from $222_3$ to $1000_3$ in terms of a succession of regrouping actions that occurred as a result of adding one to $222_3$: Adding one created a group of three ones, which was passed on the threes place. Then there were three threes, which formed a group of nine, and it was passed on to the nines place. Then there were three nines, which formed a group of twenty-seven, and it was passed on to the twenty-sevens place. In this way, Molly argued effectively that the numeral after $222$ in base three was $1000$. Molly’s argument involved using and interpreting place-value notation in base three, together with the idea of forming groups, and groups of groups, of size three.

Summary of CMP

Students’ participation in activities involving various bases contributed to more explicit attention to reference units. This attention was evident in students’ arguments as they made the place value of digits explicit and evoked ideas of groups and groups of groups. This explicit reasoning in terms of place value and reference units is important for PTs to make sense of and use nonstandard strategies. The rules involved in the standard algorithms account for place value implicitly; As long as the numerals are lined up correctly and the right steps are performed, the algorithm will lead to the correct answer. Nonstandard strategies do not provide the sorts of conventional constraints that are embedded in the standard algorithms. Thus, students’ abilities to sensibly use such strategies go hand-in-hand with their aware-ness of the place values of digits.
in numerals. The activity that characterized CMP1 moved beyond the levels of reasoning about place value that are typical of elementary PTs (Thanheiser, 2009, 2010). This reasoning contributed to the foundation from which the class made sense of standard algorithms and Transition strategies in CMP2.

Vignette 1 also illustrates the discursive theme of distinguishing numbers from numerals. Activities involving different bases brought these issues to the fore, because the same numeral can represent different amounts, depending on the base. In the arguments belonging to CMP1, class members made these distinctions explicit. For example, in justifying the transition in counting from 222three to 1000three, Molly attended to the meaning of each digit (as numbers of ones and groups of three and groups of nine).

Classroom math practice 2: Applying place-value ideas to justify standard algorithms and Transition strategies

Strategies and activity

Classroom math practice 2 (CMP2) involved justifying standard algorithms and Transition strategies for addition and subtraction. The unsubstantiated authority of the standard algorithms ceased when students articulated justifications as to why the algorithms worked. Students also used and justified their own invented (transition) strategies, and their arguments involved a common set of as-if shared ideas. New ideas were established that were integral to these justifications, and previously established ideas were employed in productive new ways. The fact that collective activity came to involve justifying standard algorithms and Transition strategies is noteworthy because researchers have found that it is difficult for PTs to produce these kinds of justifications (e.g., Harkness & Thomas, 2008; Lo, Grant, & Flowers, 2008; Thanheiser, 2009, 2010).
Discourse

Two discursive themes stand out in this classroom math practice: The first is that arguments made reference units explicit. Building on CMP1, in CMP2, students brought grouping ideas to bear in reasoning about and computing sums and differences. Activity related to addition came to involve ideas of aggregation and regrouping, grounded in counting in the given base. The addends and sum were recorded as numerals in the base. Regrouping moves were notated by writing a 1 above the digit in the next place to the left. Similarly, in subtraction activity, regrouping involved unpacking a group of a certain size. The minuend, subtrahend, and difference were recorded as numerals in the given base. Subtraction regrouping moves were notated in one of two ways: (a) by writing 1 to the left of the appropriate digit of the minuend, or (b) by writing 10 above that digit. These notations were used in a variety of bases. The second theme is that arguments featured the application of as-if-shared ideas. That is, ideas that came to function as if shared were used to justify new claims that were central to advancing the mathematical activity.

Vignette: Viewing the little as representing a group

On day 8, students reasoned about addition and subtraction in various bases. Their work with multilink cubes had occurred exclusively in base three. After discussion of base-three computations that involved relating actions with the cubes to place-value notation and the standard algorithms, the instructor gave students a set of computations to be performed in bases other than three. In the following vignette, the class discussed the solution to a base-five addition problem:
Instructor: Three four one base five [writes on screen]. Now we’re in a new special group, right? Five. Okay. You ready? So, here’s the issue that I heard as I came around. Everybody said, great, one plus zero: one. [Instructor writes 1 in the ones place.] Four plus two: six. How am I going to record that?

Students: Write a 1 and carry a 1.

Instructor [writes 1 in the fives place and “carries” 1 to the twenty-fives place]: Why would you write a 1 and record a 1 in the column to the left?

Students: You have five.

Students: That’s a group of five.

[Instructor said that she could not hear students’ suggestions because too many were talking at once. She asked for one student to speak at a time.]

Student: Because you have to move a group of five because you can’t have five groups. So, you have to move it over. Instructor: So, you imagine that if you had six, you said you can’t—once you have a group of five, it moves over here. Right?

Student: And then you have one left over.

Instructor: So, these are our ones, our strips, it’s a square now and it moves over here now, right? And then you had five here.

Student: So, then you have to move that one over.

Instructor: Now you don’t get to hold onto it, you hand it to the person who holds the cubes, right?

In Vignette 2.1 class members coconstructed an argument concerning the solution to an addition problem in base five. Base five had not been discussed previously, but students generalized from their work in other bases. Significantly, justifications that had previously
dropped off when working in base three and using the cubes resurfaced in the context of doing written addition work and dealing with an unfamiliar base. Every step that the class described in the solution to the base-five addition problem made reference to regrouping actions involving people as place values manipulating physical objects. In this way, place-value notation and the standard algorithms took on these meanings for the class.

In subsequent activities, students attended to reference units in base ten. As the class proceeded, students shifted away from the standard algorithms and instead used Transition strategies that were founded on now-established ideas of place value. The following vignette illustrates such activity.

Vignette: Decomposing numbers into tens and ones

In the following vignette from Day 9, Muriel used a transition strategy for addition in base ten. (See Figure 2.) Specifically, she used Left-to-right separation. Muriel wrote her work on the screen, showing her data (the steps she had performed) and her claim that $88 + 47 = 135$.

Nancy and another student contributed the warrant of the argument:

Muriel: [Writes on board: $8 + 7 = 15$, $80 + 40 = 120$, $15 + 120 = 135$]

Instructor: [pointing to board] Okay, and so, what did she do here? What do you see her doing?

Student: She broke it into ones and tens.

Nancy: … And then combined it.

Student: Added the tens, added the ones, and then combined it.

Instructor: She broke it into ones and tens? And added the tens up, added the ones up, and combined. We’ve done that before in here, yeah? We’ve had people share this.
In her argument, Muriel claimed that 88 plus 47 equaled 135. As data, she presented three computations: $8 + 7 = 15$, $80 + 40 = 120$, and $15 + 120 = 135$. Two other students supplied the warrant in the argument: Muriel broke the numbers into tens and ones and then combined these. Once this warrant had been voiced, the instructor asserted that Muriel’s strategy had been previously established (“We’ve done that before in here, yeah? We’ve had people share this.”). This assertion made explicit the acceptability of the warrant that the students had provided. It was true that Muriel’s strategy had been previously discussed and justified. On day 8, Amelia had used the same strategy for a different computation and had shared her thinking with the class. Amelia’s strategy had similarly been justified in terms of combining tens and ones. More broadly, the class had engaged in a great deal of activity during days 6–8 that involved grouping or decomposing numbers canonically and collecting up and counting items. Thus, Muriel’s strategy was consistent with the activity in which students had engaged over the past three days of class. The idea of separating numbers into tens and ones (for purposes such as combining them) functioned as-if shared at this point.

Summary of CMP

In CMP2, students applied ideas related to place value and addition and subtraction to justify both the standard algorithms and their own Transition strategies. The standard addition and subtraction algorithms were familiar to students at the outset of the course, and these algorithms had functioned with assumed authority in CMP1. In CMP2, the same (or similar)
procedural steps and notational conventions took on new meanings for the class as connections were made to students’ grouping activities and arguments related to computing sums and differences.

Vignette 2.1 illustrated both themes of making reference units explicit and applying as-if-shared ideas. Class members reasoned through how to notate regrouping in an unfamiliar base and were specific about the reference units for the digits involved, especially the use of a little 1 to represent a group (in this case, in base five). Vignette 2.2 illustrated the PTs’ use of Transition strategies in base ten. At this point, Muriel’s strategy involved the application of an as-if-shared idea as warrant, indicating that these strategies now stood on firm footing. In contrast to the unprecedented justifications that were characteristic of the initial mathematical practice, the justifications of strategies in CMP2 involved the application of ideas that were established and now functioned normatively.

Classroom math practice 3: Reasoning flexibly about sums Strategies and activity

Classroom math practice 3 (CMP3) involved reasoning flexibly about sums. Students engaged in using, justifying, and representing nonstandard addition strategies. Warrants concerning place value dropped off, and these arguments focused on mathematical ideas that were specific to the nonstandard aspects involved in aggregation and compensation strategies. Discourse

The main discursive theme in CMP3 was that strategies became objects of discourse. Activity became more focused on reasoning about strategies, rather than using these in service of solving broader problems. Nonstandard addition strategies were used, discussed, justified, named, and compared. In discussions of nonstandard strategies, ideas related to canonical number composition figured less prominently in students’ arguments. Students’ justifications
were grounded in as-if shared ideas related specifically to properties of addition and to noncanonical number composition. Strategies were given names, and these names afforded discussions in which the strategies became objects of discourse (Cobb, Boufi, McClain, & Whitenack, 1997).

Vignette: “Borrowing to build.”

On day 8, students solved addition and subtraction problems in a variety of bases, including base ten. Before discussing the algorithmic work for computing 95 + 27, the instructor asked the students to perform this computation mentally. A few strategies were discussed. Trina shared her levelling strategy. Aaron and another student contributed to the justification of Trina’s strategy:

Trina: You take—you want to make 95 into 100. So, you just take away 5 from 27, making the 95 a hundred. And then you’ll have 22 left from the 27. And then you’ll add 100 and 22.

Instructor: What do you think? Is this strategy something that’s going to work for whatever addition problem she picks? Or, is it just peculiar to this problem? Go ahead.

Trina: Because 95 is really close to 100, and 100’s an easy number to work with.

Instructor: So, it occurred to you because 95 is very close to 100, and 100 is a really easy number to add onto. Yeah? So, there might be other kinds, other numbers like this that you’d use this with, right? So, it sounds like Trina can see usefulness there.

Instructor: Um, why does this work? Somebody besides Trina, anybody else use this strategy? Somebody else here? Aaron used this strategy. Aaron, why does this work?
Aaron: You’re just taking 5 from 27 and giving it to 95. So, you still—you’re just moving the 5.

Instructor: Okay, so you’re saying you moved the 5.

Instructor: You were raising your hand. Did you want to add something?

Student: It’s kind of like taking a part and putting it in a different place.

Instructor: What are you taking apart here?

Student: From 27, you’re taking a part and giving it to the 95. You’re still adding all the same numbers, just in different places.

The class discussion of Trina’s strategy involved two aspects: mathematical justification and strategy selection. Class members discussed why the strategy worked, in terms of its mathematical validity. They also discussed characteristics of the problem that made the strategy a nice choice—in particular, the proximity of 95 to 100. Trina claimed that 95 + 27 = 122. She presented as data three computations: 27 – 5 = 22, 95 + 5 = 100, and 100 + 22 = 122. Aaron offered a warrant for Trina’s strategy: moving, or giving, part of one addend to the other addend. Another student supplied backing for this warrant: reasoning about addition as an associative operation (“You’re still adding all the same numbers, just in different places.”).

Between days 8 and 9, students were given a follow-up homework assignment that involved analyzing Trina’s strategy, applying it to a different problem, and suggesting a name for it. On day 9, these ideas were discussed, and Trina’s strategy was named “Borrow to Build.”

Vignette: Comparing strategies and making distinctions

On day 9, the class followed up on the discussion of Trina’s strategy. In that context, Valerie shared her own strategy:
Valerie: I solved the problem a different way than what you just described.

Instructor: Can you put it on the board up here?

Valerie: Sure. So, for example, I used [writes “88 + 47” on screen]. So, that was my example.

Instructor: Could you hold on a second? Hey, you guys. This is a new problem. Can you do this mentally? And just make a note about how you thought about it, before you listen to her. Okay? 88 plus 47 is what she has written up here. Do it mentally yourselves before you hear about Valerie’s explanation. Think about how you’d do that problem.

[Pause while students think]

Instructor: Okay, alright. Go ahead, Valerie. Did you have time? Everybody had a chance? Great.

Valerie: [standing at screen] So, I know in the first problem, Trina, she made the numbers whole, I mean, one of the numbers whole, so it would be easier to deal with [writes “88 + 2 = 90”], but I made both of them that way [writes “47 + 3 = 50”]. So, I added 2 and 3 [draws a rectangle around the “+2” and “+3”], so I made both of them whole numbers, so I didn’t have to go through addition. And so I added 90 and 50 and got 140. And then, because I added a 3 and a 2, 2 plus 3 is 5, so I needed to take 5 away from 140, and the answer is 135.

In presenting her strategy, Valerie introduced a new problem, 88 + 47. She claimed that this sum was 135. She presented several computations as data: 88 + 2 = 90, 47 + 3 = 50, 90 + 50 = 140, 2 + 3 = 5, and 140 − 5 = 135. To provide a warrant, Valerie explained her reasoning in terms of rounding and then compensating (“I added 2 and 3, so I made both of them whole numbers. … And then … I need to take 5 away.”). The justification for compensating in the way
that she did was provided by the backing, which involved reasoning about the balance of two rounding moves (adding 2 and adding 3 resulted in a total of 5 that she had added to the sum), together with reasoning about compensation as requiring the inverse operation (using subtraction to undo addition).

In her argument, Valerie introduced and justified a double compensation strategy for addition. It was evident that Valerie saw her strategy as distinct from Borrow to Build (“I solved the problem a different way.”). In particular, she made a distinction regarding forming friendly numbers: “Trina, she made the numbers whole, I mean, one of the numbers whole, so it would be easier to deal with, but I made both of them that way.” Valerie’s use of the word whole seemed to indicate round numbers, particularly decades (i.e., whole numbers of ten). In Valerie’s view, Trina (solving 95 + 27) had made one of the numbers “whole” by making 95 into 100. Valerie, by contrast, had made both of the addends “whole” by rounding 88 to 90 and 47 to 50. For this reason, Valerie saw her strategy as distinct from Trina’s. However, it was not clear whether other students would agree or disagree with Valerie’s assessment or for what reasons.

The instructor directed the class to consider whether Valerie’s strategy was the same as Borrow to Build or different. Zelda made an argument that Valerie’s strategy was different. In this sense, she agreed with Valerie. However, Zelda saw the distinction differently:

Zelda: You’re not borrowing from one of the numbers. You’re kind of adding two different numbers to both of them. So, it’s not really like the-borrow-and-the-build method.

Instructor: Maybe this isn’t Borrow to Build, because what would you be borrowing from? Is that what you’re saying? You’re not borrowing from one of the numbers.

Zelda: Right.
Zelda argued that Valerie’s strategy was different from Borrow to Build. Like Valerie, she noted that both numbers, rather than just one, were rounded. However, Zelda’s argument focused on the origins of the amounts added. In Borrow to Build, part of one addend was given to the other addend. In Valerie’s strategy, by contrast, the amounts added to the given numbers came from elsewhere. By increasing the given addends, Valerie had temporarily altered the sum and thus she had to compensate by subtracting. In Trina’s strategy, end compensation was unnecessary because the sum was unchanged.

Summary of CMP

In CMP3, students reasoned flexibly about computing sums. Part of this activity was the use of a popular, nonstandard strategy that the class named Borrow to Build. Students reasoned about numbers in terms of noncanonical composition (e.g., viewing 100 as being composed of 95 and 5). Although not described here, aggregation strategies were also used and discussed. Arguments related to these were grounded in reasoning about addition as a cumulative process of increase.

Vignettes 3.1 and 3.2 illustrated a shift in the discourse concerning strategies, which was characteristic of CMP3. Strategies became objects of discourse as the class engaged in giving names to established strategies and reasoning about fine-grained distinctions between strategies. Such activity indicates significant progress, given that students were initially reliant on the standard algorithms and not attuned to subtle distinctions between nonstandard strategies (Whitacre, 2012; Whitacre, 2015).

Classroom math practice 4: Reasoning flexibly about differences Strategies and activity

Classroom math practice 4 (CMP4) involved reasoning flexibly about differences. Analogous to CMP3, activity belonging to CMP4 involved using, justifying, naming, comparing,
and representing non-standard subtraction strategies. Students performed subtraction mentally, reasoned about each other’s strategies, interpreted children’s thinking, and created and discussed drawings for the purposes of representing and justifying strategies. The previously established idea of reasoning about subtraction as a take-away process was central to students’ justifications of subtractive aggregation and compensation strategies. Reasoning about subtraction as a cumulative process of decrease also played an important role. Students used the empty number line (Anghileri, 2000; Gravemeijer et al., 2003) and justified subtractive compensation strategies by reasoning about movement along the number line, as well as by reasoning about differences as distances between number-locations.

Discourse

Two discursive themes characterized CMP4: (a) Arguments involved interpreting children’s strategies. In keeping with CMP3, strategies continued to be objects of discourse. In particular, many arguments involved PTs interpreting the reasoning involved in samples of children’s written work, as well justifying the validity of the children’s novel strategies. (2) Arguments involved using familiar ideas and tools in new ways. In CMP4, students used and reasoned about nonstandard subtraction strategies, some of which are known to be difficult for PTs (Kazemi et al., 2011; Whitacre & Nickerson, 2012). This evolution in collective activity was accomplished using previously established ideas in new and productive ways. These ideas included reasoning about subtraction as a take-away process and as a cumulative process of decrease, as well as reasoning about differences as distances between locations on the empty number line.
In the following vignette, a student justified a nonstandard subtraction strategy by building on a much more basic and commonplace way of reasoning about subtraction. Other students related the same strategy to the number line and emphasized the idea of the difference as a distance between number-locations.

Vignette: Using familiar ideas to justify subtrahend compensation

On day 12, students were given three examples of children’s reasoning about the computation $364 - 79$. Students worked in their groups to make sense of this and other children’s strategies. In the whole-class discussion of the first child’s strategy (see Figure 3), two students offered justifications.

Betty: From what it looks like, they subtracted 100, just thinking, you know, that’s really easy to do, to get 264. But then because they had subtracted 100, they found the difference between 100 and 79, and that’s 21. So, then they added that in at the end.

Instructor: So, then they added that back in? What do you think? Is that mathematically valid? Why did she add instead of subtracting that? Why add 21 at the end? She’s subtracting, then you’re adding.

Torrin: She subtracted more than she needed.

Instructor: She subtracted more than she needed.

Torrin: To make it like easier to do mentally, to like visually see. And then she had to add back the 21 because she took away an extra 21 that wasn’t necessary.

In this exchange, students constructed an argument for the validity of the child’s strategy. The data in the argument was a description (both written and verbal) of the computational steps the child had performed. The warrant was the idea of compensating for rounding. Backing involved two ideas, reasoning about subtraction as a take-away process, and reasoning about
compensation based on the effects of rounding. Maybee presented her illustration for the child’s strategy discussed previously. The class discussed Maybee’s illustration and suggested modifications to it. Melinda argued that Maybee’s illustration would be improved if it showed the difference explicitly as the distance between 285 and 364. (See Figures 4a and 4b.)

\[
\begin{align*}
364 - 100 &= 264 \\
100 - 79 &= 21 \\
264 + 21 &= 285
\end{align*}
\]

Figure 3. Given representation of the first child’s strategy.

Figure 4. (A). Maybee’s illustration to record the child’s work. (B) Melinda’s gesturing highlights the difference of as a distance between number-locations.

Melinda: Um, I think it would be—I really like that one, but I think that it would be even better if, like, somewhere the 79 was shown, or like, the difference between 285 and 364. Because otherwise it kinda looks a little bit more random. But, so maybe somewhere show the distance to show that it’s 79.

In her argument, Melinda equated difference and distance between. She did not justify this meaning for the difference but, rather, took it as a given. Melinda’s argument was accepted. In subsequent discussion of Maybee’s illustration, a line segment was added that spanned the distance from 285 to 364. In response to another student’s argument, an arrow was added to this
segment to show that the net change depicted in the diagram was decrease, or movement to the left. Figure 5a depicts the final version of Maybee’s illustration. The class then discussed names for this strategy, and Back and Forth Subtraction became the official name. Instructor made an argument for the appropriateness of this name, using gesturing to highlight the back-and-forth nature of the strategy as depicted on the number line (Figure 5b).

Vignette: Shifting the difference

Three students made arguments for the validity of the second child’s strategy (Figure 6). Valerie’s argument involved reasoning about the difference as a distance between:

Valerie: Okay, so we thought about it in terms of, when you’re subtracting, you’re trying to find the distance between two numbers. So, we thought of it kind of in terms of a number line. … So, you started off with 79 and 364. So, 364 moved up one to 365 and also, likewise the 79 moved up to 80. So, the distance didn’t change between the numbers. So, originally it was right here, and they both moved up one on a number line. So, the distance between them is the same. So, similarly when you have 385 and 100, you just added 21. So, if you took the numbers from their original position and moved them each up 21 spaces, the shift would be the same and the distance between both numbers is the same.

[Valerie uses her hands as number-locations. She moves both hands to her right as she talks about the numbers “moving up.”]
In Valerie’s argument, reasoning about the difference as a distance between number-locations served as backing for the warrant that adding the same amount to both the minuend and subtrahend maintained the difference. In previous arguments, the idea of the difference as distance between had been used as data. The shift in its argumentative role coincided with and afforded the justification of a new idea: shifting the difference. In subsequent activity, the idea of shifting the difference, in turn, was used to justify the equal-additions algorithm. Empty-number-line inscriptions and gesturing that illustrated distances spanned and shifted were integral to this classroom math practice.
Summary of CMP

In CMP4, students engaged in sophisticated reasoning about differences. Tracing the evolution of arguments shows that this reasoning was built upon previously established—even commonplace—ideas related to subtraction and the number line. For example, reasoning about subtraction as a take-away process is a commonplace mathematical idea that may not be highly valued by mathematics teacher educators (Nickerson & Whitacre, 2007); however, in CMP4, this idea was integral to PTs’ justifications of nonstandard subtraction strategies, and thus contributed to the class reasoning flexibly about differences. Vignettes 4.1 and 4.2 also illustrated the discursive theme of PTs interpreting children’s strategies. The discourse shifted from the PTs performing computations and justifying their own strategies to the PTs interpreting and justifying children’s strategies. Although it certainly would have been possible for the class to discuss children’s strategies earlier, the nature of the discussions that took place was indicative of the progress that the class had made. The PTs were able to make sense of children’s nonstandard strategies by building upon a foundation of as-if-shared ideas.

Discussion and conclusion

The purpose of this study was to investigate the process of number sense development in a Number and Operations course for PTs. The specific learning goals that were the focus of this study concerned PTs’ improved flexibility in reasoning about whole-number composition, sums, and differences. Such flexibility, supported by deep understanding of number composition and properties of operations, is essential preparation for the meaningful teaching of elementary mathematics. It is especially necessary given con-temporary standards, which emphasize flexible strategy application and looking for and making use of structure (NGA & CCSSO, 2010).
This study has documented a case of PTs’ number sense development viewed in terms of the evolution of collective activity. The class progressed from the initial mathematical practice of using standard algorithms by default to the practices of reasoning flexibly about sums and differences. This evolution in collective activity constitutes a viable learning trajectory. In the paragraphs that follow, I highlight note-worthy characteristics of this learning trajectory. I also use the findings to raise questions concerning the role of learners’ prior knowledge in the evolution of collective activity, particularly in terms of the starting place for instruction and which knowledge is valued and focused upon in analyses of learning trajectories.

Number sense development as a cumulative learning process

In contrast to the differential equations class studied by Stephan and Rasmussen (2002), the CMPs in the Number and Operations course were found to be more or less sequential in both time and structure.: The CMPs emerged and became established in chronological order, and for the most part they consisted of distinct as-if-shared ideas (see Appendix A). Collective activity in the Number and Operations course proceeded from more familiar to less familiar ideas and strategies—all related to familiar topics in elementary number and operations.

Number sense development in this class for PTs was not a process of “unlearning,” as some authors have suggested (Utley & Reeder, 2012, p. 1). Rather, it involved building upon prior knowledge to establish as-if-shared ideas, which came to be used in novel ways. The salient collective activity tended to progress from the standard to nonstandard end of the strategy continuum. This is not to say that the PTs left their standard and Transition strategies behind, but instead that progress was cumulative and there was considerable diversity in the reasoning of individual PTs as they participated in each CMP. These individual differences are explored in a separate article (Whitacre, 2015), which documents individual changes in terms of qualitatively
different ways in which PTs became more flexible and expanded their repertoires of strategies. The analysis of collective activity, especially focusing on whole-class discussions, may tend to highlight new or particularly salient ideas that merited discussion at specific points in the instructional sequence. This focus should not be mistaken for the position of the class along a developmental trajectory in which older, less mature strategies are more or less abandoned in favor of more advanced ones (e.g., Carpenter et al., 1999; Clements & Sarama, 2009).

Student population and starting place in the study of learning trajectories

Researchers have documented viable learning trajectories for elementary students that involve the gradual development of informal and invented strategies prior to learning the standard algorithms (e.g., Carpenter et al., 1999), and such recommendations are reflected in the Common Core State Standards for Mathematics (NGA & CCSSO, 2010). PTs in mathematics content courses approach their learning from a fundamentally different starting place because they have long since learned the standard algorithms and often grown dependent upon them. Rather than ignoring that fact and attempting to transplant and condense a learning trajectory for children to one for PTs, the instructor in this course met her students where they were. She allowed them to use their prior mathematical knowledge, including their knowledge of algorithms, in their problem solving. From this initial mathematical practice, the collective activity evolved to practices involving reasoning flexibly about sums and differences and using a variety of strategies. Thus, the goal of flexibility development was accomplished, but the learning trajectory was faithful to the student population.

It is convenient to study learning trajectories in contexts in which students have not already learned the relevant algorithms. In practice, however, reality may not cooperate. In the elementary grades and beyond, students often learn algorithms prematurely or without
meaningful connections, and they become dependent upon those algorithms (Reys et al., 1999). The study of learning trajectories for goals such as flexibility development necessitates that students’ relevant prior knowledge be taken into consideration, even if researchers regard that knowledge as constituting an undesirable starting place. It is a challenge for researchers, instructional designers, and educators to take such a realistic perspective and choose not to sweep certain of students’ relevant prior knowledge under the rug.

In terms of being familiar with, and often dependent upon, the standard arithmetic algorithms, PTs may be similar to other student populations. A major difference is that elementary PTs, given their career path, take courses that involve revisiting elementary mathematics from a new perspective. These courses would likely be beneficial to a much wider range of students, but it is not typical for other students to take such courses.

Building on prior knowledge of standard algorithms and common-sense ideas

As also discussed in Whitacre and Nickerson (2016b), standard algorithms are not the enemy. Instead, PTs can make productive use of their prior knowledge of standard algorithms as their number sense develops and they come to reason more flexibly about numbers and operations. Likewise, commonsense ideas, such as reasoning about subtraction as a take-away process, prove to be valuable resources for use in PTs’ arguments, despite the fact that such ideas may not appear especially sophisticated or desirable. Finally, the progress in activity related to place value that enabled the class to justify Transition strategies also afforded justification of the standard addition and subtraction algorithms. Once these shifted from authoritative status to being established on the basis of as-if-shared ideas, the class progressed further to activity that involved reasoning flexibly about sums and differences. Thus, it is misleading to say that PTs are dependent on the standard algorithms despite not understanding them; rather, I believe they are
dependent on these algorithms because they do not understand them. The ability to justify the standard algorithms may go hand in hand with PTs’ improved flexibility. This goal does not require that the use of standard algorithms be outlawed at any time.

Unprecedented justifications in the evolution of CMPs

In classrooms that are characterized by accountable argumentation (Horn, 2008), it is fitting to view progress in collective activity as being propelled by a process of establishing and building upon as-if-shared ideas (Rasmussen & Stephan, 2008). This evolution in collective activity involves a bootstrapping process. Once this process gets going, as-if-shared ideas are used as warrants or backings in arguments that lead to the establishment of new ideas (used initially as claims), which in turn may come to function as if shared. To call this a bootstrapping process means that no further external input is necessary: As-if-shared ideas beget further as-if-shared ideas. However, this process has to start somewhere. The initial justifications that students offer depend on warrants that are likely familiar to class members, not because they were established earlier in the semester, but because they belong to common experiences (often, though not necessarily, from school mathematics). For example, the take-away meaning for subtraction did not need to have been previously discussed in the class in order to be used to justify a nonstandard subtraction strategy, because it is a generally familiar mathematical idea for US students (e.g., Whitacre, Schoen, Champagne, & Goddard, 2016).

Any bootstrapping process requires an initial input. Unprecedented justifications provide that initial input. By drawing on experiences that are likely to have been shared by their class members, students make arguments that begin to establish mathematical ideas that become important as the course progresses. These ideas might be regarded as analogous to a set of axioms. By assuming a small number of reasonable statements, mathematicians have been able
to construct elaborate theories (e.g., Euclid’s Elements). Analogously, by assuming a few common-sense mathematical ideas or appealing temporarily to the authority of familiar algorithms, a class may then develop the momentum needed to make dramatic progress by establishing further as-if-shared ideas and participating in increasingly sophisticated CMPs.

Limitations

The knowledge required to teach mathematics effectively is multifaceted (Ball et al., 2008). This study focused on PTs’ number sense development. Although the activities in the class included analysis of children’s mathematical thinking, these activities were done primarily to leverage PTs’ interest in children to contribute to the learning of mathematics content (Philipp, 2008). These PTs were several years away from becoming teachers, and their subsequent experiences (in further content courses, methods courses, field experience, and elsewhere) can be expected to exert considerable influence over their preparation to teach elementary mathematics. It would be naïve to suppose a deterministic relationship between PTs’ experience in the Number and Operations course and the quality of their teaching years into the future. There are necessarily limitations to the lasting changes that may be accomplished in a single semester and that may be captured in research over such duration. The evolution of collective activity in the course that was studied may be seen as the beginning of a longer journey. Evidence from this and other analyses indicates that the PTs were headed in a productive direction (Whitacre, 2015; Whitacre & Nickerson, 2016a; 2016b), but where they might end up is beyond the scope of this research.

Conclusion
This article contributes to the growing body of literature concerning the mathematical preparation of PTs, as well as the literature concerning number sense and its development. It answers the call of Mewborn (2001) for videotape studies of PTs’ mathematical learning by documenting the process of PTs’ number sense development in a Number and Operations course. This research is motivated by findings that many PTs in the United States and elsewhere are not well prepared to teach mathematics (Ball, 1990; Ma, 1999; Newton, 2008; Tsao, 2005; Yang et al., 2009). In content courses like the one that was studied, mathematics educators have the opportunity to foster PTs’ number sense development and thus help PTs become better prepared to teach elementary mathematics effectively. For this reason, analyses that illuminate the processes by which PTs can develop improved number sense are valuable to the field. The results reported here can help to inform instruction in mathematics courses for PTs and, thus, are useful to the mathematics teacher education community.
References


Kazemi, E., Elliott, R., Mumme, J., Carroll, C., Lesseig, K., & Kelley-Petersen, M. (2011). Noticing leaders’ interactions with videocases of teachers engaged in mathematics tasks in


Thanheiser, E. (2010). Investigating further preservice teachers’ conceptions of multidigit whole


Appendix A

As-if-shared Ideas: Occurrences and Criteria Satisfied

<table>
<thead>
<tr>
<th>Mathematical Idea</th>
<th>Occurrences as Claim</th>
<th>Occurrences as Data</th>
<th>Occurrences as Warrant</th>
<th>Occurrences as Backing</th>
<th>Criteria Satisfied</th>
</tr>
</thead>
<tbody>
<tr>
<td>Appealing to the authority of the standard algorithms (Initial Math Practice)</td>
<td>A, b, p, c</td>
<td></td>
<td></td>
<td></td>
<td>Criteria and</td>
</tr>
<tr>
<td>Forming groups, and groups of groups of eight items (CMP)</td>
<td>p, p, p</td>
<td></td>
<td>p</td>
<td></td>
<td>Criteria and</td>
</tr>
<tr>
<td>Packing baskets, boxes, and trucks Andrew’s way (CMP)</td>
<td>p, p</td>
<td></td>
<td>p</td>
<td></td>
<td>Criteria and</td>
</tr>
<tr>
<td>Using and interpreting Andrew’s bookkeeping notation (CMP)</td>
<td>p, p, p</td>
<td></td>
<td>p</td>
<td></td>
<td>Criterion</td>
</tr>
<tr>
<td>Forming groups, and groups of groups, of three items (CMP)</td>
<td>p, p, p, p</td>
<td></td>
<td>p</td>
<td></td>
<td>Criteria and</td>
</tr>
<tr>
<td>Using and interpreting place-value notation in base three (CMP &amp; )</td>
<td>p, p, p</td>
<td></td>
<td>p</td>
<td></td>
<td>Criteria and</td>
</tr>
<tr>
<td>Using and interpreting place-value notation in base ten (CMP &amp; )</td>
<td>p, p, p</td>
<td></td>
<td>A, a</td>
<td></td>
<td>Criterion</td>
</tr>
<tr>
<td>Regrouping from right to left (CMP)</td>
<td>a, p, a</td>
<td></td>
<td>p</td>
<td></td>
<td>Criteria and</td>
</tr>
<tr>
<td>Regrouping in order to subtract (CMP)</td>
<td>p, p, p</td>
<td></td>
<td>d/c</td>
<td></td>
<td>Criterion</td>
</tr>
<tr>
<td>Reasoning about subtraction as a take-away process (CMP)</td>
<td>A, c, A</td>
<td></td>
<td>A, b</td>
<td></td>
<td>Criteria and</td>
</tr>
<tr>
<td>“Borrowing to Build” (CMP)</td>
<td>A, a, a</td>
<td></td>
<td>A</td>
<td></td>
<td>Criterion</td>
</tr>
<tr>
<td>Reasoning about numbers in terms of noncanonical composition (CMP)</td>
<td>A, A, A</td>
<td></td>
<td>A, c/d, b</td>
<td></td>
<td>Criteria and</td>
</tr>
<tr>
<td>Reasoning about addition as a cumulative process A, b (CMP)</td>
<td>A, A, A</td>
<td></td>
<td>A, b</td>
<td></td>
<td>Criteria</td>
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<tr>
<td>Reasoning about subtraction as a cumulative process (CMP)</td>
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<td></td>
<td>A, b</td>
<td></td>
<td>Criteria and</td>
</tr>
<tr>
<td>Reasoning about adding and subtracting in terms of movement along a number line (CMP)</td>
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<td>A, A</td>
<td></td>
<td>Criterion</td>
</tr>
<tr>
<td>Reasoning about the difference as a distance between number-locations (CMP)</td>
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<td></td>
<td>A</td>
<td></td>
<td>Criterion</td>
</tr>
</tbody>
</table>

Argument numbers are used to identify occurrences of ideas. For example, A3.1 refers to the first argument related to addition or subtraction ideas that was made on Day 3. A stands for additive (including both addition and subtraction). P stands for place value.