DISTANCE SPACES

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CHAPTER I
INTRODUCTION

1. Purpose. The purpose of this paper is to record the results of a study of an abstract set upon which a distance function, having certain properties, has been defined. It is assumed that the reader is familiar with the fundamental concepts of set theory.

2. Definitions and Theorems. If in the stating of a certain definition or theorem the wording follows closely that of a particular author, a reference will be given.

Definition 1. A distance function $\rho$ is said to be defined on a set $S$ if and only if $\rho$ is a mapping of $S \times S$ into another set known as a distance set. In this paper the distance set will always be the set of real numbers.

Definition 2. A set $S$ is known as a distance space if a distance function has been defined on $S$.

Definition 3. If $S$ is a distance space with $\rho$ as distance function and $p$ is any element (point) of $S$, then for every real number $r > 0$ there is a subset of $S$ (known as a spherical neighborhood of $p$) denoted by $N_r(p)$ and defined by $N_r(p) = \{ x \in S \mid \rho(p, x) < r \}$.

Definition 4. Let $S$ be an abstract set whose elements
are called points. Let there be some convention according to which certain subsets of $S$ are called open. Then $S$ together with the collection of all open subsets is called a topological space if and only if the collection of open sets satisfies the following axioms:

(a) The null set (denoted by $\emptyset$) is open.

(b) $S$ is open.

(c) The union of any collection of open sets is open.

(d) The intersection of any two (and hence any finite number) open sets is open.

The collection of open sets is called the topology of the topological space.

Definition 5. Let $S$ be a set and $\mathcal{J}$ a collection of subsets of $S$. Then $\mathcal{J}$ is said to generate the collection $\mathcal{V}$ of subsets of $S$ defined as follows: A subset $K$ of $S$ is an element of $\mathcal{V}$ if and only if $K$ is the union of a collection of elements of $\mathcal{J}$. The collection $\mathcal{J}$ is said to be a basis for the collection $\mathcal{V}$ which it generates [2, p. 53].

Theorem 1. Let $S$ be a set and $\mathcal{V}$ a collection of subsets of $S$ called open sets. Let $\mathcal{J}$ be a basis for the collection $\mathcal{V}$. Then $S$ is a topological space with the topology $\mathcal{V}$ if and only if the following conditions hold:

(i) Given $p \in S$, there exists $U \in \mathcal{J}$ such that $p \in U$.

(ii) Given $U, V \in \mathcal{J}$, and any point $p \in U \cap V$, there exists an element $W$ of $\mathcal{J}$ such that $p \in W \subseteq U \cap V$ [2, p. 54, Theorem 1.6].

Definition 6. A topological space $S$ is said to be a
Hausdorff space if and only if, given any two distinct points p, q of S, there exist disjoint open sets U and V of S such that p ∈ U, q ∈ V [2, p. 61].

Definition 7. A subset H of a topological space S is said to be closed if and only if S − H is open [2, p. 62].

Definition 8. A point p of a topological space S is said to be a limit point of a subset H of S if and only if every open set containing p contains a point of H distinct from p. The set of all limit points of H is denoted by H' [2, p. 63].

Theorem 2. A subset H of a topological space S is closed if and only if every limit point of H belongs to H [2, p. 63, Theorem 3.7].

Definition 9. Let A be a subset of a topological space S, and \{A_\alpha\} the collection of all closed subsets of S, each of which contains A. Then the closure of the set A is denoted by \overline{A}, and defined by the equation \overline{A} = \bigcap A_\alpha [2, p. 63].

Theorem 3. Let A be a subset of a space S. Then \overline{A} is a closed set containing A and \overline{A} = A \cup A' [2, p. 63, Theorem 3.9].

Definition 10. Two sets H and K are said to be disjoint if and only if H \cap K = \emptyset [2, p. 35].

Definition 11. A topological space S is regular if and only if, given any closed subset F of S and any point p of S not in F, then there exist disjoint open subsets
Theorem 4. A space $S$ is regular if and only if, given any point $p$ of $S$, and any open set $U$ containing $p$, there exists an open set $V$ containing $p$ such that $p \in V \subseteq U$ [2, p. 65, Theorem 3.16].

Proof. Assume that $S$ is regular. Let $p$ be any point of $S$ and $U$ be any open set such that $p \notin U$. Then $S - U = F$ is a closed set not containing $p$. Since $S$ is regular there exist disjoint open sets $V$ and $G$ such that $p \in V$ and $F \subseteq G$. We know that $V \cap G = \emptyset$ and so that $V \subseteq (S - G)$. Now $S - G$ is closed and so we must have $V \subseteq (S - G)$. Since $F \subseteq G$ it is clear that $(S - G) \subseteq (S - F) = U$. Thus $p \in V \subseteq U$.

Now assume that the condition holds. Let $p$ be any point of $S$ and $F$ be any closed subset of $S$ not containing $p$. Then $S - F = U$ is an open set of $S$ containing $p$. Thus there exists a neighborhood $V$ of $p$ such that $p \in V \subseteq U$. It is clear that the sets $V$ and $S - V$ are disjoint open subsets of $S$ containing $p$ and $F$ respectively. This completes the proof.

Definition 12. A topological space $S$ is said to be normal if and only if, given any two disjoint closed subsets $F_1$ and $F_2$ of $S$, there exist disjoint open subsets $G_1$ and $G_2$ of $S$ containing $F_1$ and $F_2$ respectively [2, p. 110].

Definition 13. Two subsets $A$ and $B$ of a topological space $S$ are said to be separated if and only if $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$, $A \cap \overline{B} = \emptyset$ [2, p. 80].
Definition 14. A topological space $S$ is said to be completely normal if and only if, given any two separated subsets $F_1$ and $F_2$ of $S$, there exist disjoint open subsets $G_1$ and $G_2$ of $S$ containing $F_1$ and $F_2$ respectively [2, p. 110].

Theorem 5. Every completely normal space is normal.

Proof. Let $S$ be a completely normal space. Let $F_1$ and $F_2$ be any two disjoint closed subsets of $S$. Suppose one of the sets, say $F_1$, is the null set. Let $G_1 = \emptyset$ and $G_2 = S$. We thus have disjoint open subsets $G_1$ and $G_2$ of $S$ containing $F_1$ and $F_2$ respectively. Hence the theorem holds for this case.

Now suppose $F_1 \neq \emptyset$ and $F_2 \neq \emptyset$. Since $F_1$ and $F_2$ are closed we know that $F_1 = \overline{F_1}$ and $F_2 = \overline{F_2}$. We know that $F_1 \cap F_2 = \emptyset$ and hence that $\overline{F_1} \cap \overline{F_2} = \emptyset$ and $F_1 \cap \overline{F_2} = \emptyset$. Thus $F_1$ and $F_2$ are separated and since $S$ is completely normal there exist disjoint open subsets $G_1$ and $G_2$ of $S$ containing $F_1$ and $F_2$ respectively. We have now proved that $S$ is also normal.

3. Remarks on Metric Spaces. Let $S$ be a set. $S$ is said to be a metric set if and only if associated with $S$ is a mapping $\mathcal{d} : S \times S \rightarrow \mathbb{R}$ (where $\mathbb{R}$ is the set of real numbers) having the following properties for every $x, y, z$ which are elements of $S$.

(A) $\mathcal{d}(x, y) \geq 0$.

(B) $\mathcal{d}(x, y) = 0$ implies $x = y$.

(C) $x = y$ implies $\mathcal{d}(x, y) = 0$.

(D) $\mathcal{d}(x, y) = \mathcal{d}(y, x)$. 
Throughout the remainder of this paper I will refer to the above as metric properties A, B, C, D and E.

It is clear that a metric set is a distance space and so we may speak of the spherical neighborhoods of the set.

Some of the properties that a metric set $S$ has are the following:

1. $S$, with the collection of subsets of $S$ consisting of all spherical neighborhoods of $S$ as a basis for the topology of $S$, is a topological space.
2. $S$ is a Hausdorff space.
3. $S$ is regular.
4. $S$ is normal.
5. $S$ is completely normal.

Suppose now that we have a set $S$ and associated with it is a mapping $\rho : S \times S \to R$ having only metric properties B, C and E. It can be shown $[4, \text{ p. 26}]$ that these three properties imply the other two.

For if we put $z = y$ in E, we have $\rho (y, y) \leq \rho (x, y) + \rho (x, y)$. From C we know that $\rho (y, y) = 0$ and so we have $0 \leq 2 \rho (x, y)$. Dividing both sides by two, we have that $0 \leq \rho (x, y)$. This is property A.

If we put $z = x$ in E we have that $\rho (y, x) \leq \rho (x, y) + \rho (x, x)$. Since $\rho (x, x) = 0$, the last inequality is just $\rho (y, x) \leq \rho (x, y)$. Now put $x = y$, $y = x$, and $z = y$ in E. We have that $\rho (x, y) \leq \rho (y, x) + \rho (y, y)$ which is
\( \rho(x, y) \leq \rho(y, x) \) since \( \rho(y, y) = 0 \).

Since \( \rho(y, x) \leq \rho(x, y) \) and \( \rho(x, y) \leq \rho(y, x) \),
we must have \( \rho(x, y) = \rho(y, x) \) which is property D.

Thus any set \( S \) with a mapping \( \rho : S \times S \to R \) having
only metric properties B, C and E is still a metric set.
CHAPTER II

PSEUDO-METRIC SPACES

Let \( S \) be a set which has associated with it a mapping \( \rho : S \times S \to \mathbb{R} \) having all the metric properties except \( B \). Such a mapping is known as a pseudo-metric [3, p. 119]. It is clear that \( S \) is a distance space with distance function \( \rho \).

Theorem 1. Let \( S \) be a set with \( \rho \) as pseudo-metric. Denote by \( \mathcal{C} \) the collection of subsets of \( S \) consisting of all spherical neighborhoods of \( S \). Then \( S \), with \( \mathcal{C} \) as a basis for the topology of \( S \), is a topological space.

Proof. I will show that the conditions of Theorem 1 in Chapter I hold.

Let \( p \in S \). Let \( r \) be any real number such that \( r > 0 \). Then \( N_r(p) \) is an element of \( \mathcal{C} \) and \( p \) is an element of \( N_r(p) \) by definition. Thus condition (i) is satisfied.

Now let \( N_r(t) \) and \( N_\varepsilon(q) \) be any two elements of \( \mathcal{C} \). Let \( p \in N_r(t) \cap N_\varepsilon(q) \). Let \( \delta = \min \left( r - \rho(p,t), \varepsilon - \rho(p,q) \right) \). Consider \( N_\delta(p) \). Clearly this is an element of \( \mathcal{C} \) which contains \( p \). Let \( a \) be any element of \( N_\delta(p) \). Then \( \rho(a,p) < \delta \).

By the triangle inequality we know that \( \rho(a,t) \leq \rho(a,p) + \rho(p,t) \). Hence \( \rho(a,t) < \delta + \rho(p,t) \). Since \( \delta \leq r - \rho(p,t) \) we have that \( \rho(a,t) < r - \rho(p,t) + \rho(p,t) = r \). Thus \( a \in N_r(t) \). Likewise \( \rho(a,q) \leq \rho(a,p) + \rho(p,q) \). Since \( \rho(a,p) \)
< \delta < \varepsilon - \rho(p, q), we have that \rho(a, q) < \varepsilon - \rho(p, q) + \rho(p, q) = \varepsilon and so a \in N_{\varepsilon}(q). Since a \in N_{\varepsilon}(t) and a \in N_{\varepsilon}(q) it is clear that a \in N_{\varepsilon}(t) \cap N_{\varepsilon}(q). Thus condition (ii) is satisfied and so S together with the collection \sigma is a topological space.

In general a pseudo-metric space is not Hausdorff since there is a possibility of having two distinct points p and q such that \rho(p, q) = 0.

Theorem 2. Let S be a pseudo-metric space. Then S is regular.

Proof. Let \rho be the pseudo-metric for S. Let F be any closed subset of S and p be any point of S not in F. Let \rho(F, p) = \text{glb} \rho(x, p) for x in F. Then there is a real number \delta > 0 such that \rho(F, p) = 2\delta. For suppose \rho(F, p) = 0. Then for each \varepsilon > 0, there is an x in F such that \rho(x, p) < \varepsilon. Since each point of F is distinct from p, this would make p a limit point of F. This is impossible since F is closed and so \rho(F, p) = 2\delta for some \delta > 0.

Let V = N_{\delta}(p) and W = \bigcup_{x \in F} N_{\delta}(x). Clearly V and W are open sets such that p \in V and F \subseteq W. Suppose that V \cap W \neq \emptyset. Then there is a point q in V \cap W. Since q \in V we know that \rho(q, p) < \delta and since q \in W we know that \rho(q, x) < \delta for some x \in F. Then \rho(x, p) \leq \rho(q, x) + \rho(q, p) < \delta + \delta = 2\delta which is a contradiction. Hence we must have that V \cap W = \emptyset and that S is regular.
It is also true that a pseudo-metric space is normal and completely normal.

Theorem 3. Every pseudo-metric space $S$ is completely normal.

Proof. Let $\rho$ be the pseudo-metric for $S$. Let $A$ and $B$ be any two separated subsets of $S$. Let $p$ be any point of $A$. We know that $p$ is not a limit point of $B$ and so we can find a real number $r_p > 0$ such that $N_{2r_p}(p)$ contains no point of $B$. Let $H = \bigcup_{p \in A} N_{r_p}(p)$. Clearly $H$ is an open set containing $A$.

Similarly we know that if $q$ is a point of $B$, $q$ is not a limit point of $A$. We can then find a real number $r_q$ such that $N_{2r_q}(q)$ contains no point of $A$. Let $G = \bigcup_{q \in B} N_{r_q}(q)$. It is clear that $B \subseteq G$ and that $G$ is an open set.

Suppose $H \cap G \neq \emptyset$. Let $p$ be any point in $H \cap G$. Then there exist points $a \in A$ and $b \in B$ such that $p \in N_{r_a}(a)$ and $p \in N_{r_b}(b)$. This implies that $\rho(a, p) < r_a$ and that $\rho(b, p) < r_b$. By the triangle inequality we know that $\rho(a, b) \leq \rho(a, p) + \rho(p, b) < r_a + r_b$. Suppose that $r_a \leq r_b$. Then $\rho(a, b) < 2r_b$ and so $a \notin N_{2r_b}(b)$ which is impossible. If $r_b \leq r_a$, then $\rho(a, b) < 2r_a$ and so $b \notin N_{2r_a}(a)$ which is also impossible. Thus we know that $H \cap G = \emptyset$ and that the theorem is true.
By Theorem 5 of Chapter I we know that every completely normal space is normal and so it follows that a pseudo-metric space is also normal.
CHAPTER III
SEMIMETRIC SPACES

If a set $S$ has associated with it a mapping $\rho : S \times S \to R$ having all the metric properties except $E$, $S$ is called a semimetric space \[1, \text{p. 7}\]. It is clear that $S$, with $\rho$ as distance function, is also a distance space.

In general a semimetric space $S$, with the collection of subsets of $S$ consisting of all spherical neighborhoods of $S$ as a basis for the open sets of $S$, is not a topological space. This is shown by the following example.

Consider the set $S$ formed by the points of the closed interval $[0, \frac{1}{2}]$ together with the point 1. Define a mapping $\rho : S \times S \to R$ as follows:

- $\rho(x, x) = 0$ for $x \in S$,
- $\rho(x, y) = |y - x|$ for $x, y \in [0, \frac{1}{2}]$,
- $\rho(1, y) = \rho(y, 1) = 1$ for $y \in [0, \frac{1}{2}]$ and $y$ rational,
- $\rho(1, y) = \rho(y, 1) = \rho(0, y)$ for $y \in [0, \frac{1}{2}]$ and $y$ irrational.

It is obvious that metric properties A and C are true for the set $S$. Suppose that $\rho(x, y) = 0$ for some $x, y \in S$. Since $\rho(1, y) > 0$ for $y \in [0, \frac{1}{2}]$, we must have $x = y = 1$ or $x, y \in [0, \frac{1}{2}]$. If the latter case is true, we know that $\rho(x, y)$
\(|y - x| = 0\) implies that \(y = x\). Thus metric property B is true. We know that \(\varrho(1, y) = \varrho(y, 1)\) for all \(y \in S\). For \(x, y \in [0, \frac{1}{4}]\), we know that \(\varrho(x, y) = |y - x| = |x - y| = \varrho(y, x)\). Thus metric property D is also true and we have shown that \(S\) is a semimetric space.

However \(S\), with the spherical neighborhoods of \(S\) as a basis for the open sets of \(S\), is not a topological space. To see this, consider the subset \(W\) of \(S\) defined by the equation \(W = G \cap H\) where \(G = N_{\frac{1}{4}}(0)\) and \(H = N_{\frac{1}{4}}(1)\). Clearly \(G\) and \(H\) are open sets and so if \(S\) is a topological space, \(W\) is also an open set. This means that \(W\) is a spherical neighborhood of a point in \(S\) or that it is the union of spherical neighborhoods of points in \(S\). We know that each spherical neighborhood of each point in \(S\) contains a rational number. But \(W\) is the set of all irrational numbers in the closed interval \([0, \frac{1}{4}]\). Thus \(W\) is not a spherical neighborhood nor is it the union of spherical neighborhoods and so it is not open. Hence \(S\) cannot be a topological space.

In order that a semimetric space be a topological space, it is necessary to place an additional requirement on the distance function. To do this we must define what is meant by the limit of an infinite sequence of points of a semimetric space and what is meant by the continuity of the distance function. Both of these concepts are already defined for topological spaces in general.
Thus, after we have proved that the additional requirement mentioned above is sufficient to ensure that a semimetric space is a topological space, we must also show that there is no discrepancy between the definitions of limit and continuity that we have made for a semimetric space and those of the general topological space.

In the following definitions, S will be a semimetric space with \( \rho \) as semimetric.

Definition 1. A point \( p \) of \( S \) is called a limit of an infinite sequence \( \{ p_n \} \) of points of \( S \) if and only if \( \lim_{m \to \infty} \rho(p_{n_m}, p) = 0 \).

Definition 2. Let \( a = (p, q) \) be an element of \( S \times S \) and let \( \{ p_n \} \) and \( \{ q_n \} \) be any two sequences of points of \( S \) such that \( \lim_{n \to \infty} p_n = p \) and \( \lim_{n \to \infty} q_n = q \). Then \( \rho \) is said to be continuous at \( a \) if and only if \( \lim_{n \to \infty} \rho(p_n, q_n) = \rho(p, q) \).

Theorem 1. Let \( S \) be a semimetric space with \( \rho \) as semimetric. Denote by \( \mathcal{U} \) the collection of all spherical neighborhoods of \( S \). Then \( S \), with \( \mathcal{U} \) as a basis for the topology of \( S \), is a topological space if \( \rho \) is continuous at each point of \( S \times S \).

Proof. I will show that the conditions of Theorem 1 in Chapter I are met.

Let \( p \) be any point of \( S \). Let \( r \) be any real number such that \( r > 0 \). Then \( N_r(p) \) is an element of \( \mathcal{U} \) which
contains p. Thus condition (i) is true.

Let G and H be any two elements of $\mathcal{D}$ and let p be any point such that $p \in G \cap H$. Since G is an element of $\mathcal{D}$ we know that $G = N_{\varepsilon}(t)$ for some $t \in S$ and for some real number $\varepsilon > 0$. For the same reason $H = N_{\delta}(q)$ for some $q \in S$ and for $\delta > 0$.

There must exist some real number $a > 0$ such that $N_a(p) \subset G$. For suppose this is not true. Then for each integer $n$, $N_{1/n}(p)$ contains a point $p_n$ such that $p_n$ is not an element of G. Clearly $\lim_{n \to \infty} p_n = p$. Now $\rho(p_n, t) \geq \varepsilon$ for every $n$ and so $\lim_{n \to \infty} \rho(p_n, t) \geq \varepsilon$. But by the continuity of the distance function, we know that $\lim_{n \to \infty} \rho(p_n, t) = \rho(p, t)$. We then have that $\varepsilon \leq \lim_{n \to \infty} \rho(p_n, t) = \rho(p, t) < \varepsilon$ which is a contradiction.

In a similar fashion it can be proved that there is a real number $b > 0$ such that $N_b(p) \subset H$. There is no loss of generality in assuming that $a \leq b$. Then $N_a(p) \subset N_b(p)$. For if $z \in N_a(p)$, $\rho(z, p) < a \leq b$ and so $z \in N_b(p)$.

Let $W = N_a(p)$. Then W is an element of $\mathcal{D}$ and $W \subset N_b(p) \subset H$. We know that $W \subset G$. Thus $p \in W \subset G \cap H$. This shows that condition (ii) is satisfied and the proof is complete.

I will now show that definition 1 is equivalent to the definition of the limit of a sequence of points of a general topological space which I will call definition 3.
Definition 3. A point $p$ of a topological space $S$ is called a limit of an infinite sequence $\{p_n\}$ of points of $S$ if and only if, given any open set $U$ of $p$, there exists an integer $M$ such that $p_n \notin U$ for every $n \geq M$.

Let $S$ be a semimetric space with $\rho$ as semimetric. Let $\{p_n\}$ be a sequence of points of $S$ which has $p \in S$ as a limit according to definition 1. Let $U$ be any open set containing $p$. We can then find a real number $\varepsilon > 0$ such that $N_{\varepsilon}(p) \subseteq U$. Since $\lim_{n \to \infty} \rho(p_n, p) = 0$, there exists an integer $M$ such that $0 < \rho(p_n, p) < \varepsilon$ whenever $n \geq M$. This means that $p_n \in N_{\varepsilon}(p)$ whenever $n \geq M$.

Since $N_{\varepsilon}(p) \subseteq U$, it is true that $p_n \notin U$ for every $n \geq M$.

Thus the condition of definition 3 is satisfied.

Now assume that $\{p_n\}$ is a sequence of points of $S$ which has $p \in S$ as a limit according to definition 3. Let $\varepsilon$ be any positive number greater than zero. Then $N_{\varepsilon}(p)$ is an open set containing $p$ and so there exists an integer $M$ such that $p_n \in N_{\varepsilon}(p)$ whenever $n \geq M$. Thus $0 < \rho(p_n, p) < \varepsilon$ for each $n \geq M$. Hence $\lim_{n \to \infty} \rho(p_n, p) = 0$ and the condition of definition 1 is satisfied.

It is now necessary to show that definition 2 is equivalent to the definition of continuity as related to the mapping of one topological space into another. The latter definition will be stated as definition 4.

Definition 4. Let $A$ and $B$ be topological spaces and $f : A \to B$ a mapping. Then $f$ is said to be continuous at the point $a$ of $A$ if and only if, given any open subset $G$
of \( B \) such that \( a \in f^{-1}(G) \), there exists an open subset \( V \) of \( A \) such that \( a \in V \subseteq f^{-1}(G) \) \([2, \text{p. 71}]\).

It should be noted here that the cartesian product of a semimetric space \( S \) with itself is a topological space with \( \sigma \times \sigma \) as a basis for its topology, where \( \sigma \) is the collection of all spherical neighborhoods of \( S \) \([2, \text{p. 56, Theorem 1.10}]\). It is also true that \( R \) is a topological space with the collection of all open intervals as a basis for the topology of \( R \) \([2, \text{p. 52}]\).

Let \( S \) be a semimetric space with \( \rho \) as semimetric. Suppose that \( f \) is continuous at \( a = (p, q) \) according to definition 4. Let \( \{p_n\} \) and \( \{q_n\} \) be any two sequences of points of \( S \) such that \( \lim_{n \to \infty} p_n = p \) and \( \lim_{n \to \infty} q_n = q \). I must show that for each \( \varepsilon > 0 \), there exists an \( M \) such that \( |\rho(p_n, q_n) - \rho(p, q)| < \varepsilon \) when \( n \geq M \). Let \( \varepsilon > 0 \) be assigned. Then \( G = (\rho(p, q) - \varepsilon, \rho(p, q) + \varepsilon) \) is an open set in \( R \) and \( a \in f^{-1}(G) \). We know there exists an open subset \( V = W \times H \) of \( S \times S \) such that \( p \in W \), \( q \in H \) and \( f(b, c) \in G \) whenever \( b \in W \) and \( c \in H \). Since \( W \) and \( H \) are open subsets of \( S \), we can find real numbers \( r_p \) and \( r_q \) greater than zero such that \( N_{r_p}(p) \subseteq W \) and \( N_{r_q}(q) \subseteq H \). Since \( \lim_{n \to \infty} p_n = p \) and \( \lim_{n \to \infty} q_n = q \), we can find integers \( M_1 \) and \( M_2 \) such that \( \rho(p_n, p) < r_p \) whenever \( n \geq M_1 \) and \( \rho(q_n, q) < r_q \) whenever \( n \geq M_2 \). Let \( M = \max(M_1, M_2) \). Then \( n \geq M \) implies that \( p_n \in N_{r_p}(p) \subseteq W \) and \( q_n \in N_{r_q}(q) \subseteq H \) so \( (p_n, q_n) \in V \). But \( (p_n, q_n) \in V \).
implies that $\rho(p, q) \leq G$. Hence $n \geq M$ implies that $|\rho(p, q) - \rho(p, q)| < \varepsilon$. Thus $\lim_{n \to \infty} \rho(p_n, q_n) = \rho(p, q)$ and so the condition of definition 2 is satisfied.

Now suppose $\rho$ is continuous at $a = (p, q)$ according to definition 2. Let $G$ be any open subset of $\mathbb{R}$ such that $\rho(p, q) \in G$. I must show that there exists an open subset $V = W \times H$ of $S \times S$ such that $p \in W$, $q \in H$ and $\rho(b, c) \in G$ whenever $b \in W$ and $c \in H$. Since $G$ is an open subset of $\mathbb{R}$ and $\rho(p, q) \not\in G$, we can find a real number $r$ such that $(\rho(p, q) - r, \rho(p, q) + r) \subset G$. Now suppose the condition is not true. Then for every integer $n$, we can find points of $S$, $p_n$ and $q_n$, such that $p_n \in N_{1/n}(p)$, $q_n \in N_{1/n}(q)$ and $\rho(p_n, q_n) \not\in G$. Clearly $\lim_{n \to \infty} p_n = p$, $\lim_{n \to \infty} q_n = q$ and so $\lim_{n \to \infty} \rho(p_n, q_n) = \rho(p, q)$. Since $\rho(p_n, q_n) \not\in G$, we have that $|\rho(p_n, q_n) - \rho(p, q)| \geq r$ for each $n$. This is a contradiction since we know that there is an $M$ such that $n \geq M$ implies $|\rho(p_n, q_n) - \rho(p, q)| < r$. Thus the condition of definition 4 must be satisfied.

Theorem 2. Let $S$ be a semimetric space with semimetric $\rho$. Then $S$ is a Hausdorff space if $\rho$ is continuous at each point of $S \times S$.

Proof. Let $p$ and $q$ be any two distinct points of $S$. Consider $N_{\varepsilon}(p)$ and $N_{\varepsilon}(q)$. Suppose that for every $\varepsilon > 0$, $N_{\varepsilon}(p) \cap N_{\varepsilon}(q) \neq \emptyset$. Then for each positive integer $n$, 

there is an element \( r_\eta \) such that \( \rho(p, r_\eta) < \frac{1}{\eta} \) and \( \rho(q, r_\eta) < \frac{1}{\eta} \). Then the sequence of points \( \{r_\eta\} \) has both \( p \) and \( q \) as limits. But from \( \lim_{\eta \to \infty} r_\eta = p \) and \( \lim_{\eta \to \infty} r_\eta = q \) it follows by the continuity of the distance function that \( \lim_{\eta \to \infty} \rho(r_\eta, r_\eta) = \rho(p, q) \).

Now \( \rho(r_\eta, r_\eta) = 0 \) for every \( n \) and so \( \lim_{n \to \infty} \rho(r_\eta, r_\eta) = 0 \). Thus \( \rho(p, q) = 0 \) which is a contradiction. Thus \( S \) is a Hausdorff space.

**Theorem 3.** Let \( S \) be a semimetric space with \( \rho \) as semimetric. Then \( S \) is regular if \( \rho \) is continuous at each point of \( S \times S \).

**Proof.** Let \( p \) be any point of \( S \) and \( U \) be any open set containing \( p \). We can find \( \varepsilon > 0 \) such that \( N_{\frac{\varepsilon}{2}}(p) \subseteq U \). Let \( G = N_{\frac{\varepsilon}{2}}(p) \) and \( V = N_{\varepsilon}(p) \). Clearly \( V \subseteq G \) for \( q \in V \) implies \( \rho(q, p) < \varepsilon < 2\varepsilon \) and so \( q \in G \). Suppose \( V \) is not contained in \( G \). Then there is a limit point \( q \) of \( V \) such that \( q \notin G \). That is, \( \rho(q, p) \geq 2\varepsilon \). Since \( q \) is a limit point of \( V \) we know that for every integer \( n \) there is a \( q_\eta \in V \) such that \( \rho(q_\eta, q) < \frac{1}{\eta} \). Thus \( \lim_{\eta \to \infty} q_\eta = q \). Now \( q_\eta \in V \) for every \( i \) and so \( \rho(q_\eta, p) < \varepsilon \) for every \( i \). Since the sequence \( \{q_\eta\} \) has \( p \) as a limit, we know by the continuity of the distance function that the sequence \( \rho(q_\eta, p) \) has \( \rho(p, q) \) as limit. But \( \rho(q_\eta, p) < \varepsilon \) for every \( n \) and so \( \lim_{\eta \to \infty} \rho(q_\eta, p) \leq \varepsilon \). We have then that \( \varepsilon \geq \lim_{\eta \to \infty} \rho(q_\eta, p) = \rho(q, p) \geq 2\varepsilon \) which is impossible. Thus \( V \subseteq G \subseteq U \) and so \( S \) is regular.
We have seen that a semimetric space $S$ with continuous distance function is a topological space and that $S$ as a topological space is Hausdorff and regular. One of the questions this paper will leave unanswered is whether or not the continuity of the distance function is sufficient to ensure that $S$ is completely normal or even normal. However, with an additional restriction, $S$ becomes completely normal and hence normal also.

Definition 5. Let $S$ be a semimetric space with $\rho$ as distance function. Then $\rho$ is said to be regularly continuous if and only if for each closed set $B$ of $S$ and any sequence $\{p_n\}$ of points of $S$ such that $\lim p_n = p \in S$, $\lim \rho(p_n, B) = \rho(p, B)$.

It should be noted here that it is possible to have a semimetric space which has a continuous distance function that is not regularly continuous. Consider the following example.

Let $A_0$, $A_1$, and $B$ be sets with elements in $\mathbb{R}$ defined by

- $A_0 = \{a_0 = 0\}$,
- $A_1 = \{a_n | a_n = \frac{1}{n} \text{ for } n = 1, 2, \ldots\}$
- $B = \{b_n | b_n = n \text{ for } n = 2, 3, \ldots\}$

If $A = A_0 \cup A_1$, then the set $S$ defined by $S = A \cup B$ is a semimetric space with semimetric $\rho$ defined by

$$\rho(a_n, b_m) = \rho(b_m, a_n) = \begin{cases} \frac{1}{n} & \text{if } n = m, \\ 1 & \text{if } n \neq m, \end{cases}$$

$$\rho(a_0, b_m) = \rho(b_m, a_0) = 1$$

$$\rho(x, y) = |y - x| \text{ in all other cases.}$$

Clearly $\rho(x, y) \geq 0$ for $x, y \in S$. If $x = y$, then $\rho(x, y)$
If \( \rho(x, y) = 0 \), then we must have \( \rho(x, y) = |y - x| \) and hence that \( x = y \). If \( x \in A \) and \( y \in B \), it is clear that \( \rho(x, y) = \rho(y, x) \) and in all other cases \( \rho(x, y) = |y - x| = |x - y| = \rho(y, x) \). Thus it is true that \( S \) is a semimetric space.

Before proving that \( \rho \) is continuous, it is beneficial to note that for every point \( p \) of \( S \), except a_{ao}, there is a spherical neighborhood of \( p \) which contains no other point of \( S \). For if \( p = b_m \), \( N_{1/\sqrt{m}}(p) \) contains only \( b_m \) and if \( p = a_n \), then \( N_{1/\sqrt{n(n+1)}}(p) \) contains only \( a_n \). It is clear then that \( a_{ao} \) is the only point of \( S \) that is a limit point of \( S \). This implies that if \( \{p_\eta\} \) is a sequence of points of \( S \) such that \( \lim \rho(p_\eta, q_\eta) = 0 \), then \( \rho(p_\eta, q_\eta) = 0 \).

Proof that \( \rho \) is continuous. Let \( \{p_\eta\} \) and \( \{q_\eta\} \) be sequences of points of \( S \) such that \( \lim \rho(p_\eta, q_\eta) = 0 \). There are four cases to consider.

Case 1. Suppose \( p = q = a_{ao} \). Then there exists an integer \( M \) such that \( n > M \) implies \( p_\eta \) and \( q_\eta \) are elements of \( A \) and so \( \rho(p_\eta, q_\eta) = |q_\eta - p_\eta| \) when \( n > M \). Let \( \varepsilon > 0 \) be assigned. Since \( \lim \rho(p_\eta, q_\eta) = 0 \), we can find an integer \( M_1 > M \) such that \( n > M_1 \) implies \( |p_\eta - a_{ao}| < \frac{\varepsilon}{2} \) and \( |q_\eta - a_{ao}| < \frac{\varepsilon}{2} \). Then \( |q_\eta - p_\eta| \leq |q_\eta - a_{ao}| + |p_\eta - a_{ao}| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \) when \( n > M_1 \). Hence \( \lim \rho(p_\eta, q_\eta) = 0 = \rho(p, q) \).
Case 2. Suppose $p = a_0$ and $q_i = q = b_m$ for all $i$ greater than some integer $M$. There exists an integer $M_1$ such that $n > M_1$ implies $p_n \in A$ and $p_n = a_j$ for some $j > m$. Choose $M' = \max (M_1, M)$. Then $n > M'$ implies $\rho (p_n, q_n) = 1$. Hence $\lim_{n \to \infty} \rho (p_n, q_n) = 1 = \rho (p, q)$.

Case 3. Suppose $p = a_0$ and $q_i = q = a_j$ for all $i$ greater than some integer $M$. Let $\varepsilon > 0$ be assigned. There exists an integer $M'$, such that $n > M'$ implies $p_n \in A$ and $p_n = a_k$ for some $k > \max (j, \frac{1}{\varepsilon})$. Let $M'' = \max (M', M)$. Then $n > M''$ implies $|\rho (p_n, q_n) - a_j| = |a_j - a_k| = a_j - a_k \cdot$ Now $k > j$ implies $a_j > a_k$ and so that $|a_j - a_k| = a_j - a_k$. Thus $|\rho (p_n, q_n) - a_j| = |a_j - a_k - a_j| = a_k = \frac{1}{k} < \varepsilon$ and so $\lim_{n \to \infty} \rho (p_n, q_n) = a_j = \rho (a_0, a_j) = \rho (p, q)$.

Case 4. Suppose $p_i = p$ and $q_i = q$ for all $i$ greater than some integer $M$. Then $n > M$ implies $\rho (p_n, q_n) = \rho (p, q)$. Thus $\lim_{n \to \infty} \rho (p_n, q_n) = \rho (p, q)$. We have now shown that $\rho$ is continuous.

Proof that $\rho$ is not regularly continuous. The subset $B$ of $S$ is closed since no point of $S$ is a limit point of $B$. The sequence $\{a_n\}$ of points of $S$ has the property that $\lim a_n = a_0 \in S$. It is clear that $\rho (a_0, B) = 1$ since $\rho (a_0, b_n) = 1$ for each $n$. But $\lim_{n \to \infty} \rho (a_n, B) = 0$. To see this, let $\varepsilon > 0$ be assigned. I must show that there exists an integer $M$ with the property that $n > M$ implies that $\rho (a_n, B) < \varepsilon$. Let $M \neq 1$ be any integer such that $M > \frac{1}{\varepsilon}$. 
Let \( a_m \) be an element of \( \{a_n\} \) such that \( m > M \). Then 
\[
\rho(a_m, B) \leq \rho(a_m, b_m) = \frac{1}{m} < \frac{1}{M} < \epsilon.
\]
This completes the proof that \( \rho \) is not regularly continuous.

**Theorem 4.** Let \( S \) be a semimetric space with continuous distance function \( \rho \). If \( \rho \) is regularly continuous, then \( S \) is completely normal.

**Proof.** Let \( A \) and \( B \) be two separated subsets of \( S \).
For each point \( p \in A \), it is true that \( \rho(p, B) = r_p > 0 \).
If this were not so, \( p \) would be a limit point of \( B \) which is impossible since a closed set contains all of its limit points.

There must be an integer \( n = n_p \geq 2 \) such that \( q \in N_{\frac{r_p}{n_p}}(p) \) implies that \( \rho(q, B) \geq \frac{r_p}{2} \). Suppose that this is not true. Then for each integer \( n \) there exists a point \( q_n \) such that \( q_n \in N_{\frac{r_p}{n_p}}(p) \) and \( \rho(q_n, B) \leq \frac{r_p}{2} \).

It is clear that \( \lim_{n \to \infty} q_n = p \) and since \( \rho \) is regularly continuous, \( \lim_{n \to \infty} \rho(q_n, B) = \rho(p, B) \).
But \( \rho(q_n, B) \leq \frac{r_p}{2} \) for each \( n \) and so \( \lim_{n \to \infty} \rho(q_n, B) \leq \frac{r_p}{2} \).
We now have that \( \frac{r_p}{2} \leq \lim_{n \to \infty} \rho(q_n, B) = \rho(p, B) = r_p \) which is impossible.

Let \( A_p = N_{\frac{r_p}{n_p}}(p) \) and let \( G = \bigcup_{p \in A} A_p \). It is clear that \( G \) is an open set containing \( A \).

Similarly, for each \( q \in B \), we can find a spherical
neighborhood \( B_q = N_{\frac{r_q}{n_q}}(q) \) such that \( p \in B_q \) implies

\( \rho(p, \overline{A}) > \frac{r_q}{n_q} \). As before \( n_q \) is an integer such that

\( n_q \geq 2 \) and \( r_q = \rho(q, \overline{A}) > 0 \). Then \( H = \bigcup_{q \in S} B_q \) is an open

set containing \( B \).

Suppose that \( G \cap H \neq \emptyset \). Let \( t \) be any point such

that \( t \in G \cap H \). Then there must exist points \( a \in A, b \in B \)
such that \( t \in A_a \) and \( t \in B_b \). Now \( t \in A_a \) implies that

\( \rho(t, a) < \frac{r_a}{n_a} \) and that \( \rho(t, \overline{B}) > \frac{r_b}{n_b} \). Likewise, \( t \in B_b \)
implies that \( \rho(t, b) < \frac{r_b}{n_b} \) and that \( \rho(t, \overline{A}) > \frac{r_a}{n_a} \). It
is clear that \( \rho(t, \overline{A}) \leq \rho(t, a) \) and that \( \rho(t, \overline{B}) \leq \rho(t, b) \).

Thus we have

\( \frac{r_b}{n_b} < \rho(t, \overline{A}) \leq \rho(t, a) < \frac{r_a}{n_a} \leq \frac{r_a}{n_a} \) and

\( \frac{r_a}{n_a} < \rho(t, \overline{B}) \leq \rho(t, b) < \frac{r_b}{n_b} \leq \frac{r_b}{n_b} \).

Adding these inequalities we can obtain

\[ \frac{r_b + r_a}{2} < \frac{r_a + r_b}{2} \]

which is a contradiction. Thus \( G \cap H = \emptyset \) and so \( S \) is com-
pletely normal.

By Theorem 4 of Chapter I, we know that any topological
space which is completely normal is also normal. Thus a semi-
metric space \( S \) with continuous distance function \( \rho \) is normal if

\( \rho \) is also regularly continuous.

Now let \( S \) be a semimetric space with \( \rho \) as distance
function. Denote by $\mathscr{J}$ the collection of all spherical neighborhoods of $S$. Suppose that $S$, with $\mathscr{J}$ as a basis for the topology of $S$, is a topological space and that $\mathcal{O}$ is regularly continuous. It is a very interesting but unanswered question as to whether or not these conditions imply that $\mathcal{O}$ is continuous.

