Combinatory Geometry

If ~~one is~~ you are given a theorem in geometry to prove, ~~one~~ you would usually first try to do it by descriptive methods, ~~and~~ if these failed, ~~one~~ you would have to resort to analytic methods.

The theorems to be considered in this lecture are all what may be called incidence theorems, i.e. ~~if~~ given that certain pts lie on certain lines ~~and~~ or conics etc planes, to prove that a certain other pt. lies on a certain other line, conic etc. The method about to be described, which considers how incidences must be combined to form a true theorem, enables one to obtain many such theorems by simple, almost mechanical procedures.

We shall throughout use capital letters to represent points and small letters to represent lines ~~if we are working in 2 dimns and planes if we are working in 3 dimensions~~

Take homog. co-ords $x_1\, x_2\, x_3$ in two dimensions

The invariants may conveniently be written out in the form of a matrix, and to ~~[illegible]~~ prove any ~~geometrical~~ incidence theorem we have to show that if certain determinants of this matrix vanish, then a certain other determinant of the matrix also vanishes.

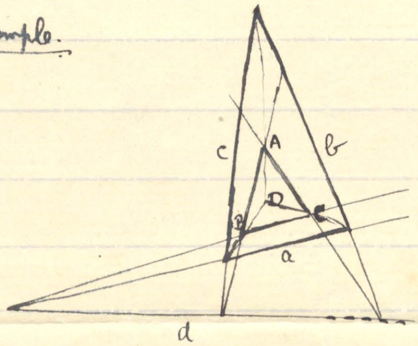
The application of this method is specially easy when all the our conditions are the vanishing of determinants of the 2nd order. Simple example. Theorem of perspective Δs.
The process is quite straightforward but there is rather a lot of writing down.
This may be avoided by the following method. Quad method.

The quads may overlap.

So far we have used only determinants of the 2nd order. The algebra is not so simple when determinants of other orders are used, but the method can in this case ~~to enable~~ easily be adapted to enable us to obtain one theorem from another:
The procedure is like this.

The condition for an incidence of the r^{th} kind between r points and r hyps. is independent of n provided $n > r_r$. ~~This holds for all proj. relations~~. We are, in fact, only interested in the sections of the hyps. with the space which is determined by the points: any other dimensions are irrelevant. We can therefore ~~use the idea of~~ let our small letters represent hyperplanes without stating the no. of dims. in which the hyperplanes lie, it being ~~assumed~~ understood that this no. is greater than the no. of dimensions determined by the points in which we are interested. A hyperplane in this sense is just something which meets any line in a point, any plane in a line, any V_3 in a V_2 etc.

Example.



$$\begin{vmatrix} Ba & Bc \\ Da & Dc \end{vmatrix} = 0 \qquad \frac{Ba.Dc}{Bc.Da} = 1 \qquad \frac{Cb.Da}{Ca.Db} = 1 \qquad \frac{Ac.Db}{Ab.Dc} = 1$$

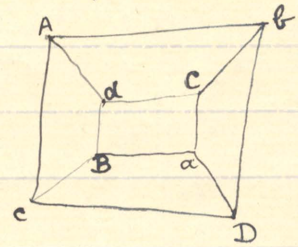
$$\frac{Bd.Ca}{Ba.Cd} = 1 \qquad \frac{Ad.Bc}{Ac.Bd} = 1$$

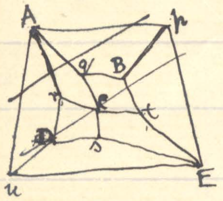
Multiplying we get $\dfrac{Cb.Ad}{Ab.Cd} = 1$

We have not used the fact that the 4 points are coplanar.
In 3 dims. we get theorem of persp. tetrahedra.

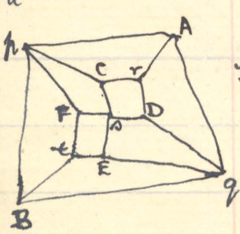
General Theorem which can be stated entirely in terms of incidences of 2^{nd} kind.

The incidences may be represented diagramatically by →



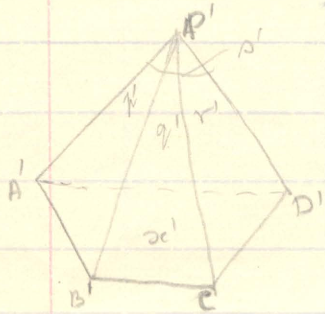


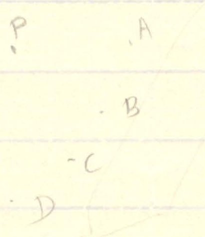
At the ends of each side of one of the quads. ~~there~~ one of the letters represents a point, the other a hyp. Each quad. ~~represents~~ can be regarded as representing one incidence of 2^{nd} kind, the letters at its vertices representing the ~~points~~ elements taking part in the incidence.



The incidences are related in such a way that If all the incidences except one are true the last one is also true, as can be seen from long fraction. Each such diagram gives a theorem. The theorem will, of course, still be true if we make the elements satisfy any additional incidences (Part. base). Suppose for simplicity that the points all lie in 3 dimensions. We may regard the points ~~of~~ as forming the vertices of a polyhedron, the edges of the polydron being formed by joining each pair ~~of~~ of points that are represented by ~~letters~~ ~~at~~ opposite corners of a quad, and each face of the polyhedron containing all the points that correspond to vertices joined to a given vertex (these points are assumed to be coplanar when there are more than 3 of them). ~~In a similar way~~

The theorem may be represented diagramatically by a polyhedron, each vertex of the poly. cor. to a point, each face to a hyp. and each edge to the incidence of 2^{nd} kind between the two points ~~and~~ the ~~2 hyp.~~ which correspond to the two vertices at the end of the edge, and the two hyp. which correspond to the two faces bounded by the edge. Prove that If for any such polyhedron the incidences corresponding to all the edges except one are satisfied, the incidence corresponding to the remaining edge is satisfied. ~~Line~~ Edge P'A' gives represents the incidence ~~which is rep~~ whose analytical





condition is $\dfrac{Pp \cdot Aa}{Pa \cdot Ap} = 1$

Edge P'B' gives $\dfrac{Pq \cdot Bp}{Pp \cdot Bq} = 1$

Edge A'B' gives $\dfrac{Ap \cdot Bx}{Ax \cdot Bp} = 1$

A special case of the foregoing is the theorem of perspective polyhedra, which states—

Line Geometry in 3 Dimensions

1. If co-ords of line $\underline{a}$ are $a_{12}\ a_{23}\ a_{31}\ a_{14}\ a_{24}\ a_{34}$

   define $ab = a_{12}b_{34} + a_{23}b_{14} + a_{31}b_{24} + a_{14}b_{23} + a_{24}b_{31} + a_{34}b_{12}$

2. $ab = 0$ when lines a and b intersect. $aa = 0$ identically.

3. Each line occurs in one row and one column of the matrix.

4. Any projective relation between the lines is represented by an equation which is homogeneous in each ~~line~~ letter, and conversely.

<u>meaning of vanishing of determinants</u>

5. Any determinant of 7^{th} order vanishes identically, but there is no other identity between the terms besides $aa = 0$.

6. A determinant of 6^{th} order vanishes when either the lines that form the rows or those that form the columns belong to a linear complex.

5^{th} order determinants

7. Determinant of 5^{th} order formed by the 5 lines with any other lines vanishes when the 5 lines have two common crossers. Consider $\begin{vmatrix} a & b & c & d & e & x & y \\ k_1 & k_2 & \text{any} & & & & \end{vmatrix}$ $\begin{vmatrix} a & b & c & d & e & x & y \\ \text{any 5 lines} & & & & & k_1 & k_2 \end{vmatrix}$ k_1 & k_2 are crossers

8. Det. of 5^{th} order formed by 5 lines with themselves consider $\begin{vmatrix} a\,b\,c\,d\,e\,k_1\,y \\ a\,b\,c\,d\,e\,k_1\,y \end{vmatrix}$

9. Prove converses of 7 and 8 by considering $\begin{vmatrix} a\,b\,c\,d\,e\,k_1\,k_2 \\ a\,b\,c\,d\,e\,k_1\,k_2 \end{vmatrix}$ & $\begin{vmatrix} a\,b\,c\,d\,e\,k_1\,y \\ a\,b\,c\,d\,x\,k_1\,k_2 \end{vmatrix}$ where k_1 & k_2 meet abcd

<u>4 th order det.</u>

10. Det. formed by 4 lines with any other lines vanishes when the 4 lines lie on a quadric.

   " " " " " themselves vanishes when the lines have co-incident crossers.

   Consider $\begin{vmatrix} a\,b\,c\,d\,k_1\,k_2\,k_3 \\ a\,b\,c\,d\,k_1\,k_2\,k_3 \end{vmatrix}$ where k_3 meets a b and c.

11. The lines $a_1\,a_2\,a_3 \dots$ are projectively related to the lines $b_1\,b_2\,b_3 \dots$ if

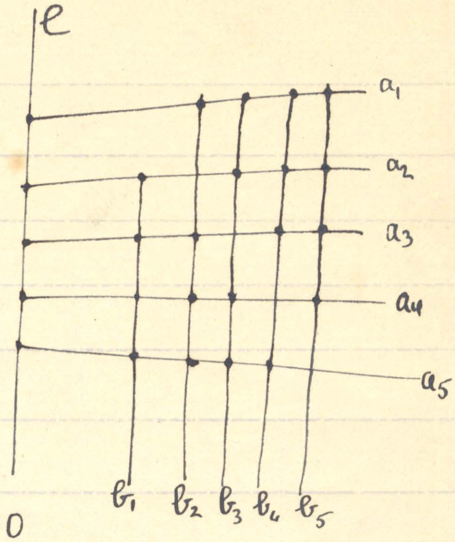
   $\dfrac{a_p a_q . a_r a_s}{a_p a_r . a_q a_s} = \dfrac{b_p b_q . b_r b_s}{b_p b_r . b_q b_s}$ etc. i.e. if $a_p a_q = f(p)\,f(q)\,b_p b_q$

   (~~collineation~~)

12. The projectivity is involuntary if $a_p b_q = f'(p)\,f'(q)\,b_p a_q$

<u>The Double Six.</u>

$$
\cancel{b}\quad
\begin{array}{c|ccccccc|}
 & b_1 & b_2 & a_1 & a_2 & a_3 & a_4 & a_5 \\
b_1 & 0 & \cdot & & 0 & 0 & 0 & 0 \\
b_2 & \cdot & 0 & 0 & & 0 & 0 & 0 \\
a_1 & & 0 & 0 & & & & \\
a_2 & 0 & & & 0 & & & \\
a_3 & 0 & 0 & & & 0 & & \\
a_4 & 0 & 0 & & & & 0 & \\
a_5 & 0 & 0 & & & & & 0
\end{array} = 0
$$



$$-2\, b_1 b_2 . b_1 a_1 . b_2 a_2 \begin{vmatrix} a_1 a_3 a_4 a_5 \\ a_2 a_3 a_4 a_5 \end{vmatrix} + (a_1 b_1)^2 (a_2 b_2)^2 \begin{vmatrix} a_3 a_4 a_5 \\ a_3 a_4 a_5 \end{vmatrix} = 0$$

$$\cancel{b_1 b_2 = \frac{a_1 b_1 . a_2 b_2}{\begin{vmatrix} a_1 a_3 a_4 a_5 \\ a_2 a_3 a_4 a_5 \end{vmatrix}} \frac{\Pi a_i a_k . a_1 a_2}{\Pi a_1 a_k \, \Pi a_2 a_k} = f(1) f(2) . a_1 a_2} \quad \checkmark$$

$$\cancel{b_1 b_2 = a_1 a_2 . a_1 b_1 \, a_2 b_2}$$

Any theorem concerning points and hyperplanes lines gives a theorem in algebra which states that if certain determinants of a matrix vanish, then another determinant vanishes. If we now take a new set of points and hyperplanes lines, and form a new matrix in which each term is the ~~square~~ s^{th} power of an ~~invariant~~ expression (Aa), and apply the algebraic theorem to this matrix then a new geometric theorem will be obtained. A different theorem will be obtained for each different value of n. Most interesting are $s=2$ and $s=-1$.

Any relationship between certain elements will corresponding to a different relationship between the corresponding elements for each different s.

$s=1$		$s=2$		$s=-1$	
$Ap=0$	A incident to p	~~$Ap=0$~~ $A'p'=0$	A' incident to p'	$A''p''=\infty$	must not occur
$\begin{vmatrix} Ap & Aq \\ Bp & Bq \end{vmatrix}=0$	Incidence of 2nd kind between AB pq	$\begin{vmatrix} \overline{A'p'}^2 & \overline{A'q'}^2 \\ \overline{B'p'}^2 & \overline{B'q'}^2 \end{vmatrix}=0$	same relation between $A'B'$ $p'q'$	$\begin{vmatrix} \frac{1}{A''p''} & \frac{1}{A''q''} \\ \frac{1}{B'p''} & \frac{1}{B'q''} \end{vmatrix}=0$	same.
$\begin{vmatrix} Ap & Aq & Ar \\ Bp & Bq & Br \\ Cp & Cq & Cr \end{vmatrix}=0$	Incidence of 3rd kind between ABC pqr.	$\begin{vmatrix} \overline{A'p'}^2 & \overline{A'q'}^2 & \overline{A'r'}^2 \\ \overline{B'p'}^2 & \cdot & \cdot \\ \overline{C'p'}^2 & \cdot & \end{vmatrix}=0$	(Take $p'q'r'$ as axes) $A'B'C'$ lie on a conic self polar to $p'q'r'$ (or dual relation)	$\begin{vmatrix} \frac{1}{A''p''} & \frac{1}{A''q''} & \frac{1}{A''r''} \\ \frac{1}{B''p''} & & \\ \frac{1}{C''p''} & & \end{vmatrix}=0$	$A''B''C''$ lie on a conic circumscribed to $p''q''r''$ (or dual)
$\begin{vmatrix} Ap & Aq & Ar & As \\ Bp & & & \\ Cp & & & \\ Dp & & & \end{vmatrix}=0$	I.c. of 4th kind between $ABCD$ $pqrs$	$\begin{vmatrix} \overline{A'p'}^2 & \overline{A'q'}^2 & \overline{A'r'}^2 & \overline{A's'}^2 \\ \overline{B'p'}^2 & & & \\ \overline{C'p'}^2 & & & \\ \overline{D'p'}^2 & & & \end{vmatrix}=0$	$A'B'C'D'$ lie on a quadric self polar to $p'q'r's'$	$\begin{vmatrix} \frac{1}{A''p''} & \cdot & \cdot & \frac{1}{A''s''} \\ \vdots & & & \\ \frac{1}{D''p''} & \cdot & \cdot & \end{vmatrix}=0$	$A''B''C''D''$ lie on a cubic which passes through the edges of tetr. $p''q''r''s''$

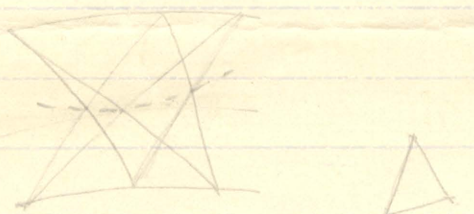
etc.

Different results will be obtained by choosing different elements ~~of~~ in the original theorem to represent the rows and columns of the matrix (it must, of course, be possible to state all the incidences in the ~~fig~~ theorem in terms of the elements chosen). Hence it is possible to obtain several different theorems from any given theorem for each value of s.

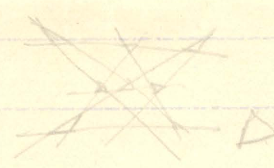
A certain general method~~s~~ can be obtained by this

(1) Suppose ~~the or the relations~~ all in the original theorem is regarded as consisting entirely of relations of ~~a~~ the type; — 3 points are collinear.

Then the matrix will be formed by all the points in the figure and 3 arbitrary hyps, and the analytical condition which expresses that 3 points are collinear will be the vanishing of the det. ~~with~~ having those three points as columns and the 3 arb. hyps. as rows. Then applying the ~~method~~ correspondence for $s=2$ we find that the theorem will still be true when each straight line joining three points is replaced by a conic through the 3 points self conj. to ~~the~~ any fixed triangle. ~~formed by any 3 lines.~~ Similarly for $s=-1$ - - - - . The points and conics need not be coplanar provided we replace the three ~~pla~~ sides of the triangle by 3 planes, and the conics are self conj. to the triangles formed by ~~their~~ the intersections of their planes with the 3 fixed planes.

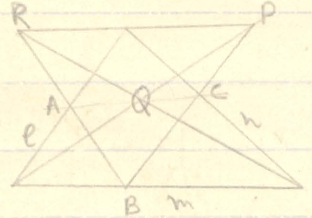


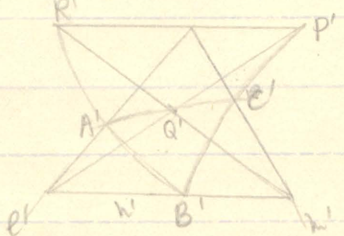
Dual



If conics can be circumscribed to 8 of the little Δ's and a fixed Δ, the a conic can be inscribed to 9th & the fixed Δ

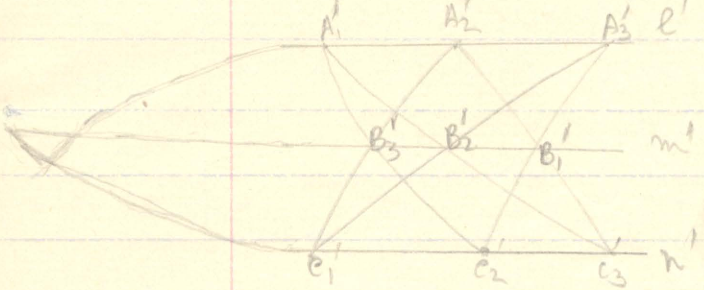
Other theorems obtained from Pappus' Theorem





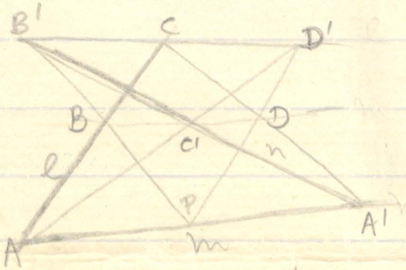
Al Bm Cn
PQlm QRmn RPnl
P'B'C' Q'C'A' R'A'B'

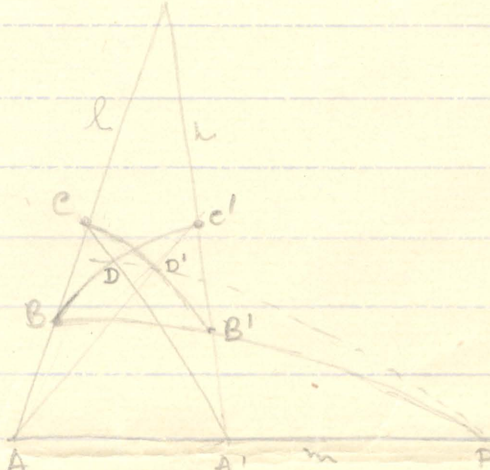
If we have two triangles one inscribed to the other (lmn), an 3rd Δ can be drawn circumscribed to the 1st, inscribed to 2nd PQR - - - - - - - - - , a 'conical Δ' to - - - - - - - - . each side of the conical Δ being a conic self conj. to the 1st Δ

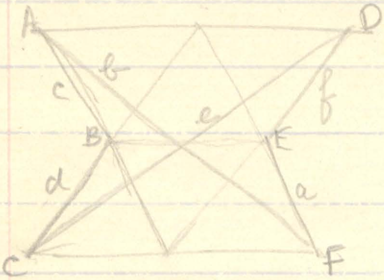


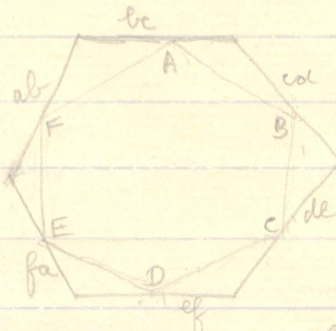
If sets of 3 points on two sides of a triangle are joined by ~~lines~~ conics self conj. to the Δ in such a way that the intersections of two pairs of conics lie on the 3rd side of Δ, then the intersection of the 3rd pair of conics also lies on this side

$A_1 l$, $A_2 l$, $A_3 l$, $B_1 m$ ... $C_3 n$
$A_1B_3C_2$, $A_1B_2C_3$ ...



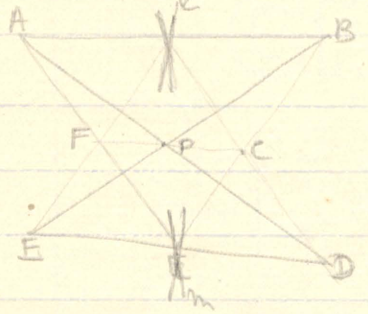


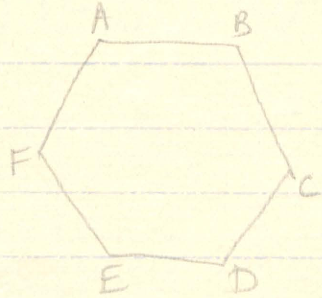




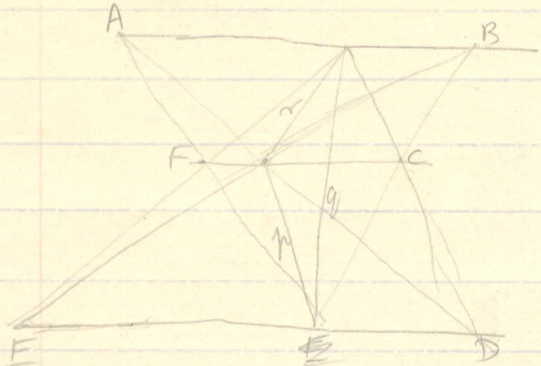
AD meets ad
BE meets be
CF " cf

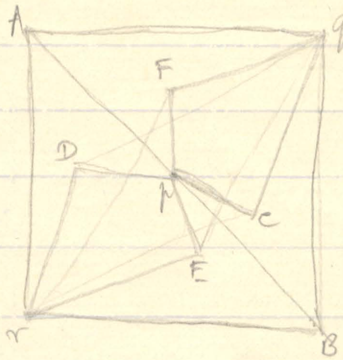
If a skew hex^n A in 5 dimension has its vertices on the edges of the dual of a skew hexagon B, and corresponding diagonals of the two hexagons intersect. If 3 quadrics can be drawn touching the ~~dual~~ hexagon B and self conj to 3 of the tetrahedra formed by 3 sets of 4 vertices of the ~~original~~ ~~quad~~ hex A, then a quadric can be drawn touching the ~~dual~~ hex. B and self conj. to any tetrahedron formed by 4 vertices of the hex. A

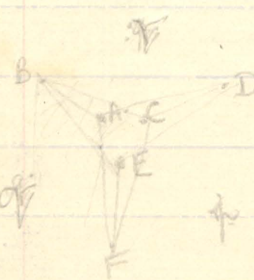


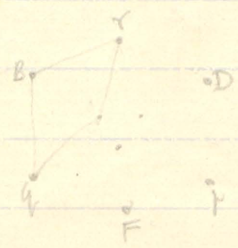


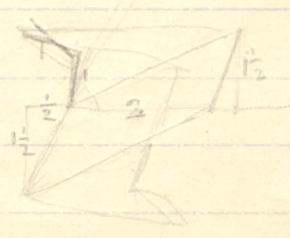
We have a cubic surface passing through the edges of a and a skew hexagon ABCDEF with its vertices on the cubic and each edge passing through one or other of two non-intersecting edges of the tetrahedron. If a point P is chosen such that conics can be drawn through P, A, D, and any 3 intersecting edges of the, and conics can be drawn through PBE and — — — —, then conics can be drawn through PCF & — — — —











$2\frac{1}{2}$, $1\frac{1}{2}$