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Algorithms for Solving Linear Differential Equations with Rational Function Coefficients

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ALGORITHMS FOR SOLVING LINEAR DIFFERENTIAL EQUATIONS WITH RATIONAL FUNCTION COEFFICIENTS

By

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LIST OF SYMBOLS

The following list of symbols are commonly used throughout this thesis.

R	a commutative ring with identity
K	a field
∂	a derivation, generally $\partial = \frac{d}{dx}$
$K(x)[\partial]$	the ring of differential operators over $K(x)$
GHDO	Gauss hypergeometric differential operator
${}_2\mathrm{F}_1(a_1, a_2; b_1; x)$	a Gauss hypergeometric function with parameters $a_1, a_2, b_1 \in \mathbb{Q}$
L	a differential operator
L_{inp}	an input differential operator
L_B	a Gauss hypergeometric differential operator (GHDO)
t_s	the local parameter of the point s
$V(L)$	the solution space of the operator L
$\Delta(L, p)$	an exponent-difference of L at p
α_0, α_1, α_∞	the exponent-differences of a GHDO at 0, 1, ∞ respectively
$\xrightarrow{f}_C$	a change of variable transformation with a pullback function f
$\xrightarrow{r}_E$	an exp-product transformation with a parameter r
$\xrightarrow{r_0, r_1, \ldots, r_{n-1}}_G$	a gauge transformation with parameters $r_0, r_1, \ldots, r_{n-1}$
$\mathcal{G}$	the operator which corresponds to the gauge transformation $\xrightarrow{r_0, r_1, \ldots, r_{n-1}}_G$
$L_{1/x}$	the operator obtained from L via $\xrightarrow{1/x}_C$
ℓ	a good prime number
d_f	the degree of a pullback function f
a_f	the algebraic degree of a pullback function f
$a_f max$	a bound for the algebraic degree a_f of a pullback function f
e_p	the ramification order of p
$\text{TrueSing}(L)$	the set of all non-removable singularities (true singularities) of L
n_{true}	the number of non-removable singularities of a differential operator
n_{diss}	the number of disappeared singularities of a differential operator
e_{inp}	a list of exponent-differences of L_{inp} at its non-removable singularities
e_{app}	a list of exponent-differences of L_{inp} at its apparent singularities
$v_s(y)$	the valuation of the function y at s
$\text{LCLM}(L_1, L_2)$	the least common left multiple of L_1 and L_2
$\text{GCRD}(L_1, L_2)$	the greatest common right divisor of L_1 and L_2

ABSTRACT

This thesis introduces two new algorithms to find hypergeometric solutions of second order regular singular differential operators with rational function or polynomial coefficients. Algorithm 3.2.1 searches for solutions of type

$$\exp\left(\int r\,dx\right) \cdot {}_2\mathrm{F}_1(a_1, a_2; b_1; f)$$

and Algorithm 5.2.1 searches for solutions of type

$$\exp\left(\int r\,dx\right) \left(r_0 \cdot {}_2\mathrm{F}_1(a_1, a_2; b_1; f) + r_1 \cdot {}_2\mathrm{F}_1'(a_1, a_2; b_1; f)\right)$$

where $f, r, r_0, r_1 \in \overline{\mathbb{Q}(x)}$, $a_1, a_2, b_1 \in \mathbb{Q}$, and ${}_2\mathrm{F}_1$ denotes the Gauss hypergeometric function. The algorithms use modular reduction, Hensel lifting, rational function reconstruction, and rational number reconstruction to do so. Numerous examples from different branches of science (mostly from combinatorics and physics) showed that the algorithms presented in this thesis are very effective. Presently, Algorithm 5.2.1 is the most general algorithm in the literature to find hypergeometric solutions of such operators.

This thesis also introduces a fast algorithm (Algorithm 4.2.4) to find integral bases for arbitrary order regular singular differential operators with rational function or polynomial coefficients. A normalized integral basis for a differential operator (output of Algorithm 4.3.1) provides us transformations that convert the differential operator to its standard forms (output of Algorithm 5.1.1) which are easier to solve.

CHAPTER 1

INTRODUCTION

1.1 Closed Form Solutions

Linear homogeneous differential equations with rational function or polynomial coefficients are very common and they play an important role in mathematics, physics, combinatorics, and other branches of science. Many scientists depend on computer algebra systems (Maple, Mathematica, Matlab, etc.) to solve the equations that they encounter during their research. Nowadays, computer algebra systems contain numerous effective procedures to match an equation with a well-known equation in databases and textbooks such as [19, 29, 30, 48]. A huge advantage of these programs is that they take much less time than searching the library. Nevertheless, there are also classes of well-known functions for which current solvers are incomplete. For example, computer algebra systems often fail to find hypergeometric solutions of differential equations.

Mathematicians and computer scientists have been developing algorithms to find solutions of linear differential equations with rational function coefficients. Finding closed form solutions, which are solutions that are expressible in terms of well-studied elementary and special functions, of such differential equations is a challenging and an intriguing research area in computer algebra and computational differential algebra. With the virtue of effective factorization and simplification methods [38, 42, 43], it is a reasonable approach to focus on developing algorithms to solve second order equations. Through the years, several effective algorithms were developed to find Liouvillian solutions, Bessel solutions, Kummer solutions, and hypergeometric solutions of second order linear differential equations with rational function coefficients [17, 18, 20, 21, 32, 33, 34, 35, 44, 46, 50].

Finding hypergeometric solutions of a second order regular singular (Fuchsian) differential equation with rational function coefficients is an active and interesting research area. Although a general algorithm is still lacking, there are powerful algorithms to find hypergeometric solutions under some restrictions. For example, [20, 21] find hypergeometric solutions if there is a so-called 2-descent, [35] finds hypergeometric solutions if the so-called pullback function has degree 3, [44] finds hypergeometric solutions if the input equation has four singularities, and [34] finds hypergeometric solutions if the input equation has five singularities with at least one of them being logarithmic.

In this thesis, we present two new algorithms to find hypergeometric solutions of second order regular singular differential equations with rational function coefficients. Algorithm 3.2.1 searches for hypergeometric solutions of type

$$\exp\left(\int r\,dx\right)\cdot {}_2\mathrm{F}_1(a_1,a_2;b_1;f)$$

and Algorithm 5.2.1 searches for hypergeometric solutions of type

$$\exp\left(\int r\,dx\right)\left(r_0\cdot {}_2\mathrm{F}_1(a_1,a_2;b_1;f)+r_1\cdot {}_2\mathrm{F}_1'(a_1,a_2;b_1;f)\right)$$

where $f, r, r_0, r_1 \in \overline{\mathbb{Q}(x)}$ and $a_1, a_2, b_1 \in \mathbb{Q}$. Algorithm 5.2.1 is more general than previous algorithms [20, 21, 34, 35, 44] due to the fact that it has no restrictions neither on the degree of the pullback function f nor the number of singularities of the input equation.

Paper [31] introduced the notion of integral bases for differential equations. In this thesis, we also introduce a fast algorithm (Algorithm 4.2.4) to find an integral basis for an arbitrary order regular singular differential equation with rational function coefficients and an algorithm (Algorithm 4.3.1) to normalize an integral basis at infinity. Normalization procedure is a new addition and it was not mentioned in [31]. A normalized integral basis for a differential equation gives us transformations which simplify the original differential equation to, what we called, its standard forms (Algorithm 5.1.1). It helped us to generalize Algorithm 3.2.1. Algorithm 5.2.1 is a combination of Algorithms 3.2.1, 4.2.4, 4.3.1, and 5.1.1.

1.2 Motivation: CIS Solutions and Globally Bounded Equations

Definition 1.2.1.

- A *Convergent Integer power Series solution (CIS solution)* of a differential equation is a solution $y = \sum_n u(n)x^n \in \mathbb{Z}[x]$ with a positive radius of convergence.
- A differential equation with rational function coefficients which (after a simple scaling) has a CIS solution is called a *globally bounded equation* [14].

Example 1.2.1. The differential equation

$$x\,(1+x)\,(8\,x-1)\,\frac{d^2}{dx^2}y(x)+\left(24\,x^2+14\,x-1\right)\frac{d}{dx}y(x)+(2+8\,x)y(x)=0 \qquad (1.1)$$

is a globally bounded equation because it has a CIS solution

$$y(x) = 1 + 2\,x + 10\,x^2 + 56\,x^3 + 346\,x^4 + 2252\,x^5 + 15184\,x^6 + 104960\,x^7 + 739162\,x^8 + \ldots$$

where the coefficients of $y(x)$ are "Franel numbers" (A000172 in the on-line encyclopedia of integer sequences [1]). Note that (1.1) also admits a hypergeometric solution,

$$y(x) = \frac{1}{1-2\,x} \cdot {}_2\mathrm{F}_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{27\,x^2}{(1-2\,x)^3}\right).$$

Mark van Hoeij and Vijay J. Kunwar observed that many (all?) second order globally bounded equations (coming from [1] and many other sources) have hypergeometric solutions or algebraic solutions. This significant observation gave rise to the following conjecture.

Conjecture 1.2.1. *Every second order globally bounded equation has a hypergeometric solution or an algebraic solution.*

Algebraic solutions of second order globally bounded equations can be found using Kovacic's algorithm [32]. Therefore, it is natural to ask how to find hypergeometric solutions. Currently, the answer of this question is not known. However, there are methods to search hypergeometric solutions of regular singular equations under some restrictions on the number of singularities of the equation or the degree of the so-called pullback function.

There is no regular singular differential equation with only one non-removable singularity. Regular singular equations with two non-removable singularities have Liouvillian solutions and [32] can find them. Equations with three singularities were treated in [20, 21], four singularities were discussed in [44], and five singularities (with at least one logarithmic singularity) were examined in [34]. The paper [35] treats the case where the so-called pullback function has degree 3.

Remark 1.2.1. If we restrict to univariate hypergeometric functions (${}_3\mathrm{F}_2$, ${}_4\mathrm{F}_3$, …) then, Conjecture 1.2.1 does not hold for higher orders (there are examples for order three globally bounded equations [49]). For higher orders, we need to consider A-hypergeometric functions [8].

1.3 Objectives

We want to develop a general procedure, with no restriction on the number of singularities and the degree of the pullback function, to find hypergeometric solutions of second order regular singular differential equations with rational function coefficients. This is motivated by Conjecture 1.2.1 which says that such solutions are common. In the case that Conjecture 1.2.1 turns out to be false, our algorithms could be very helpful for finding a counter example.

1.4 Contributions

The main contributions of this thesis are Algorithms 3.2.1 and 5.2.1 which can find hypergeometric solutions of second order differential equations with rational function coefficients. Presently, Algorithm 5.2.1 is the most general algorithm in the literature. To illustrate the value of hypergeometric solutions, the physics paper [4] gave hypergeometric solutions to differential equations from the Ising Model. At the time, such solutions were hard to find, in fact, their existence was surprising enough for [4] to end up in the "Journal of Physics A highlights of 2012". The algorithms presented in this thesis can solve these equations automatically. Numerous examples coming from different sources (including [1]) showed that the Algorithms 3.2.1 and 5.2.1 are very effective and that hypergeometric solutions are very common in many areas of physics (the Ising model [4], the Feynman diagrams [2]) and combinatorics [12].

Another contribution of this thesis is the fast Algorithm 4.2.2 to compute an integral basis for an arbitrary order regular singular differential equation and Algorithm 4.3.1 to normalize the integral basis at the point at infinity. A normalized integral basis for a differential equation gives us transformations that transforms the equation to another equation (one of its standard forms, Algorithm 5.1.1) which is easier to solve.

1.5 Plan of the Thesis

The plan of this thesis is as follows: Chapter 2 introduces the notion of differential operators and investigates the theory developed around their singularities, exponent-differences, solutions, and transformations between them. Chapter 3 is dedicated to describe our algorithm (Algorithm 3.2.1) to find hypergeometric solutions of second order linear differential equations using quotient of formal solutions around a non-removable singular point. In Chapter 4, integral bases for arbitrary order differential operators are investigated and a fast algorithm (Algorithm 4.2.2) is given to compute these bases. Moreover, Chapter 4 introduces an algorithm (Algorithm 4.3.1) to normalize integral bases for differential operators at the point at infinity. The last chapter, Chapter 5, presents two applications of the softwares given in Chapters 3 and 4. First part of Chapter 5 introduces standard forms of a differential operator (Algorithm 5.1.1). Second section of this chapter, Section 5.2, introduces a very efficient algorithm (Algorithm 5.2.1) to find hypergeometric solutions of second order linear differential equations with rational function coefficients. Maple implementations of our algorithms [23, 24, 25] accompany this thesis.

CHAPTER 2

PRELIMINARIES

Differential operators are the central mathematical objects of interest of this thesis. In this chapter we briefly recall some fundamental definitions and facts about differential operators. In the first section, we investigate solutions of differential operators, singularities, and exponent-differences. The second section is devoted to Gauss hypergeometric differential operator and its solutions. In the third section, we discuss transformations between two second order differential operators. The last section defines hypergeometric solutions and states an important theorem (Theorem 2.4.1) about exponent-differences of a Gauss hypergeometric differential operator and a second order differential operator which are connected via a change of variables transformation. Omitted proofs and more comprehensive information can be found at [7, 16, 17, 19, 20, 29, 34, 40, 48, 50]

2.1 Differential Operators

Definition 2.1.1. Let R be ring and $\partial : R \longrightarrow R$ be an additive homomorphism which satisfies the Leibniz product rule,

$$\partial(rs) = r\partial(s) + s\partial(r)$$

for all $r, s \in R$.

- ∂ is called a *derivation* on R.
- A ring R with a derivation ∂ is called a *differential ring*.
- A field K with a derivation $\partial : K \longrightarrow K$ is called a *differential field*.

Definition 2.1.2. Let R be a differential ring or a differential field with a derivation ∂.

- If $\partial(c) = 0$ for $c \in R$, then c is called a *constant* of R.
- The set of all constants of R is given by

  $$C_R = \ker(\partial : R \longrightarrow R)$$

  and it forms a ring. C_R is called the *ring of constants* of R.

- If R is a differential field, then C_R is a field, the *field of constants* of R.

Example 2.1.1.

1. Let $R = \mathbb{Z}$ or $R = \mathbb{Q}$. Then R is a differential ring with the *trivial derivation* $\partial = 0$. This is the only derivation on R and $C_R = R$.

2. The field of rational functions with complex coefficients $\mathbb{C}(x)$ is a differential field with the derivation $\partial = \frac{d}{dx}$. The field of constants are $C_{\mathbb{C}(x)} = \mathbb{C}$.

Definition 2.1.3. Let K be a differential field with a derivation ∂.

- A *differential operator* over K is defined by

$$L = \sum_{i=0}^{n} A_i \partial^i$$

  where $A_i \in K(x)$. The order of L is given by $\mathrm{ord}(L) = \max(i : A_i \neq 0)$.

- The set of all differential operators over K form a non-commutative Euclidean domain. It is denoted by $K[\partial]$ and it is called the *ring of differential operators over* K.

Remark 2.1.1. There is a one-to-one correspondence between elements of $K[\partial]$ and ordinary linear differential equations over K.

Example 2.1.2. Let $\partial = \frac{d}{dx}$. An n-th order differential operator $L = \sum_{i=0}^{n} A_i \partial^i \in \mathbb{C}(x)[\partial]$ corresponds to an n-th order ordinary linear differential equation over $\mathbb{C}(x)$, which is

$$L(y) = A_n \frac{d^n}{dx^n} y(x) + A_{n-1} \frac{d^{n-1}}{dx^{n-1}} y(x) + \cdots + A_0 y(x) = 0.$$

An n-th order ordinary linear differential equation over $\mathbb{C}(x)$ also corresponds to an element of $\mathbb{C}(x)[\partial]$. So, we can use the terms "differential operator" and "differential equation" interchangeably.

Notation 2.1.1. For the remaining part of this chapter, K will denote a field such that $K \subseteq \mathbb{C}$ and ∂ will denote the derivation $\frac{d}{dx}$.

Definition 2.1.4. A *universal extension* Ω of $K(x)$ is a $K(x)[\partial]$-module such that

1. Ω is a $K(x)$ algebra,

2. for every $L \in K(x)[\partial]$, the differential equation $L(y) = 0$ has $\mathrm{ord}(L)$ linearly independent solutions in Ω,

3. for every $f \in \Omega$ there exist a non-zero differential operator $L \in K(x)[\partial]$ such that $L(f) = 0$.

Universal extension Ω exists and is unique up to isomorphism if K is algebraically closed [40]. Then, the *solution space* of L is defined as

$$V(L) = \ker(L : \Omega \longrightarrow \Omega).$$

$V(L)$ is a vector space over $\overline{K}$ with dimension $\mathrm{ord}(L)$.

Definition 2.1.5. Let $L = \sum_{i=0}^{n} A_i \partial^i \in K(x)[\partial]$ with $\mathrm{ord}(L) = n$ and $s \in \overline{K} \cup \{\infty\}$.

- s is called a *singularity* (or a *singular point*) of L if $A_n(s) = 0$ or s is a pole of one of the non-leading coefficients of L, which are $A_1, \ldots, A_{n-1}$.
- If s is not a singularity, then it is called a *regular point* of L.
- The point $s = \infty$ is a singularity (respectively regular point) of L if 0 is a singularity (respectively regular point) of $L_{1/x}$. Here $L_{1/x}$ is the operator obtained from L via substituting $(x, \partial) = \left(x, \frac{d}{dx}\right) \mapsto \left(\frac{1}{x}, \frac{d}{d\frac{1}{x}}\right) = \left(\frac{1}{x}, -x^2\partial\right)$.

Remark 2.1.2. A singularity of a solution of a differential operator is also a singularity of the differential operator, however, the converse is in general not true.

Definition 2.1.6. A singularity s of of an operator $L \in K(x)[\partial]$ is called an *apparent singularity* if all solutions of L are analytic at s.

Definition 2.1.7. Let $L = \sum_{i=0}^{n} A_i \partial^i \in K(x)[\partial]$ and $s \in \overline{K} \cup \{\infty\}$ be a singularity of L.

- The *local parameter* of s is defined by

$$t_s = \begin{cases} x - s, & \text{if } s \neq \infty \\ \frac{1}{x}, & \text{if } s = \infty. \end{cases}$$

- A finite point s is called a *regular singularity* if

$$\frac{A_{n-i}}{A_n} t_s^i$$

  is analytic at s for every $i \in \{1, \ldots, n\}$.

- If $s = \infty$, then s is a regular singularity if 0 is a regular singularity of $L_{1/x}$.
- If s is not a regular singularity, then it is called an *irregular singularity.*
- A differential operator L which has only regular singularities is called a *regular singular operator* or *Fuchsian operator.*

Remark 2.1.3. For the remaining of this chapter we will only consider second order regular singular differential operators over $K(x)$.

Theorem 2.1.1 ([17, 20, 34, 48, 50])**.** *Let $L \in K(x)[\partial]$ be a second order regular singular operator and s be a regular singularity or a regular point of L. Then, in the neighborhood of s the set $\{Y_1, Y_2\}$ forms a basis of the solution space $V(L)$ as a vector space over $\overline{K}$, where:*

1.

$$Y_1 = t_s^{e_1} \sum_{i=0}^{\infty} a_i t_s^i$$
$$Y_2 = t_s^{e_2} \sum_{i=0}^{\infty} b_i t_s^i$$

with $a_0 \neq 0$, $b_0 \neq 0$, $e_1, e_2, a_i, b_i \in \overline{K}$, and $e_1 \neq e_2$. In this case $\Delta(L, s) = |e_1 - e_2| \in \overline{K}$, or,

2.

$$Y_1 = t_s^{e_1} \sum_{i=0}^{\infty} a_i t_s^i$$
$$Y_2 = t_s^{e_2} \sum_{i=0}^{\infty} b_i t_s^i + Y_1 \cdot \log(t_s)$$

with $a_0 \neq 0$ and $e_1, e_2, a_i, b_i \in \overline{K}$. In this case $\Delta(L, s) = |e_1 - e_2| \in \mathbb{Z}$.

3. If s is an apparent singularity or a regular point, then Y_1 and Y_2 are analytic at s. In this case, for the corresponding e_1 and e_2 we have $\Delta(L, s) = |e_1 - e_2| \in \mathbb{Z}$.

Definition 2.1.8. In the Theorem 2.1.1,

- e_1 and e_2 are called the *exponents* of L at s and the absolute value of the difference of them

$$\Delta(L, s) = |e_1 - e_2|$$

  is called the *exponent-difference* of L at s.
- In Theorem 2.1.1 Item 2, it is said that Y_2 is a *logarithmic solution* of L.
- If L has a logarithmic solution at a singularity s, then s is called a *logarithmic singularity.*

2.2 Gauss Hypergeometric Differential Operator and Hypergeometric Function

This section is devoted for Gauss hypergeometric differential operator and its solutions. For more details see [7, 19, 48].

Definition 2.2.1. Let $a_1, a_2, b_1 \in \mathbb{Q}$ such that $b_1 \notin \{0\} \cup \mathbb{Z}^-$.

- The differential operator

$$L_B = \partial^2 + \frac{b_1 - (a_1 + a_2 + 1)}{x(1-x)}\partial - \frac{a_1 a_2}{x(1-x)} \in \mathbb{Q}(x)[\partial] \tag{2.1}$$

  is called *Gauss hypergeometric differential operator* (or *GHDO* in short).

- Since L_B is a second order operator, it has two linearly independent solutions in the universal extension of $\mathbb{Q}(x)$. One of the solutions of L_B at the regular singular point $x = 0$ is called *Gauss hypergeometric function* and it is given by the infinite Gauss series

$${}_2\mathrm{F}_1(a_1, a_2; b_1; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b_1)_k} \frac{x^k}{k!}.$$

  Here, for $\lambda \in \mathbb{Q}$ the notation $(\lambda)_k$ denotes the *Pochhammer Symbol* which is defined by raising factorials,

$$(\lambda)_k = \begin{cases} 1, & \text{if } k = 0 \\ \lambda(\lambda+1)\dots(\lambda+k-1), & \text{if } k > 0. \end{cases}$$

  Another solution of L_B at $x = 0$ is

$$x^{1-b_1} \cdot {}_2\mathrm{F}_1(a_1 - b_1 + 1, a_2 - b_1 + 1; 2 - b_1; x).$$

Remark 2.2.1. GHDO L_B in (2.1) is regular singular (Fuchsian) and it has three regular singularities at 0, 1, and ∞ with exponents $\{0, 1 - b_1\}$, $\{0, b_1 - a_1 - a_2\}$, and $\{a_1, a_2\}$ respectively. We denote the exponent-differences of L_B as

- $\alpha_0 = \Delta(L_B, 0) = |1 - b_1|$,
- $\alpha_1 = \Delta(L_B, 1) = |b_1 - a_1 - a_2|$,
- $\alpha_\infty = \Delta(L_B, \infty) = |a_2 - a_1|$.

2.3 Transformations Between Differential Operators

Notation 2.3.1. In this section we only consider differential operators in $K(x)[\partial]$ where $K \subseteq \mathbb{C}$ and $\partial = \frac{d}{dx}$.

Definition 2.3.1. Let $L_1 \in K(x)[\partial]$ be an n-th order differential operator. A *transformation* is an operation which converts L_1 to another n-th order differential operator $L_2 \in K(x)[\partial]$, with an equipped map $V(L_1) \to V(L_2)$.

Definition 2.3.2. There are three known type of order-preserving transformations that can be expressed in terms of operators as well as in terms of solutions:

1. *Change of Variables Transformation*:
   Suppose $f \in K(x)$ and $y(x)$ is a solution of an equation $L_1(y) = 0$ with $L_1 \in K(x)[\partial]$. Then the composition $y(f)$ is a solution of an operator L_2 obtained from L_1 by substituting $(x, \partial) \mapsto \left(f, \frac{1}{f'}\partial\right)$. We denote this *change of variables transformation* as

   $$L_1 \xrightarrow{f}_C L_2.$$

   Here f is called a *pullback function.*

2. *Exp-Product Transformation*:
   Suppose $f \in K(x)$ and $y(x)$ is a solution of an equation $L_1(y) = 0$ with $L_1 \in K(x)[\partial]$. Then $y(x) \cdot \exp\left(\int r\, dx\right)$ is a solution of an operator $L_2 \in K(x)[\partial]$ where L_2 is obtained from L_1 by substituting $\partial \mapsto \partial - r$. We denote this *exp-product transformation* as

   $$L_2 = L_1 \circledS (\partial - r)$$

   or

   $$L_1 \xrightarrow{r}_E L_2.$$

3. *Gauge Transformation*:
   Let $r_0, r_1, \ldots, r_{n-1} \in K(x)$ and $y(x)$ be a solution of an equation $L_1(y) = 0$. Let $\mathcal{G} = r_{n-1}\partial^{n-1} + \cdots + r_0$. Then $\mathcal{G}(y)$ is a solution of an operator $L_2 \in K(x)[\partial]$ where L_2 is obtained from L_1 by right-dividing $\mathrm{LCLM}(L_1, \mathcal{G})$ by $\mathcal{G}$. We denote this *gauge transformation* by

   $$L_1 \xrightarrow{\mathcal{G}}_G L_2$$

   or

   $$L_1 \xrightarrow{r_0, r_1, \ldots, r_{n-1}}_G L_2.$$

   We only allow cases with $\mathrm{GCRD}(L_1, \mathcal{G}) = 1$ so that $\mathrm{ord}(L_1) = \mathrm{ord}(L_2)$.

Theorem 2.3.1 ([17])**.**

- *Exp-product and gauge transformations are reflexive, symmetric, and transitive.*
- *Change of variables transformation is reflexive and transitive. If we allow algebraic pullback functions, then change of variables becomes symmetric.*

Theorem 2.3.2 ([17])**.** *Let $L_1, L_2 \in K(x)[\partial]$ be of order n such that*

$$L_1 \xrightarrow{r}_E \xrightarrow{r_0,r_1,\ldots,r_{n-1}}_G L_2.$$

Then, there exist $t, t_0, t_1, \ldots, t_{n-1} \in K(x)$ such that

$$L_1 \xrightarrow{t_0,t_1,\ldots,t_{n-1}}_G \xrightarrow{t}_E L_2.$$

Theorem 2.3.3 ([17])**.** *Let $L_1, L_2 \in K(x)[\partial]$ be of order n such that L_2 is obtained from L_1 with change of variables, exp-produt, and gauge transformations in any order, namely*

$$L_1 \rightarrow L_2.$$

Then, there exist $f, r \in \overline{K(x)}$, $r_0, r_1, \ldots, r_{n-1} \in K(x)$ such that

$$L_1 \xrightarrow{f}_C \xrightarrow{r}_E \xrightarrow{r_0,r_1,\ldots,r_{n-1}}_G L_2.$$

Definition 2.3.3. If a singularity s of $L \in K(x)[\partial]$ becomes a regular point under a combination of exp-product or gauge transformations, then s is called a *removable singularity* (or *false singularity*). Otherwise it is called a *non-removable singularity* (or *true singularity*).

Remark 2.3.1. Logarithmic singularities are non-removable singularities, they stay singular under any combination of all three transformations. Apparent singularities are removable singularities.

2.4 Hypergeometric Solutions

Definition 2.4.1. Let $L \in \mathbb{Q}(x)[\partial]$ be a second order operator. If it exists, a *hypergeometric solution* of L is a solution of the form

$$\exp\left(\int r\,dx\right)\left(r_0 \cdot {}_2\mathrm{F}_1(a_1, a_2; b_1; f) + r_1 \cdot {}_2\mathrm{F}_1'(a_1, a_2; b_1; f)\right)$$

where $f, r, r_0, r_1 \in \overline{\mathbb{Q}(x)}$ and $a_1, a_2, b_1 \in \mathbb{Q}$.

Remark 2.4.1. If a second order $L \in \mathbb{Q}(x)[\partial]$ has a hypergeometric solution, then (by Theorem 2.3.3) there exists a GHDO

$$L_B = \partial^2 + \frac{b_1 - (a_1 + a_2 + 1)}{x(1-x)}\partial - \frac{a_1 a_2}{x(1-x)} \in \mathbb{Q}(x)[\partial]$$

such that

$$L_B \xrightarrow{f}_C \xrightarrow{r}_E \xrightarrow{r_0, r_1}_G L$$

where $f, r, \in \overline{\mathbb{Q}(x)}$ and $r_0, r_1 \in \mathbb{Q}(x)$.

Theorem 2.4.1 ([10])**.** *Let a GHDO $L_B \in \mathbb{Q}(x)[\partial]$ have exponent-differences α_0 at 0, α_1 at 1, and α_∞ at ∞. Suppose*

$$L_B \xrightarrow{f}_C L.$$

Let $p \in \mathbb{C}$ is a point such that $f(p) \in \{0, 1, \infty\}$. Then L has exponent-difference at p

- *$e_p \alpha_0$ if f has a zero at p with multiplicity e_p,*
- *$e_p \alpha_1$ if $f - 1$ has a zero at p with multiplicity e_p,*
- *$e_p \alpha_\infty$ if f has a pole at p with order e_p.*

CHAPTER 3

COMPUTING HYPERGEOMETRIC SOLUTIONS VIA QUOTIENT METHOD

Let $\partial = \frac{d}{dx}$. In this chapter we introduce a heuristic algorithm to compute hypergeometric solutions of a second order regular singular differential operator $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ in the form of

$$\exp\left(\int r\,dx\right) \cdot {}_2\mathrm{F}_1(a_1, a_2; b_1; f) \tag{3.1}$$

where $f, r \in \overline{\mathbb{Q}(x)}$ and $a_1, a_2, b_1 \in \mathbb{Q}$. If a given operator L_{inp} has hypergeometric solutions in the form of (3.1), then there exists a GHDO $L_B \in \mathbb{Q}(x)[\partial]$ such that

$$L_B \xrightarrow{f}_C \xrightarrow{r}_E L_{\text{inp}}. \tag{3.2}$$

Therefore, finding solutions of L_{inp} in the form of (3.1) is equivalent to finding a GHDO L_B and parameters f, r of the transformations such that (3.2) holds. There are formulas and algorithms [41] to recover r, so the crucial part is to find L_B and the pullback function f.

Remark 3.0.1. The contents of this chapter (apart from some new materials and adoptions) have been published in [28].

Notation 3.0.1. Throughout this chapter we use $\partial = \frac{d}{dx}$.

3.1 Quotients of Formal Solutions

Example 3.1.1 (Rational Pullback Function)**.** The differential operator

$$L_{\text{inp}} = 147x(x-1)(x+1)\partial^2 + (266x^2 - 42x - 98)\partial + 20x - 5 \in \mathbb{Q}(x)[\partial] \tag{3.3}$$

has a hypergeometric solution in the form of (3.1) and it is

$$Y(x) = \exp\left(\int r\,dx\right) \cdot {}_2\mathrm{F}_1\left(\frac{5}{42}, \frac{11}{42}; \frac{2}{3}; f\right)$$

where

$$\exp\left(\int r\,dx\right) = (x+1)^{-\frac{5}{21}} \tag{3.4}$$

and

$$f = \frac{4x}{(x+1)^2}.$$

Section (3.6) shows how to find the parameters of the ${}_2\mathrm{F}_1$ function, $a_1, a_2, b_1 = \frac{5}{42}, \frac{11}{42}, \frac{2}{3}$. Then f is computed with the quotient method illustrated in the below remark.

Remark 3.1.1 (Overview of the Quotient Method). The hypergeometric function

$$y_1(x) = {}_2\mathrm{F}_1\left(\frac{5}{42}, \frac{11}{42}; \frac{2}{3}; x\right) \tag{3.5}$$

is a solution of the GHDO

$$L_B = \partial^2 + \frac{(29\,x - 14)}{21x\,(x-1)}\partial + \frac{55}{1764\,x\,(x-1)} \in \mathbb{Q}(x)[\partial].$$

L_B has two formal solutions at $x = 0$ and they are

$$y_1(x) = {}_2\mathrm{F}_1\left(\frac{5}{42}, \frac{11}{42}; \frac{2}{3}; x\right) = 1 + \frac{55}{1176}x + \frac{27401}{1382976}x^2 + \ldots,$$
$$y_2(x) = x^{\frac{1}{3}} \cdot {}_2\mathrm{F}_1\left(\frac{19}{42}, \frac{25}{42}; \frac{4}{3}; x\right) = x^{\frac{1}{3}} \cdot \left(1 + \frac{475}{2352}x + \frac{1941325}{19361664}x^2 + \ldots\right).$$

The exponents of L_B at 0 are $e_{0,1} = 0$ and $e_{0,2} = \frac{1}{3}$. The minimal operator for $y(f)$ has the following solutions at $x = 0$,

$$y_1(f) = 1 + \frac{55}{294}x - \frac{4939}{86436}x^2 + \ldots,$$
$$y_2(f) = c \cdot x^{\frac{1}{3}}\left(1 + \frac{83}{588}x + \frac{6805}{1210104}x^2 + \ldots\right)$$

for some constant c that depends on f. Here the exponents are again 0 and $\frac{1}{3}$ because $x = 0$ is a root of f with multiplicity 1 (see Theorem 2.4.1). Let

$$Y_1(x) = \exp\left(\int r\,dx\right) y_1(f) = 1 - \frac{5}{98}x + \frac{439}{9604}x^2 + \ldots, \tag{3.6}$$
$$Y_2(x) = \exp\left(\int r\,dx\right) y_2(f) = c \cdot x^{\frac{1}{3}}\left(1 - \frac{19}{196}x + \ldots\right). \tag{3.7}$$

(3.6) and (3.7) form a basis of solutions of L_{inp} in (3.3). Here $\exp(\int r\,dx)$ is as the same as in (3.4). Denote the quotients of the formal solutions of L_B and L_{inp} by

$$q(x) = \frac{y_1(x)}{y_2(x)}$$

and

$$Q(x) = \frac{Y_1(x)}{Y_2(x)} = \frac{\exp\left(\int r\,dx\right) y_1(f)}{\exp\left(\int r\,dx\right) y_2(f)} = \frac{y_1(f)}{y_2(f)} = q(f)$$

respectively. It follows that

$$q^{-1}(Q(x))$$

gives an expansion of f at $x = 0$. Given enough terms of y_1, y_2, Y_1, Y_2 we can compute f with rational function reconstruction.

Remark 3.1.2. The method given in the Remark (3.1.1) works, however, the following questions are needed to be answered:

1. How many terms are needed to reconstruct the pullback function f? This is equivalent to finding a degree bound for f.

2. How can we find the parameters a_1, a_2, b_1 (the three rational numbers in (3.5)) that defines the GHDO $L_B \in \mathbb{Q}(x)[\partial]$? This is the combinatorial part of our algorithm.

3. The exponents $0, \frac{1}{3}$ of L_{inp} at $x = 0$ only determine $\frac{Y_1}{Y_2}$ up to a constant factor. This means the quotient $\frac{y_1(f)}{y_2(f)}$ is only known up to a constant factor c. How to find this constant?

4. What if L_{inp} has a logarithmic solutions at $x = 0$?

5. What if f is an algebraic function?

6. What if L_{inp} does not have solutions in the form of (3.1), but have solutions in the form of

$$\exp\left(\int r\,dx\right)\left(r_0 \cdot {}_2\mathrm{F}_1(a_1, a_2; b_1; f) + r_1 \cdot {}_2\mathrm{F}_1'(a_1, a_2; b_1; f)\right)$$

   with $f, r, r_0, r_1 \in \overline{\mathbb{Q}(x)}$?

Answers of the first five questions are given in this chapter. Last question will be discussed in Chapter 5.2, after the notion of integral bases for differential operators is introduced in Chapter 4.

Example 3.1.2 (Algebraic Pullback Function)**.** The differential operator

$$L_{\text{inp}} = \partial^2 + \frac{x^4 - 44\,x^3 + 1206\,x^2 - 44\,x + 1}{4x^2\left(x^2 - 34\,x + 1\right)^2}$$

has a hypergeometric solution in the form of (3.1), which is

$$Y(x) = \exp\left(-\frac{1}{2}\int r\,dx\right) \cdot {}_2\mathrm{F}_1\left(\frac{1}{3}, \frac{2}{3}; 1; f\right)$$

where

$$r = \frac{-x^5 + 22\,x^4 - 55\,x^3 - 343\,x^2 + 58\,x - 1 + 6\,x\left(x^2 - 7\,x + 1\right)\sqrt{x^2 - 34\,x + 1}}{x\,(x^4 - 41\,x^3 + 240\,x^2 - 41\,x + 1)\,(x + 1)}$$

and

$$f = \frac{1}{2}\,\frac{1 + 30\,x - 24\,x^2 + x^3 - \left(x^2 - 7\,x + 1\right)\sqrt{x^2 - 34\,x + 1}}{1 + 3\,x + 3\,x^2 + x^3}.$$

Here the pullback function f is an algebraic function. $\mathbb{Q}(x, f)$ is an algebraic extension of $\mathbb{Q}(x)$ of algebraic degree $a_f = 2$. Note that $a_f = 1$ if and only if f is a rational function, as in Example 3.1.1.

3.2 General Outline of Quotient Method Algorithm

Problem Statement 3.2.1. Given a second order linear differential operator $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$, irreducible and regular singular, we want to find a hypergeometric solution of L_{inp} in the form of (3.1). This problem is equivalent to finding parameters f and r of the change of variables and exp-product transformations from a specific GHDO L_B to L_{inp} such that

$$L_B \xrightarrow{f}_C \xrightarrow{r}_E L_{\text{inp}}.$$

Therefore, we need to find

1. a Gauss Hypergeometric Differential Operator L_B (i.e., the rational numbers $a_1, a_2, b_1 \in \mathbb{Q}$),
2. parameters f and r.

Algorithm 3.2.1 ([23])**.** General Outline of `find_2f1`.

Input(s):

- $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ = A second order regular singular irreducible differential operator.
- $a_f max$ = A bound for the algebraic degree a_f of f (if omitted, then $a_f max = 2$ which means our implementation tries $a_f = 1$ and $a_f = 2$ only).

Output:

- Solutions of L_{inp} in the form of (3.1), or an empty list.

1. Try Kovacic's algorithm [32]. This algorithm finds Liouvillian solutions of L_{inp}. If there exist Liouvillian solutions, then return them. For an irreducible L_{inp}, the algorithm in [45] computes Liouvillian solutions in form (3.1).

2. For each $a_f \in \{1, \ldots, a_f max\}$:

   (a) Use Section 3.6 to compute candidates for a GHDO L_B and a candidate degree d_f for f. This becomes a combinatorial problem and Theorem 3.5.1 helps us to eliminate vast majority of the candidates.

   (b) For a candidate pair (L_B, d_f), compute formal solutions y_1, y_2 of L_B and Y_1, Y_2 of L_{inp} at a non-removable singularity up to precision a where

   $$a \geq (a_f + 1)(d_f + 1) + 6.$$

   Here we add six extra terms to reduce the number of false positives. Then take the quotients of the formal solutions, $q = \frac{y_1}{y_2}$ and $Q = \frac{Y_1}{Y_2}$, and compute power series expansions for q^{-1} and Q in order to compute

   $$f(x) \equiv q^{-1}(c \cdot Q(x)) \tag{3.8}$$

   in the next step. Here $c \in \mathbb{C}$ is an unknown constant.

   (c) Choose a suitable prime number ℓ (see Remark 3.2.2) and try to find c modulo ℓ by looping $c = 1, \ldots, \ell - 1$ as explained in Sections 3.7.1 and 3.7.2.

   (d) For each c:

      i. Compute f mod (x^a, ℓ) from equation (3.8) and use it to reconstruct f mod ℓ (the image of f in $\mathbb{F}_\ell(x)$) [47]. If it fails for every c, then proceed with the next candidate GHDO (if any) in Step 2b. If no candidates remain, then return an empty list.

      ii. If rational reconstruction in Step 2(d)i succeeds for some c values, then apply Hensel lifting (Section 3.8) to find f mod a power of ℓ. Then try rational number reconstruction [47]. If it does not fail for at least one c value, then we have f. If no solution is found (see Remark 3.8.1 in Section 3.8), then proceed with the next candidate GHDO (if any) in Step 2b. If no candidates remain, then return an empty list.

      iii. Use Section 3.8.3 to compute the parameter r of the exp-product transformation.

      iv. Return a basis of hypergeometric solutions of L_{inp} in form (3.1).

Remark 3.2.1. If f in (3.8) has non-integer powers of x and $a_f = 2$, then Algorithm 3.2.1 separates f mod (x^a, ℓ) into two parts as $f = f_0 + x^{\frac{1}{2}} f_1$ mod (x^a, ℓ). Then it computes the *trace* $\text{Tr}(f) = 2f_0$ of f and the *norm* $\text{N}(f) = f_0^2 - x f_1^2$ of f. Then it lifts $\text{Tr}(f)$ mod ℓ and $\text{N}(f)$ mod ℓ to a power of ℓ (Section 3.8). After each lifting it tries rational reconstruction. If rational reconstruction succeeds for both $\text{Tr}(f)$ and $\text{N}(f)$, then Algorithm 3.2.1 computes the algebraic pullback function f from its minimal polynomial $y^2 - \text{Tr}(f)y + \text{N}(f)$.

Remark 3.2.2. Numerous examples (which can be found at [23]) verified that Algorithm 3.2.1 is very effective. For completeness of the algorithm, we still need a theorem for *good prime numbers* (a *good prime* is a prime for which rational reconstructions in Step 2d will work).

3.3 Good Prime Numbers

A good prime number ℓ is a prime for which rational function and rational number reconstruction in Step 2d of Algorithm 3.2.1 will work. There are certain prime numbers that we need to avoid. Algorithm 3.2.1 chooses a good prime number ℓ with respect to following main criteria:

- We take the quotients q and Q of the formal solutions of L_B and L_{inp} in Step 2b and then we compute q and Q modulo a prime ℓ. If the formal solutions of L_B and L_{inp}, computed to precision $O(x^a)$, are undefined modulo ℓ, then ℓ is not a good prime number.
- For a prime ℓ, if the number of singularities of L_{inp} is not equal to the number of singularities of L_{inp} modulo ℓ, then ℓ is not a good prime. Equivalently, if ℓ divides the discriminant, or the leading coefficient, of the polynomial whose zeros are the singularities of L_{inp}, then ℓ is not a good prime.

Example 3.3.1. Let

$$\begin{aligned} L_{\text{inp}} &= \partial^2 \\ &+ \frac{34553639952\,x^6 - 3934841352\,x^5 + 1197258796\,x^4 - 57806358\,x^3 - 53494036\,x^2 + 2168214\,x - 1379960}{(78\,x-1)\,(1156\,x^2 - 204\,x + 49)\,(11271\,x^2 + 680\,x + 362)\,(34\,x^2+1)}\partial \\ &+ \frac{4335\,(78\,x-1)^2}{(4624\,x^2 - 816\,x + 196)\,(11271\,x^2 + 680\,x + 362)\,(34\,x^2+1)^2}. \end{aligned}$$

Algorithm 3.2.1 finds a solution of L_{inp} in the form of (3.1), which is

$$\frac{\sqrt[4]{\frac{34\,x^2+1}{x^2}}}{\sqrt[12]{\frac{1156\,x^2-204\,x+49}{x^2}}\,\sqrt[6]{\frac{578\,x^2+51\,x+20}{x^2}}} \cdot {}_2\mathrm{F}_1\left(\frac{1}{12}, \frac{7}{12}; 1; \frac{304317\,x^2 + 18360\,x + 9774}{(1156\,x^2 - 204\,x + 49)\,(578\,x^2 + 51\,x + 20)^2}\right).$$

Finite singularities of L_{inp} are the roots of the polynomial (denominator of L_{inp})

$$\begin{aligned} P_{\text{sing}} &= 34553639952\,x^7 - 4456012392\,x^6 + 3274278676\,x^5 \\ &\quad - 266828498\,x^4 + 114830750\,x^3 - 4596307\,x^2 + 1424092\,x - 17738. \end{aligned}$$

For this example $\ell = 19$ is not a good prime number because the discriminant of P_{sing}

$$\operatorname{disc}(P_{\text{sing}}) = -1 \cdot 2^{29} \cdot 3^6 \cdot 5^3 \cdot 7^{16} \cdot 11^2 \cdot 13^2 \cdot 17^{21} \cdot 19^{13} \cdot 23^{16}$$

vanishes modulo 19. This means that two distinct singularities go to the same singularity modulo 19. Algorithm 3.2.1 avoids these type of prime numbers. For this example, Algorithm 3.2.1 chooses $\ell = 29$, which is the smallest good prime number, reconstructs the pullback function, and finds hypergeometric solutions.

3.4 Degree Bounds

Theorem 3.4.1 (Riemann-Hurwitz Formula, [22])**.** *Let X and Y be two algebraic curves with genera g_X and g_Y respectively. If $f : X \longrightarrow Y$ is a non-constant morphism, then*

$$2g_X - 2 = \deg(f)(2g_Y - 2) + \sum_{p\in X}(e_p - 1). \tag{3.9}$$

Here $p \in X$ is a branching point and e_p is its ramification order. See [22] for more details.

Remark 3.4.1. If we let $X = \mathbb{P}^1$ and $Y = \mathbb{P}^1$ in the Theorem 3.4.1, then we obtain

$$\sum_{p\in\mathbb{P}^1}(e_p - 1) = 2\deg(f) - 2. \tag{3.10}$$

3.4.1 A Degree Bound for Logarithmic Case

Let $L_B \in \mathbb{Q}(x)[\partial]$ be a GHDO with at least one logarithmic singularity and assume that for $L \in \mathbb{Q}(x)[\partial]$ we have

$$L_B \xrightarrow{f:\mathbb{P}^1 \longrightarrow \mathbb{P}^1}_C \xrightarrow{r}_E L$$

and let $d_f = \deg(f)$. Singularities of L_B are $S = \{0, 1, \infty\}$. If there is no ramification, then the number of elements in the set $T := f^{-1}(S)$ is $3d_f$. So

$$\#T \leq 3d_f.$$

The number of elements in $T = f^{-1}(S)$ can be given as

$$\begin{aligned}
\#T &= \sum_{p\in T} 1 \\
&= \sum_{p\in T}(e_p - (e_p - 1)) \\
&= \sum_{p\in T} e_p - \sum_{p\in T}(e_p - 1) \\
&= 3d_f - \sum_{p\in T}(e_p - 1) \qquad (3.11)
\end{aligned}$$

where e_p is the ramification order of p. From the Riemann-Hurwitz formula (3.9) we have

$$\sum_{p\in T}(e_p-1)\le\sum_{p\in\mathbb{P}^1}(e_p-1)=2d_f-2. \tag{3.12}$$

Therefore, from (3.11) and (3.12),

$$\#T\ge 3d_f-(2d_f-1)=d_f+2.$$

Hence,

$$d_f+2\le \#T\le 3d_f.$$

Let TrueSing(L) be the set of all non-removable singularities (true singularities) of L. Then TrueSing(L) is a subset of T. In general these two sets are not equal. All of the points in T come from (p comes from s if $f(s)=p$) the points $S=\{0,1,\infty\}$. Let $p\in T$ comes from 0 and L_B has exponents 0 and $\frac{1}{3}$ at 0. If p is a root of f with multiplicity $e_p=3$, then L has exponents 0 and $1=3\cdot\frac{1}{3}$ at p. So p is a regular point of L. Such a point is called a *disappeared singularity*. The set of all disappeared singularities of L is $T\setminus$ TrueSing(L). Logarithmic singularities never disappear. If $s\in S$ is a logarithmic singularity of L_B, then every point $p\in T$ which come from s is a logarithmic singularity of L. Let n_{diss} be the number of disappeared singularities of L, namely $n_{\text{diss}}:=\#\left(T\setminus \text{TrueSing}(L)\right)$. For a GHDO L_B with exponent-differences $\alpha_0=0$, $\alpha_1=\frac{1}{2}$, and $\alpha_\infty=\frac{1}{3}$, we have

$$n_{\text{diss}}\le\frac{1}{2}d_f+\frac{1}{3}d_f. \tag{3.13}$$

Here equality occurs if and only if every point p coming from 1 and ∞ disappears. So, the total number of non-removable singularities n_{true} of L is

$$\begin{aligned}
n_{\text{true}}&=\#T-n_{\text{diss}}\\
&=\left(3d_f-\sum_{p\in T}(e_p-1)\right)-n_{\text{diss}}\\
&\ge(3d_f-(2d_f-2)-n_{\text{diss}})\\
&\ge(d_f+2)-n_{\text{diss}}\\
&\ge(d_f+2)-\left(\frac{1}{2}d_f+\frac{1}{3}d_f\right)\\
&\ge\frac{1}{6}d_f+2
\end{aligned}$$

and therefore

$$d_f \leq 6n_{\text{true}} - 12. \tag{3.14}$$

This inequality is an upper bound for d_f in all cases with at least one logarithmic singularity. This is because (3.13) is an upper bound for the number of disappeared singularities in the logarithmic case. An irreducible GHDO L_B can not have two singularities with exponent-differences $\frac{1}{2}$. This gives (3.13).

3.4.2 A Degree Bound for Non-logarithmic Case

Consider a GHDO $L_B \in \mathbb{Q}(x)[\partial]$ with no logarithmic singularity and assume that for $L \in \mathbb{Q}(x)[\partial]$ we have

$$L_B \xrightarrow{f:\mathbb{P}^1 \longrightarrow \mathbb{P}^1}_C \xrightarrow{r}_E L$$

where $f, r \in \mathbb{Q}(x)$ and let $d_f = \deg(f)$. In this case one could have all disappeared singularities coming from the singularities 0, 1, and ∞ of L_B. In the non-logarithmic case, the maximum degree bound is achieved the GHDO L_B having exponent-differences $\alpha_0 = \frac{1}{2}$, $\alpha_1 = \frac{1}{3}$, and $\alpha_\infty = \frac{1}{7}$. All of the other GHDO's are reducible or appear in Schwarz's list [37] which means they admit Liouvillian solutions. The maximum number n_{diss} of disappeared singularities for an L_B with $\alpha_0 = \frac{1}{2}$, $\alpha_1 = \frac{1}{3}$, and $\alpha_\infty = \frac{1}{7}$ is not $\frac{1}{2}d_f + \frac{1}{3}d_f + \frac{1}{7}d_f$. This is because it contradicts the formula (3.10). If we use the formula (3.10) to compute an upper for n_{diss}, then we obtain that

$$n_{\text{diss}} \leq \left(\frac{1}{2} + \frac{1}{3}\right) d_f + \frac{1}{7-1}\left(2d_f - 2 - \left(\frac{2-1}{2}d_f + \frac{3-1}{3}d_f\right)\right) = \frac{35}{36}d_f - \frac{1}{3}.$$

This bound on n_{diss} leads to

$$n_{\text{true}} \geq \frac{1}{36}d_f + \frac{7}{3}$$

which gives

$$d_f \leq 36n_{\text{true}} - 84. \tag{3.15}$$

Therefore, combining inequalities (3.14) and (3.15), we can state an a-priori degree bound for a rational pullback function f,

$$d_f \leq \begin{cases} 3n_{\text{true}} - 12 & \text{logarithmic case} \\ 36n_{\text{true}} - 84 & \text{non-logarithmic case .} \end{cases} \tag{3.16}$$

Algorithm (3.2.1) uses this bound as a starting point. Several additional restrictions computed in the run-time may lower the degree bound on d_f.

3.5 Riemann-Hurwitz Type Formula For Differential Equations

Remark 3.5.1. Let X be an algebraic curve with function field $\mathbb{C}(X)$. The ring $D_{\mathbb{C}(X)} := \mathbb{C}(X)[\partial_t]$ is the ring of differential operators on the curve X. Here $t \in \mathbb{C}(X) \setminus \mathbb{C}$. An element of $L \in D_{\mathbb{C}(X)}$ is a differential operator defined on X.

Theorem 3.5.1 (Lemma 1.5 in [6])**.** *Let X and Y be two algebraic curves with genera g_X and g_Y, function fields $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ respectively. Let $f : X \longrightarrow Y$ be a non-constant morphism. The morphism f corresponds a homomorphism $\mathbb{C}(Y) \longrightarrow \mathbb{C}(X)$, which in turn corresponds a homomorphism $D_{\mathbb{C}(Y)} \longrightarrow D_{\mathbb{C}(X)}$. If $L_1 \in D_{\mathbb{C}(Y)}$ is a second order operator and $L_2 \in D_{\mathbb{C}(X)}$ is the corresponding operator, then*

$$\mathrm{Covol}(L_2, X) = \deg(f) \cdot \mathrm{Covol}(L_1, Y) \tag{3.17}$$

where

$$\mathrm{Covol}(L, X) := 2g_X - 2 + \sum_{p \in X} (1 - \Delta(L, p)).$$

Here note that $\Delta(L, p)$ is the exponent-difference of L at p.

Proof. As in [6], take finite sets $S \subseteq Y$ and $T := f^{-1}(S) \subseteq X$ in such a way that

1. all singularities of L_1 are in S,
2. all singularities of L_2 are in T,
3. all branching points in X are in T.

$$\begin{aligned}
\#T &= \sum_{p \in T} 1 \\
&= \sum_{p \in T} e_p - \sum_{p \in T} (e_p - 1) \\
&= \deg(f) \cdot \#S - \sum_{p \in X} (e_p - 1) && (3.18) \\
&= \deg(f) \cdot \#S - (2g_X - 2 - \deg(f)(2g_Y - 2)) \,. && (3.19)
\end{aligned}$$

From (3.18) to (3.19) we use (3.9). Then,

$$\begin{aligned}\sum_{p\in X}(\Delta(L_2,p)-1) &= \sum_{p\in T}(\Delta(L_2,p)-1)\\ &= \sum_{p\in T}\Delta(L_2,p)-\sum_{p\in T}1\\ &= \deg(f)\sum_{s\in S}\Delta(L_1,s)-\#T. \end{aligned}\tag{3.20}$$

Then combine (3.19) and (3.20), and get

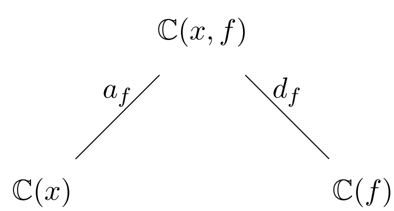
$$2-2g_X+\sum_{p\in X}(\Delta(L_2,p)-1)=\deg(f)\left(2-2g_Y+\sum_{s\in Y}(\Delta(L_1,s)-1)\right).\tag{3.21}$$

which is the same as (3.17). □

Corollary 3.5.2. *Let* $X=Y=\mathbb{P}^1$ *and suppose that* $L_B \xrightarrow{f:\mathbb{P}^1\to\mathbb{P}^1}_C \xrightarrow{r}_E L_{\text{inp}}$ *where* $L_B\in\mathbb{C}(x)[\partial]$ *is a GHDO with exponent-differences* $\alpha_0,\alpha_1,\alpha_\infty$ *at* $\{0,1,\infty\}$*. Since an exp-product transformation does not affect exponent-differences, Theorem 3.5.1 gives the following equation for* $\text{Covol}(L_{\text{inp}},\mathbb{P}^1)$*:*

$$-2+\sum_{p\in\mathbb{P}^1}(1-\Delta(L_{\text{inp}},p))=\deg(f)\left(-2+\sum_{i\in\{0,1,\infty\}}(1-\alpha_i)\right).\tag{3.22}$$

Corollary 3.5.3. *Let* L_B *and* L_{inp} *be as in Corollary 3.5.2. Both have rational function coefficients. This time, suppose that* f,r *in* $L_B\xrightarrow{f}_C\xrightarrow{r}_E L_{\text{inp}}$ *are algebraic functions. Then* $f: X\to\mathbb{P}^1$ *for an algebraic curve* X *whose function field* $\mathbb{C}(X)=\mathbb{C}(x,f)$ *is an algebraic extension of both* $\mathbb{C}(x)\cong\mathbb{C}(\mathbb{P}^1)$ *and* $\mathbb{C}(f)\cong\mathbb{C}(\mathbb{P}^1)$*. Let* a_f *and* d_f *denote the degrees of these extensions.*



Applying (3.17) *for both field extensions, then we will get*

$$\text{Covol}(L_{\text{inp}},\mathbb{P}^1)=\frac{d_f}{a_f}\left(-2+\sum_{i\in\{0,1,\infty\}}(1-\alpha_i)\right).\tag{3.23}$$

3.6 Candidate Gauss Hypergeometric Differential Operators

Let $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$. Problem Statement 3.2.1 says that, if L_{inp} has solutions in the form of (3.1), then there exists a GHDO L_B such that $L_B \xrightarrow{f}_C \xrightarrow{r}_E L_{\text{inp}}$. with $f, r \in \overline{\mathbb{Q}(x)}$. The following algorithm finds candidates for such a GHDO L_B. In order to find such a candidate, it is enough to find candidate exponent-differences $\alpha_0, \alpha_1, \alpha_\infty$ at its singularities $0, 1, \infty$.

Algorithm 3.6.1. General Outline of `find_expdiffs`.

Inputs:

- e_{inp} = The list of exponent-differences of L_{inp} at its non-removable singularities.
- e_{rem} = The (possibly empty) list of exponent-differences of L_{inp} at its removable singularities.
- a_f = A candidate algebraic degree.

Output:

- A list of all lists $e_B = [\alpha_0, \alpha_1, \alpha_\infty, d]$ of integers or rational numbers where $[\alpha_0, \alpha_1, \alpha_\infty]$ is a list of candidate exponent-differences and d is a candidate degree d_f for f such that:
  (a) For every exponent-difference m in e_{inp} there exists $e \in \mathbb{Q}$ with $e \cdot a_f \in \{1, \ldots, d\}$ such that $m = e \cdot \alpha_i$ for some $i \in \{0, 1, \infty\}$.
  (b) The multiplicities e are consistent with (3.9), and their sums are compatible with d (see the last paragraph of Step 2).

1. Let $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3 = \alpha_0, \alpha_1, \alpha_\infty$. After reordering we may assume that $\overline{\alpha}_1, \ldots, \overline{\alpha}_k \in \mathbb{Z}$ and $\overline{\alpha}_{k+1}, \ldots, \overline{\alpha}_3 \notin \mathbb{Z}$ for $k \in \{0, 1, 2, 3\}$. For each $k \in \{0, 1, 2, 3\}$ we use `CoverLogs` in [23] to compute candidates for $\overline{\alpha}_1, \ldots, \overline{\alpha}_k \in \mathbb{Z}$.

   Algorithm `CoverLogs` computes candidates that meet these requirements:

   (a) Logarithmic singularities are non-removable singularities with integer exponent-differences. If L_{inp} has at least one logarithmic singularity s with exponent-difference $\Delta(L_{\text{inp}}, s)$, then a candidate L_B must have at least one logarithmic singularity; at least one of the $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3$ must be an integer that divides $a \cdot \Delta(L_{\text{inp}}, s)$ for some $a \in \{1, \ldots, a_f\}$, and for every $\overline{\alpha}_i \in \mathbb{Z}$ there must be at least one s such that $\overline{\alpha}_i$ divides $a \cdot \Delta(L_{\text{inp}}, s)$.

   (b) $\Delta(L_{\text{inp}}, s) = 0$ for some $s \iff 0 \in \{\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3\}$.

   (c) Theorem 2.4.1.

If $\overline{\alpha}_1 + \cdots + \overline{\alpha}_k \neq 0$, then algorithm `CoverLogs` also computes the exact degree d_f of f using Theorem 2.4.1 which shows that $d_f(\overline{\alpha}_1 + \cdots + \overline{\alpha}_k)/a_f$ must be the sum of the logarithmic exponent-differences of L_{inp}. Otherwise, it uses (3.16) to compute a bound for d_f, and uses it as d_f to compute a candidate degree.

2. We will explain only the case $a_f = 1$, and only $k = 1$, which is the case $[\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3] = [\alpha_0, \alpha_1, \alpha_\infty]$, where $\alpha_0 \in \mathbb{Z}$ and $\alpha_1, \alpha_\infty \notin \mathbb{Z}$.

   Let $k = 1$. Let $\alpha_0 \in \mathbb{Z}$ be one of the candidates from algorithm `CoverLogs`. We need to find candidates for α_1 and α_∞.

   The logarithmic singularities of L_{inp} come from the point 0. Non-integer exponent-differences of L_{inp} must be multiples of α_1 or α_∞. Let S_N be the set of non-logarithmic exponent differences of L_{inp} and S_R be the set of exponent-differences of L_{inp} at its removable singularities. Consider the set

$$\Gamma_1 = \begin{cases} \Gamma_A = \{\frac{\max(S_N)}{b} : b = 1, \ldots, d_f\} & \text{if } S_N \neq \emptyset, \\ \Gamma_B = \{\frac{a}{b} : a \in S_R \cup \{1\}, b = 1, \ldots, d_f\} & \text{otherwise.} \end{cases}$$

   α_1 (or α_∞, but if so, we may interchange them) must be one of the elements of Γ_1. We loop over all elements of Γ_1. Assume that a candidate for α_1 is chosen. Let $\Omega = S_N \setminus \alpha_1\mathbb{Z}$. Now consider the set

$$\Gamma_\infty = \begin{cases} \Gamma_A \cup \Gamma_B & \text{if } \Omega = \emptyset, \\ \{\frac{g}{b} \,:\, g = \gcd(\Omega) : b = 1, \ldots, d_f\} & \text{otherwise.} \end{cases}$$

   Now take all pairs (α_∞, d) satisfying (3.23), $\alpha_\infty \in \Gamma_\infty$, $1 \leq d \leq d_f$, with additional restrictions on d, as follows:

   (a) For every potential non-zero value v for one of the α_i's we pre-compute a list of integers N_v by dividing all exponent-differences of L_{inp} by v and then selecting the quotients that are integers.

   (b) Next, let D_v be the set of all $1 \leq d \leq d_f$ that can be written as the sum of a sublist of N_v. Each time a non-zero value v is taken for one of the α_i, it imposes the restriction $d \in D_v$. This means that we need not run a loop for $\alpha_\infty \in \Gamma_\infty$, instead, we run a (generally much shorter) loop for d (taking values in the intersection of the D_v's so far) and then for each such d compute α_∞ from (3.23). We also check if $d \in D_{\alpha_\infty}$.

3. Return the list of candidate exponent-differences with a candidate degree, the list of lists $[\alpha_0, \alpha_1, \alpha_\infty, d]$, for candidate GHDOs.

3.7 Quotient Method

3.7.1 Non-logarithmic Case

Let the second order differential operator $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ be given. Let L_B be a GHDO such that $L_B \xrightarrow{f}_C \xrightarrow{r}_E L_{\text{inp}}$. Let $f : \mathbb{P}^1_x \mapsto \mathbb{P}^1_z$ and $L_1 \xrightarrow{f}_C L_2$. If $x = p$ is a singularity of L_2 and $z = s$ is a singularity of L_1, then we say that p comes from s when $f(p) = s$. After a change of variables we can assume that $x = 0$ is a singularity of L_{inp} that comes from the singularity $z = 0$ of L_B. This means $f(0) = 0$ and we can write $f = c_0 \cdot x^{v_0(f)} (1 + \dots)$ where $c_0 \in \mathbb{C}$, $v_0(f)$ is the multiplicity of 0, and the dots refer to an element in $x \cdot \mathbb{C}[[x]]$.

Let y_1 and y_2 be the formal solutions of L_B at $x = 0$. The following diagram shows the effects of the change of variables and exp-product transformations on the formal solutions of L_B:

$$y_i(x) \xrightarrow{f}_C y_i(f) \xrightarrow{r}_E Y_i(x) = \exp\left(\int r dx\right) \cdot y_i(f)$$

where $i \in \{1, 2\}$ and Y_i is a solution of L_{inp}. Let $q = \frac{y_1}{y_2}$ be a quotient of formal solutions of L_B. The change of variables transformation sends x to f, and so q to $q(f)$. Therefore, $q(f)$ will be a quotient of formal solutions of L_{inp}. The effect of exp-product transformation disappears under taking quotients. In general, a quotient of formal solutions of L_B at a point $x = p$ is only unique up to Moebius transformations

$$\frac{y_1}{y_2} \mapsto \frac{\alpha y_1 + \beta y_2}{\gamma y_1 + \eta y_2}.$$

If $x = p$ has a non-integer exponent-difference, then we can choose q uniquely up to a constant factor c. So if we likewise compute a quotient Q of formal solutions of L_{inp}, then we have

$$q(f(x)) = c \cdot Q(x)$$

for some unknown constant c. Then

$$f(x) = q^{-1} (c \cdot Q(x)) . \tag{3.24}$$

If we know the value of this constant c, then (3.24) allows us to compute an expansion for the pullback function f from expansions of q and Q. To obtain c with a finite computation, we take a prime number ℓ. Then, for each $c \in \{1, \dots, \ell - 1\}$ we try to compute f modulo ℓ in $\mathbb{F}_\ell(x)$ using series-to-rational function reconstruction. If this succeeds, then we lift f modulo a power of ℓ, and try to find $f \in \mathbb{Q}(x)$ with rational number reconstruction. Details of lifting are given in Sections 3.8.1 and 3.8.2.

Remark 3.7.1. We compute formal solutions up to a precision $a \geq (a_f + 1)(d_f + 1) + 6$. This suffices to recover the correct pullback function with a few extra terms to reduce the number of false positives.

Algorithm 3.7.1. General Outline of `case_1` (non-logarithmic case).

Inputs:

- L_{inp} = A second order regular singular differential operator.
- L_B = A candidate GHDO.
- d_f = A candidate degree for f.
- a_f = A candidate algebraic degree for f.

Outputs:

- f = Pullback function.
- r = Parameter of exp-product transformation.

1. Compute formal solutions y_1, y_2 of L_B and Y_1, Y_2 of L_{inp} up to precision $a \geq (a_f+1)(d_f+1)+6$.
2. Compute $q = \frac{y_2}{y_1}$, $Q = \frac{Y_2}{Y_1}$, and q^{-1}.
3. Select a prime ℓ for which these expansions can be reduced mod ℓ.
4. For each c_0 in $\{1, \ldots, \ell - 1\}$:
   (a) Evaluate $\overline{f}_{1,c_0} = q^{-1}(c_0 \cdot Q) \in \mathbb{Z}[x]/(\ell, x^a)$.
   (b) If $a_f = 1$ then try rational function reconstruction for $\overline{f}_{1,c_0}$ (the case $a_f > 1$ is explained in Section 3.8.2).
      i. If rational function reconstruction succeeds and produces f_{1,c_0}, then store c_0 and f_{1,c_0}.
      ii. If rational function reconstruction fails for every c_0, then return 0.
5. For n from 2 (see Remark 3.8.1 in Section 3.8):
   (a) For each stored c_0:
   (b) Using the techniques explained in Sections 3.8.1 and 3.8.2 lift f_{n-1,c_0} to f_{n,c_0}.
   (c) f_{n,c_0} is a candidate for f mod ℓ^n. Try to obtain f from this with rational number reconstruction. If this succeeds, compute M such that $L_B \xrightarrow{f}_C M$. Compute r such that $M \xrightarrow{r}_E L_{\text{inp}}$, if it exists (see Section 3.8.3). If so, return f and r.

3.7.2 Logarithmic Case

A logarithm may occur in one of the formal solutions of $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ at $x = p$ if exponents at $x = p$ differ by an integer. We may assume that L_{inp} has a logarithmic solution at the singularity $x = 0$.

Let y_1, y_2 be the formal solutions of L_B at $x = 0$. Let y_1 be the non-logarithmic solution (it is unique up to a multiplicative constant). Then

$$\frac{y_2}{y_1} = c_1 \cdot \log(x) + h$$

for some $c_1 \in \mathbb{C}$ and $h \in \mathbb{C}[[x]]$. We can choose y_2 such that

$$c_1 = 1 \qquad \text{and} \qquad \text{constant term of } h = 0. \tag{3.25}$$

That makes $\frac{y_2}{y_1}$ unique. If h does not contain negative powers of x then define

$$g = \exp\left(\frac{y_2}{y_1}\right) = x \cdot (1 + \dots) \tag{3.26}$$

where the dots refer to an element of $x \cdot \mathbb{C}[[x]]$.

Remark 3.7.2. If we choose y_2 differently, then we obtain another $\tilde{g} = \exp\left(\frac{y_2}{y_1}\right)$ that relates to g in (3.26) by $\tilde{g} = c_1 g^{c_2}$ for some constants c_1, c_2. If h contains negative powers of x, then the formula for g is slightly different (we did not implement this case, instead we use Section 4 to transform differential equations).

We do likewise for the formal solutions Y_1, Y_2 of L_{inp} and denote

$$G = \exp\left(\frac{Y_2}{Y_1}\right) = x \cdot (1 + \dots). \tag{3.27}$$

Write $f \in \mathbb{C}(x)$ as $c_0 \cdot x^{v_0(f)} (1 + \dots)$. Then

$$g(f) = c \cdot x^{v_0(f)} (1 + \dots).$$

Note that g, G are not intrinsically unique, the choices we made in (3.25) implies that

$$g(f) = c_1 \cdot G^{c_2} \tag{3.28}$$

for some constants c_1, c_2. Here $c_1 = c$ and $c_2 = v_0(f)$. If $\Delta(L_{\text{inp}}, 0) \neq 0$, then find $v_0(f)$ from $\Delta(L_B, 0) v_0(f) = \Delta(L_{\text{inp}}, 0)$. Otherwise we loop over $v_0(f) = 1, 2, \dots, d_f$. That leaves one unknown constant c. We address this problem as before, choose a good prime number ℓ, try $c = 1, 2, \ldots, \ell-1$. Then calculate an expansion for f with the formula

$$f = g^{-1}\left(c \cdot G^{v_0(f)}\right). \tag{3.29}$$

Algorithm 3.7.2. General Outline of `case_2` (logarithmic case).

Inputs:

- L_{inp} = A second order regular singular differential operator.
- L_B = A candidate GHDO.
- d_f = A candidate degree for f.
- a_f = A candidate algebraic degree for f.

Outputs:

- f = Pullback function.
- r = Parameter of exp-product transformation.

1. Compute the exponents of L_{inp} and L_B. If $\Delta(L_{\text{inp}}, 0) = 0$, then replace L_{inp} with L defined in Remark (3.7.2) above. Otherwise let $L = L_{\text{inp}}$.
2. Compute formal solutions y_1, y_2 of L_B and Y_1, Y_2 of L up to precision $a \geq (a_f+1)(d_f+1)+6$.
3. Compute $q = \frac{y_2}{y_1}$, $Q = \frac{Y_2}{Y_1}$. Compute g, G from (3.28) and (3.29) respectively, and g^{-1}.
4. Same as in Algorithm 3.7.1 Step 3.
5. Compute $v_0(f)$ and search for c_0 value(s) such that c could be $\equiv c_0 \bmod \ell$ by looping over $c_0 = 1, \ldots, \ell - 1$. If $\Delta(L_{\text{inp}}, 0) = 0$, then also simultaneously loop over $v_0(f) = 1, \ldots, d_f$ to find $v_0(f)$.
6. For each c_0 in $\{1, \ldots, \ell - 1\}$:
   (a) Evaluate $\overline{f}_{1,c_0} = g^{-1}\left(c_0 \cdot G^{v_0(f)}\right) \in \mathbb{Z}[x]/(\ell, x^a)$.
   (b) Try rational function or algebraic function reconstruction for $\overline{f}_{1,c_0}$ as in Algorithm 3.7.1 Step 4b.
7. Same as in Algorithm 3.7.1 Step 5.

3.8 Recovering Pullback Functions and Parameter of Exp-product

3.8.1 Lifting for Rational Pullback Functions

By using the formula (3.24), which is $f(x) = q^{-1}(c \cdot Q(x))$, we can recover the rational pullback function f, if we know the value of the constant c. We do not have a direct formula for c. However, if we know c_0 such that

$$c \equiv c_0 \mod \ell$$

for a good prime number ℓ, then we can recover the pullback function f. This can be done via *Hensel lifting* techniques.

Let ℓ be a good prime number and consider

$$\begin{aligned} &h : \mathbb{Q} \longrightarrow \mathbb{Q}[x]/(x^a) \\ &h(c) \equiv q^{-1}(c \cdot Q(x)) \mod x^a. \end{aligned}$$

By looping on $c_0 = 1, \ldots, \ell - 1$ and trying rational function reconstruction for $h(c_0) \bmod (\ell, x^a)$, we can compute the image of $f \in \mathbb{F}_\ell(x)$ from its image in $\mathbb{F}_\ell[x]/(x^a)$. If a is high enough, then for correct value(s) of c_0, rational function reconstruction will succeed and return a rational function $\frac{A_0}{B_0} \bmod \ell$. This c_0 is the one satisfying

$$c \equiv c_0 \mod \ell.$$

Write

$$c \equiv c_0 + \ell c_1 \mod \ell^2$$

for $0 \leq c_1 \leq \ell - 1$. Taylor series expansion of h gives us

$$h(c) = h(c_0 + \ell c_1) \equiv h(c_0) + \ell c_1 h'(c_0) \mod (\ell^2, x^a). \tag{3.30}$$

Substitute $c_1 = 0$, $c_1 = 1$, respectively, in (3.30) and compute

$$h(c_0) \mod (\ell^2, x^a), \tag{3.31}$$

$$h(c_0 + \ell) \equiv h(c_0) + \ell h'(c_0) \mod (\ell^2, x^a). \tag{3.32}$$

Subtracting (3.31) from (4.8) gives

$$\ell h'(c_0) \equiv [h(c_0 + \ell) - h(c_0)] \mod (\ell^2, x^a).$$

Let

$$E_{c_1} = h(c_0) + c_1 \ell h'(c_0) \tag{3.33}$$

where c_1 is an unknown constant. Suppose $f = \frac{A}{B}$ in characteristic 0. We do not know what A and B are. However, from applying rational function reconstruction for $h(c_0)$, we obtain A_0, B_0 with $f \equiv \frac{A_0}{B_0} \mod (\ell, x^a)$. It follows that

$$f = \frac{A}{B} \equiv \frac{A_0}{B_0} \equiv E_{c_1} \mod (\ell, x^a).$$

From this equation we have

$$A \equiv BE_{c_1} \mod (\ell, x^a). \tag{3.34}$$

Now let

$$f = \frac{A}{B} \equiv \frac{A_0 + \ell A_1}{B_0 + \ell B_1} \mod (\ell^2, x^a) \tag{3.35}$$

where

$$A_1 = a_0 + a_1 x + \cdots + a_{\deg(A_0)} x^{\deg(A_0)}$$
$$B_1 = b_1 x + \cdots + b_{\deg(B_0)} x^{\deg(B_0)}$$

are unknown polynomials. Here we are fixing the constant term of B. We need values of $\{a_i, b_j\}$ to find $f \mod (\ell^2, x^a)$. From (3.34), we have

$$(A_0 + \ell A_1) \equiv (B_0 + \ell B_1) \cdot E_{c_1} \mod (\ell^2, x^a). \tag{3.36}$$

Now, solve the linear system (3.36) for unknowns $\{a_i, b_j, c_1\}$ in $\mathbb{F}_\ell$. From (3.35) find $f \mod (\ell^2, x^a)$ and $c \equiv c_0 + \ell c_1 \mod \ell^2$. Try rational number reconstruction after each Hensel lifting. If it succeeds, then check if this rational function is the one that we are looking for as in the last step of Algorithm 3.7.1. If it is not, then lift $f \mod (\ell^2, x^a)$ to mod (ℓ^3, x^a) (or (ℓ^4, x^a) if an implementation for solving linear equations mod ℓ^n is available). After a (finite) number of steps, we can recover the rational pullback function f.

Remark 3.8.1. Our implementation gives up when the prime power becomes "too high" (a proven bound is still lacking, but would be needed for a rigorous algorithm).

3.8.2 Lifting for Algebraic Pullback Functions

We can recover algebraic pullback functions in a similar way. However, we need to know

$$a_f = [\mathbb{C}(x,f) : \mathbb{C}(x)].$$

The idea is to recover the minimal polynomial of f. Let

$$d_f = [\mathbb{C}(x,f) : \mathbb{C}(f)].$$

Consider the polynomial in y

$$\sum_{j=0}^{a_f} A_j y^j \mod (\ell, x^a) \tag{3.37}$$

with unknown polynomials

$$A_j = \sum_{i=0}^{d_f} a_{i,j} x^i$$

where $j = 0, \ldots, a_f$. First we need to find the value of c_0 such that $c_0 \equiv c \bmod \ell$. As before, by looping on $c_0 = 1, \ldots, \ell - 1$, we can compute the corresponding f_{c_0} which is a candidate for f mod (x^a, ℓ) in $\mathbb{F}_\ell[x]/(x^a)$. The polynomial (3.37) should be congruent to 0 mod (ℓ, x^a) if we plug in f_{c_0} for y. Solve the system

$$\sum_{j=0}^{a_f} A_j f_{c_0}^j \equiv 0 \mod (\ell, x^a)$$

over $\mathbb{F}_\ell$ and find the unknown polynomials A_j mod ℓ. Then let

$$c \equiv c_0 + \ell c_1 \mod \ell^2.$$

Now let E_{c_0} be as in (3.33) and consider the system

$$\sum_{j=0}^{a_f} (A_j + \ell \tilde{A}_j) E_{c_0}^j \equiv 0 \mod (\ell^2, x^a).$$

Solve it over $\mathbb{F}_\ell$ to find c_1 and the unknown polynomials $\tilde{A}_j$. After a finite number of lifting steps and rational reconstruction, we will have the minimal polynomial

$$\sum_{j=0}^{a_f} A_j y^j$$

of f in $\mathbb{Q}[x,y]$.

3.8.3 Recovering Parameter of Exp-product

After finding f, we can compute the differential operator M, such that $L_B \xrightarrow{f}_C M \xrightarrow{r}_E L_{\text{inp}}$. Then we can compare the second highest terms of M and L_{inp} to find the parameter r of the exp-product transformation. If $M = \partial^2 + B_1\partial + B_0$ and $L_{\text{inp}} = \partial^2 + A_1\partial + A_0$, then

$$r = \frac{B_1 - A_1}{2}.$$

We can also use the implementation in [41] to find the parameter $r \in \overline{\mathbb{Q}(x)}$.

CHAPTER 4

INTEGRAL BASES FOR DIFFERENTIAL OPERATORS AND NORMALIZATION AT INFINITY

In this chapter, we investigate the notion of integral bases for differential operators (first introduced in [31] and further developed in [13, 26, 27]), give a fast algorithm to compute integral bases, and an algorithm to normalize these bases at the point at infinity.

Why compute a normalized integral basis? Suppose a computation, say Feynman diagrams, leads to a differential equation $L(y) = 0$ with $L \in D = \mathbb{C}(x)[\partial]$ where $\partial = \frac{d}{dx}$. The operator L is gauge equivalent to many other operators; for any cyclic vector $\mathcal{G} \in D/DL$ the annihilator $\tilde{L}$ of $\mathcal{G}$ is gauge equivalent to L. In many computation it is not likely that the computation happened to produce the "smallest/nicest" operator L. In such situation, the natural question becomes: Given L, how to find gauge equivalent $\tilde{L}$ with "small" (close to optimal as in Remark 4.1.1) coefficients?

Suppose $L \in D$ is a differential operator with "large" coefficients and L is gauge equivalent to another operator $\tilde{L}$ with "small" coefficients. If we can find this gauge transformation and $\tilde{L}$, then it reduces a large problem to a smaller one. A gauge transformation is a D-module isomorphism $D/D\tilde{L} \to D/DL$. To find it we need the image $\mathcal{G} \in D/DL$ of the generator $1 \in D/D\tilde{L}$. How to find $\mathcal{G} \in D/DL$ with "small" annihilator $\tilde{L}$?

An analogous situation is the problem of finding an element $\mathcal{G} \in \mathbb{Q}[x]/(L)$ (for given $L \in \mathbb{Q}[x]$) whose minimal polynomial over $\mathbb{Q}$ has "small" coefficients. An algebraic number like $\mathcal{G} = \frac{x}{10^{1000}} \in \mathbb{Q}[x]/(L)$, although "small" viewed as a complex number, is "large" in terms of bit-size. However, if $\mathcal{G}$ is integral over $\mathbb{Z}$ and has small absolute values ("small" at ∞), then the bit-size of the minimal polynomial of $\mathcal{G}$ will be small.

In analogy, we search for $\mathcal{G} \in D/DL$ that is "integral" and simultaneously "small at ∞". This way we obtain $\mathcal{G}$ whose annihilator $\tilde{L}$ has "small" coefficients.

Remark 4.0.1. Some of the contents of this chapter (apart from some new materials and adoptions) have been published in [26, 27].

Notation 4.0.1. Throughout this chapter we use $\partial = \frac{d}{dx}$.

4.1 Standard Form Map

Definition 4.1.1. Let A be non-empty set, $\sim$ be an equivalence relation on A, and B be the set of non-empty finite subsets of A. A map $\Phi : A \longrightarrow B$ is called a *standard form map* if the following is true:

1. For all $a_1, a_2 \in A$, $\Phi(a_1) = \Phi(a_2)$ if and only if $a_1 \sim a_2$.
2. For all $a \in A$ and for all $b \in \Phi(a)$, $a \sim b$.

Remark 4.1.1. A standard form map Φ is useful if $\Phi(a)$ gives elements of the equivalence class of a that are close to optimal in some sense (small bit-size, height, or number of apparent singularities).

Example 4.1.1. Let $A = \{f \in \mathbb{Q}[x] \mid f \text{ is irreducible}\}$. Let for $f_1, f_2 \in \mathbb{Q}[x]$,

$$f_1 \sim f_2 \Longleftrightarrow \mathbb{Q}[x]/(f_1) \cong \mathbb{Q}[x]/(f_2).$$

Given f_1 it is easy to find many $f_2 \in A$ with $f_1 \sim f_2$ (just pick the minimal polynomial of a randomly chosen element of $\mathbb{Q}[x]/(f_1) - \mathbb{Q}$). However, random choices tend to give polynomials with large bit-size.

Goal: For given f_1, find $f_2 \in A$ with small bit-size (small height) such that $f_1 \sim f_2$.

Solution: `POLRED` algorithm [15]:

1. Compute a basis for algebraic integers of $\mathbb{Q}[x]/(f_1)$, and
2. Apply `LLL` [36] to this basis to find an algebraic integer with small absolute values.

Application: Reduce computations in $\mathbb{Q}[x]/(f_1)$ to computations in $\mathbb{Q}[x]/(f_2)$ where the bit-size of f_2 is close to optimal.

Example 4.1.2. Let $\partial = \frac{d}{dx}$, $D = \mathbb{Q}(x)[\partial]$, and $A = \{L \in D \mid L \text{ is irreducible}\}$. Let for $L_1, L_2 \in D$,

$$L_1 \sim L_2 \Longleftrightarrow D/DL_1 \cong D/DL_2 \text{ as } D\text{-modules.}$$

Goal: Given $L_1 \in A$, find one or more $L_2 \in A$ with small height such that $L_1 \sim L_2$.

Solution: Imitate `POLRED` algorithm:

1. Chapter 4, analog of the first step of `POLRED`, and
2. Chapter 5.1, analog of the second step of `POLRED`.

Application: Reduce solving an operator $L_1 \in \mathbb{Q}(x)[\partial]$ with many apparent singularities to solving another operator $L_2 \in \mathbb{Q}(x)[\partial]$ with few apparent singularities (see Example 5.1.1). For another recent application of integral bases [13].

4.2 Integral Bases

Remark 4.2.1. Let $L \in \mathbb{C}(x)[\partial]$ be a differential operator and t_s be the local parameter (Definition 2.1.7) of a point $s \in \mathbb{C} \cup \{\infty\}$. If $L \in \mathbb{C}(x)[\partial]$ is regular singular of order n, then, by Theorem 2.1.1, L has a basis of formal solutions at a point $x = s$ in the form

$$y = t_s^{\nu_s} \sum_{i=0}^{\infty} P_i t_s^i \tag{4.1}$$

where $\nu_s \in \mathbb{C}$ and $P_i \in \mathbb{C}[\log(t_s)]$ with $\deg(P_i) < \operatorname{ord}(L)$ and $P_0 \neq 0$.

Definition 4.2.1. Let y be as in (4.1) with $P_0 \neq 0$. Then the *valuation* of y at $x = s$ is defined as

$$v_s(y) = \operatorname{Re}(\nu_s).$$

Remark 4.2.2. It can be seen from Definition 4.2.1 that,

- $v_s(y) > 0$ if and only if y converges to 0 as x approaches s,
- $v_s(y_1 y_2) = v_s(y_1) + v_s(y_2)$.

Definition 4.2.2. Fix $L \in \mathbb{C}(x)[\partial]$ and let $G \in \mathbb{C}(x)[\partial]$. The operator G is called *integral for L at s* if

$$v_s(G) = \inf\{v_s(G(y)) \,|\, y \text{ is a solution of } L \text{ at } x = s\} \geq 0.$$

Remark 4.2.3. In Definition 4.2.2, without loss of generality, we may assume that $\operatorname{ord}(G) < \operatorname{ord}(L)$ because using division with remainder we can write $G = QL + R$ for some $Q, R \in \mathbb{C}(x)[\partial]$ such that $\operatorname{ord}(R) < \operatorname{ord}(L)$ and $G(y) = R(y)$ for all solutions y of L.

Definition 4.2.3. Fix $L \in \mathbb{C}(x)[\partial]$. An operator $G \in \mathbb{C}(x)[\partial]$ is called *integral for L* if $v_s(G) \geq 0$ for all $s \in \mathbb{C}$.

Definition 4.2.4. Let $L \in \mathbb{C}(x)[\partial]$ with $\mathrm{ord}(L) = n$, and let

$$\mathcal{O}_L = \{G \in \mathbb{C}(x)[\partial] \,|\, G \text{ is integral for } L \,\text{and}\, \mathrm{ord}(G) < n\}.$$

A basis of $\mathcal{O}_L$, as a $\mathbb{C}[x]$-module, is called an *(global) integral basis for L.*

Definition 4.2.5. Let $L \in \mathbb{C}(x)[\partial]$ of order n and $P \in \mathbb{C}[x]$. We say that $\{b_1, \ldots, b_n\}$ is a *local integral basis for L at P* when

$$\left\{\frac{A_1}{B_1}b_1 + \cdots + \frac{A_n}{B_n}b_n \,|\, A_i, B_i \in \mathbb{C}[x] \,\text{and}\, \gcd(P, B_i) = 1\right\}$$
$$=$$
$$\{G \,|\, G \text{ is integral for } L \text{ at every root of } P\}.$$

Remark 4.2.4. A local integral basis for L at a finite singularity $s \in \mathbb{C}$ is a local integral basis at $P = x - s$.

Remark 4.2.5. To compute a local integral basis for L at infinity, apply a change of variables transformation $x \mapsto \frac{1}{x}$ and compute a local integral basis for the resulting operator $L_{1/x}$ at 0. Applying change of variables transformation to this local basis elements with parameter $\frac{1}{x}$ gives a local integral basis for L at infinity.

Lemma 4.2.1. *If $\{b_1, \ldots, b_n\}$ is an integral basis for $L \in \mathbb{C}(x)[\partial]$ at $s \in \mathbb{C}$ and $c_1, \ldots, c_n \in \mathbb{C}(x)$ such that $v_s(c_i) = 0$ for all $i \in \{1, \ldots, n\}$, then $\{c_1b_1, \ldots, c_nb_n\}$ is also an integral basis for L at $s \in \mathbb{C}$.*

Theorem 4.2.2.

A *$\{b_1, \ldots, b_n\}$ is a local integral basis for $L \in \mathbb{C}(x)[\partial]$ at $x = 0$*

if and only if

B *for all $i, j \in \{1, \ldots, n\}$ we have $v_0(b_i(y_j)) \geq 0$ (here $\{y_1, \ldots, y_n\}$ is a basis of solutions of L at $x = 0$, each y_i is of the form (4.1)).*

and

C *for all $(c_1, \ldots, c_n) \in \mathbb{C}^n \setminus (0, \ldots, 0)$ there exists $j \in \{1, \ldots, n\}$ such that $v_0((c_1b_1 + \cdots + c_nb_n)(y_j)) < 1$.*

Proof.

1. $\mathbf{A} \Rightarrow \mathbf{B}$ and $\mathbf{C}$

   $\mathbf{B}$ is a part of Definition 4.2.5. If $\neg\mathbf{C}$, then for $(c_1, \ldots, c_n) \in \mathbb{C}^n \setminus (0, \ldots, 0)$ we have $v_0((c_1b_1 + \cdots + c_nb_n)(y_j)) \geq 1$ for all j. Then $\frac{1}{x}(c_1b_1 + \cdots + c_nb_n)$ is integral at 0 contradicting $\mathbf{A}$.

2. $\mathbf{A} \Leftarrow \mathbf{B}$ and $\mathbf{C}$

   $\mathbf{B}$ says that $b_1, \ldots, b_n$ are integral for L at 0. We will first show that $\mathbf{C}$ implies that they are linearly independent over $\mathbb{C}[x]$. Assume for $C_1, \ldots, C_n \in \mathbb{C}[x]$ with constant terms $c_1, \ldots, c_n$ such that $(c_1, \ldots c_n) \neq (0, \ldots, 0)$ we have $C_1b_1 + \cdots + C_nb_n = 0$. Then,

$$\sum_i C_ib_i - \sum_i c_ib_i = x \cdot \sum_i a_ib_i$$

   for some $a_i \in \mathbb{C}[x]$. Then for every solution y_j of L at $x = 0$,

$$v_0\left(\left(\sum_i C_ib_i - \sum_i c_ib_i\right)(y_j)\right) = v_0\left(\left(x \cdot \sum_i a_ib_i\right)(y_j)\right) \geq 1.$$

   because $v_0(b_i(y_j)) \geq 0$ for all i since $\sum_i C_ib_i = 0$. This implies $v_0\left((c_1b_1 + \cdots + c_nb_n)(y_j)\right) \geq 1$ which contradicts $\mathbf{C}$. Therefore, $b_1, \ldots, b_n$ are $\mathbb{C}[x]$-linearly independent. This means that any integral G can be written as $G = \sum_i c_ib_i$ for some $c_i \in \mathbb{C}(x)$. Suppose $\min_i\{v_0(c_i)\} < 0$. After multiplying by a power of x we may assume that $\min_i\{v_0(c_i)\} = -1$. Let r_i be the residue of c_i. Then

$$G = \frac{1}{x}\sum_i r_ib_i + \sum_i \tilde{c}_ib_i.$$

   with $v_0(\tilde{c}_i) \geq 0$. This implies $\frac{1}{x}\sum_i r_ib_i$ is integral for L at 0. This implies for each y_j, $v_0((\frac{1}{x}\sum_i r_ib_i)(y_j)) \geq 0$, so $v_0((\sum_i r_ib_i)(y_j)) \geq 1$ which contradicts $\mathbf{C}$. Therefore, $G = \sum_i c_ib_i$ with $v_0(c_i) \geq 0$ and hence $\mathbf{A}$.

□

4.2.1 Computing a Local Integral Basis at One Point

Let $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ be a given (i.e., input) regular singular differential operator with $\text{ord}(L_{\text{inp}}) = n$, and let $s \in \mathbb{C}$ be a singularity of L_{inp}. We can always move the point s to 0 via a change of variables transformation $x \mapsto x + s$. So, without loss of generality, we may assume that $s = 0$. So, to compute a local integral basis at s it is enough to have a procedure to compute a local integral basis at 0. We compute a local integral basis $\{b_1, \ldots, b_n\}$ for L_{inp} at $x = 0$ as follows: First of all we need to compute the valuations

$$v_j = v_0(y_j)$$

of the formal solutions y_j of L_{inp} at $x = 0$ in order to see that how far they are from being integral at $x = 0$. Here $j = 1, \dots, n$. Then we find the integer

$$m = -\lfloor \min(v_j) \rfloor. \tag{4.2}$$

The first element of the local integral basis for L_{inp} at 0 is given by

$$b_1 = x^m. \tag{4.3}$$

This basis element, b_1, shifts all of the valuations v_j by m and makes all of the valuations of $b_1(y_j) = x^m y_j$ non-negative. Then, for $i = 2, \dots, n$, we set the initial value for the local basis element b_i,

$$b_i = x \cdot \partial \cdot b_{i-1}.$$

At this point, $b_1, \dots, b_n$ are linearly independent and are integral. For an integral basis there is more requirement, namely Theorem 4.2.2 part **C**. So we make the ansatz

$$\mathfrak{A} = \frac{1}{x} \left(u_1 \cdot b_1 + \cdots + u_{i-1} \cdot b_{i-1} + b_i \right) \tag{4.4}$$

where $c_1, \dots, c_{i-1}$ are unknown constants. For every formal solution y_j of L_{inp} at $x = 0$ we evaluate

$$\mathfrak{A}(y_j).$$

This $\mathfrak{A}(y_j)$ may be integral or non-integral at $x = 0$. If it is non-integral, then finitely many terms of $\mathfrak{A}(y_j)$ are non-integral at $x = 0$, i.e., have negative valuations. Find the non-integral terms of $\mathfrak{A}(y_j)$ and equate their coefficients to 0. This process will give us a linear system of equations with unknown constants $c_1, \dots, c_{i-1}$. If this linear system admits a solution, then replace b_i by $\mathfrak{A}$ evaluated at that solution. We repeat this process until the system no longer has a solution. Then

$$\{b_1, \dots, b_n\}$$

satisfies condition **C** and will be a local integral basis for L_{inp} at $x = 0$ by Theorem 4.2.2.

Remark 4.2.6. The method explained in Section 4.2.1 works and it is not so different than the method given in the paper [31]. However, one can speed up it by dealing with apparent singularities and algebraic singularities separately. These two cases are discussed in Sections 4.2.2 and 4.2.3 respectively.

Algorithm 4.2.1 ([25]). General Outline of `local_basis_at_0`.

Input:

- $L_{\text{inp}} \in \mathbb{Q}(x)[\partial] =$ A regular singular differential operator of order n.

Output:

- A local integral basis for L_{inp} at $x = 0$ ($P = x$).

1. Let $b_1 = x^m$ as in (4.3), where $m = -\lfloor \min(v_k) \rfloor$ as in (4.2).
2. For $1 < i \leq n$ do:
   (a) Set $b_i = x \cdot \partial \cdot b_{i-1}$.
   (b) Make the ansatz $\mathfrak{A}$ as in (4.4) with unknown constants $c_1, \ldots, c_{i-1}$.
   (c) For every solution y of L_{inp} at $x = 0$ compute $\mathfrak{A}(y)$. Equate the coefficients[1] of all non-integral terms of $\mathfrak{A}(y)$ to 0 and form a system of equations with unknowns $c_1, \ldots, c_{i-1}$.
   (d) Find the solution (if there is one), update b_i as $b_i = \mathfrak{A}$, and go back to Step 2b. Otherwise, go to the next i in the loop.
3. Return $\{b_1, \ldots, b_n\}$.

Theorem 4.2.3. *Algorithm 4.2.1 terminates.*

Proof. The proof of Theorem 4.2.3 is very similar to the proof of Theorem 18 in [31]. □

4.2.2 Local Bases at Apparent Singularities

Definition 4.2.6. Let s be a singularity of $L \in \mathbb{Q}(x)[\partial]$ where $\text{ord}(L) = n$. If formal solutions of L at s have no logarithms and have non-negative integer valuations $0 \leq e_1 < e_2 < \cdots < e_n$ at s, then s is called an apparent singularity. Equivalently, this means that all solutions of L are analytic at s.

To compute a local integral basis for a regular singular operator $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ with $\text{ord}(L_{\text{inp}}) = n$, at an apparent singularity s, we do not need to use Algorithm 4.2.1. By Theorem 4.2.2 a local integral basis at an apparent singularity is given by

$$\{b_1 = \partial^{e_1}, \ldots, b_n = \partial^{e_n}\}. \tag{4.5}$$

[1] y_j in 4.1 contains an infinite power series, but it is not hard to bound how many terms are actually needed, see the implementation at [24] for the precise bound.

Here note that if $e_i \geq \text{ord}(L_{\text{inp}})$, then take the remainder $b_i = \text{Rem}(\partial^{e_i}, L_{\text{inp}})$ to bring b_i in standard form (see Remark 4.2.3 and Definition 4.2.4). For efficiency reasons, our implementation (Algorithm 4.2.2) only checks for apparent singularities of the most common type where $(e_1, e_2, \ldots, e_{n-1}, e_n) = (0, 1, \ldots, n-2, n)$. These are covered by (4.5) in our implementation while everything else is covered by Algorithm 4.2.1.

4.2.3 Local Bases at Algebraic Singularities

Definition 4.2.7. Let $\mathbb{Q}(s)$ be an algebraic number field of degree d over $\mathbb{Q}$. There are field embeddings $\sigma_i : \mathbb{Q}(s) \to \mathbb{C}$ for $i = 1, \ldots, d$.

- The *trace* of an element $\alpha \in \mathbb{Q}(s)$ is

$$\text{Tr}(\alpha) = \sum_{i=1}^{d} \sigma_i(\alpha).$$

- The numbers $s_i = \sigma_i(s)$ are called the *conjugates* of s, they are the roots of the minimal polynomial P of s.

Let $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ be a regular singular operator with $\text{ord}(L_{\text{inp}}) = n$. Let s be an algebraic singularity of L_{inp} with minimal polynomial $P \in \mathbb{Q}[x]$ and let $[\mathbb{Q}(s) : \mathbb{Q}] = d$. Each conjugate of s (each root of P) is also a singularity of L_{inp}. We could compute a local integral basis for L_{inp} at every conjugate of s with Algorithm 4.2.1. The primary reason of the huge time differences, given in Table 4.1, between our algorithm and the algorithm in paper [31] is that, while the algorithm in paper [31] computes at every conjugate of s in $\mathbb{Q}(s_1, \ldots, s_d)$, we only[2] compute a local integral basis at s and then modify this basis in such a way that it becomes an integral basis for L_{inp} at every conjugate of s. To do that, first we scale the local integral basis and then we use the trace map. Details are as follows: First, we compute a local integral basis for L_{inp}

$$\{b_1, \ldots, b_n\} \subset \mathbb{Q}[x, \frac{1}{x-s}][\partial]$$

at the algebraic singularity s by using Algorithm 4.2.1. Here $\text{ord}(b_i) = i - 1$. We want to scale b_i to $\tilde{b}_i = c_i b_i$ in such a way that

1. $\tilde{b}_1, \ldots, \tilde{b}_n$ is still an integral basis at $x = s$ (Lemma 4.2.1),

[2]Computations over $\mathbb{Q}(s)$ (degree d) take much less time than computations over $\mathbb{Q}(s_1, \ldots, s_d)$ (degree $\leq d!$).

2. $\tilde{b}_1, \ldots, \tilde{b}_n$ have valuations at least 1 at all roots of $\frac{P}{x-s}$.

After that

$$\{\mathrm{Tr}(\tilde{b}_1), \ldots, \mathrm{Tr}(\tilde{b}_n)\} \subset \mathbb{Q}(x)[\partial]$$

is a local integral basis at all conjugates of s, by Theorem 4.2.2. This scaling factor is

$$c_i = \left(\frac{P}{x-s}\right)^{a+i}$$

where

$$a = v_s(b_1).$$

4.2.4 Integral Basis for a Differential Operator at a Polynomial

Sections 4.2.1, 4.2.2, and 4.2.3 lead to the following algorithm:

Algorithm 4.2.2 ([25])**.** General Outline of `local_basis_minpoly`.

Inputs:

- $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ = A monic regular singular differential operator of order n.
- $P \in \mathbb{Q}[x]$ = An irreducible factor of P_{sing}.
- P_{sing} = Denominator of L_{inp} ($\{$roots of $P_{\text{sing}}\} = \{$singularities of $L_{\text{inp}}\} - \{\infty\}$).
- Δ = Greatest common divisor of P_{sing} and the denominator of $\mathrm{LCLM}(L_{\text{inp}}, \partial - r)$ where r is a random integer.

Output:

- A local integral basis for L_{inp} at P.

1. Let s be a root of P.

   (a) If $P \nmid \Delta$ (then s is an apparent singularity) and $P^2 \nmid P_{\text{sing}}$, then s is an apparent singularity of the most common type where $(e_1, e_2, \ldots, e_{n-1}, e_n) = (0, 1, \ldots, n-2, n)$. Now, use Section 4.2.2 and return

   $$\{1, \partial, \ldots, \partial^{n-2}, \mathrm{Rem}(\partial^n, L_{\text{inp}})\}.$$

   Otherwise, compute a local integral basis $\{b_1, \ldots, b_n\}$ of L_{inp} at s with Algorithm 4.2.1.

   (b) If $\deg(P) = 1$, then return $\{b_1, \ldots, b_n\}$. Otherwise let

   $$a = v_s(b_1) \tag{4.6}$$

   and return

   $$\left\{\mathrm{Tr}\left(\left(\frac{P}{x-s}\right)^{a+1} b_1\right), \ldots, \mathrm{Tr}\left(\left(\frac{P}{x-s}\right)^{a+n} b_n\right)\right\}. \tag{4.7}$$

4.2.5 Combining Two Local Integral Bases

Theorem 4.2.4 (The Classification Theorem, [3])**.** *Let R be a principal ideal domain (PID) and M be a finitely generated R-module. Then there exist non-zero, non-unit ideals $(a_1), \ldots, (a_m)$ such that $(a_1) \supseteq \cdots \supseteq (a_n)$ and*

$$M \cong R^{\operatorname{rank}(M)} \oplus (R/(a_1) \oplus \cdots \oplus R/(a_n)) .$$

Let $L_{\text{inp}} \in \mathbb{C}(x)[\partial]$ be a differential operator with $\operatorname{ord}(L_{\text{inp}}) = n$. Let

$$\{b_1, \ldots, b_n\} \tag{4.8}$$

be a local integral basis for L_{inp} at P_1,

$$\{\beta_1, \ldots, \beta_n\} \tag{4.9}$$

be a local integral basis for L_{inp} at P_2, and $\gcd(P_1, P_2) = 1$. Assume $\operatorname{ord}(b_i) = \operatorname{ord}(\beta_i) = i - 1$, i.e., (4.8) and (4.9) are in *triangular form*. Let

$$M_1 = \operatorname{SPAN}_{\mathbb{C}[x]}\{b_1, \ldots, b_n\},$$
$$M_2 = \operatorname{SPAN}_{\mathbb{C}[x]}\{\beta_1, \ldots, \beta_n\}.$$

$M_1 + M_2$ is a module over $\mathbb{C}[x]$, which is a PID, and $M_1 + M_2 \subseteq V$ where V is the $\mathbb{C}(x)$-vector space

$$V = \{G \in \mathbb{C}(x)[\partial] \mid \operatorname{ord}(G) < n\}.$$

$M_1 + M_2$ is torsion-free because $M_1 + M_2 \subseteq V$. Moreover, $\operatorname{rank}(M_1 + M_2) = n$ because $\dim_{\mathbb{C}(x)}(V) = n$. Then, from the Classification Theorem (Theorem 4.2.4), $M_1 + M_2$ is a free module and has a $\mathbb{C}[x]$-module basis $\{B_1, \ldots, B_n\}$. We want to find this basis because it is a local integral basis at $P_1 P_2$, provided that (4.8) is integral at P_2 and (4.9) is integral at P_1. Algorithm 4.2.2 does not guarantee this condition, so we need to multiply (4.8) by suitable powers of P_2:

$$\tilde{b}_i = P_2^{i-1} \cdot \beta_1 \cdot b_i$$

is integral at all roots of P_2 and the set

$$\{\tilde{b}_1, \ldots, \tilde{b}_n\} \tag{4.10}$$

is still a local integral basis at P_1 (Lemma 4.2.1). Similarly, we need to multiply (4.9) by suitable powers of P_1:

$$\tilde{\beta}_i = P_1^{i-1} \cdot b_1 \cdot \beta_i$$

is integral at all roots of P_1 and the set

$$\{\tilde{\beta}_1, \ldots, \tilde{\beta}_n\} \tag{4.11}$$

is still a local integral basis at P_2 (Lemma 4.2.1). We want to find a basis of the $\mathbb{C}[x]$-module generated by (4.10) and (4.11). Let $B_i = \texttt{combine\_diff\_ops}(\tilde{b}_i, \tilde{\beta}_i)$ be the output of the Algorithm 4.2.3 below for inputs $\tilde{b}_i$ and $\tilde{\beta}_i$. The set

$$\{B_1, \ldots, B_n\}$$

is a local integral basis for L_{inp} at P_1P_2 in triangular form. To see this, we can use Condition b of the output of Algorithm 4.2.3 below and apply induction on i.

Algorithm 4.2.3 ([25])**.** General Outline of `combine_diff_ops`.

Inputs:

- $L_1, L_2 \in \mathbb{C}(x)[\partial]$ = Two operators of order i.

Output:

- An operator $L \in \mathbb{C}(x)[\partial]$ such that
  (a) $L = TL_1 + SL_2$ for some $T, S \in \mathbb{C}[x]$,
  (b) For any $Q_1, Q_2 \in \mathbb{C}[x]$ there is $R \in \mathbb{C}[x]$ such that $Q_1L_1 + Q_2L_2 - RL$ has order less than i.

1. Let f and g be the leading coefficients of L_1 and L_2 with respect to ∂. Let D be the least common multiple of the denominators of f and g. So $f = \frac{A}{D}$ and $g = \frac{B}{D}$ with $A, B, D \in \mathbb{C}[x]$.
2. Write $\gcd(A, B) = SA+TB$ for some $S, T \in \mathbb{C}[x]$ found by the Extended Euclidean Algorithm.
3. Return $L = SL_1 + TL_2$.

4.2.6 Computation of Global Integral Bases

We can compute a global integral basis for an $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ as follows:

Algorithm 4.2.4 ([25])**.** General Outline of `global_integral_basis`.

Input:

- $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ = A regular singular differential operator of order n.

Output:

- A global integral basis for L_{inp}.

1. If L_{inp} is monic, then let P_{sing} be the common denominator of L_{inp}. If $L_{\text{inp}} \in \mathbb{C}[x][\partial]$, let P_{sing} be the leading coefficient of L_{inp}. P_{sing} is the polynomial whose zeros are the finite singularities of L_{inp}.
2. For each irreducible factor P of P_{sing}, use Algorithm 4.2.2 to compute local integral bases for L_{inp} at P.
3. Use Section 4.2.5 to combine all local bases for L_{inp} at P and return a global integral basis for L_{inp}.

4.3 Normalization at Infinity

Definition 4.3.1. Let $L \in \mathbb{C}(x)[\partial]$ be a differential operator with $\text{ord}(L) = n$. The set $\{b_1, \dots, b_n\}$ is called *normalized* at $x = s$, if there exist rational functions $r_i \in \mathbb{C}(x)$ such that $\{r_1 b_1, \dots, r_n b_n\}$ is a local integral basis at $x = s$.

Definition 4.3.2. Let $\mathcal{G}, L \in \mathbb{C}(x)[\partial]$ be differential operators. *The degree of* $\mathcal{G}$ *at infinity (for L)* is the smallest $m \in \mathbb{Z}$ such that $\frac{1}{x^m}\mathcal{G}$ is integral for L at infinity.

Remark 4.3.1. Let $L \in \mathbb{C}(x)[\partial]$ with $\text{ord}(L) = n$.

1. If $\{b_1, \dots, b_n\}$ is a local integral basis for L at infinity, then the degree of $\mathcal{G} = r_1 b_1 + \dots + r_n b_n$ ($r_i \in \mathbb{C}(x)$) at infinity is equal to $\max_i\{-\lfloor v_\infty(r_i b_i)\rfloor\}$.
2. $\mathcal{G} \in \mathbb{C}(x)[\partial]$ is integral for L at infinity if and only if the degree of $\mathcal{G}$ at infinity is ≤ 0.

Let $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ be a given regular singular operator of order n. Let $\{B_1, B_2, \ldots, B_n\}$ be an integral basis for L_{inp}. The basis $\{B_1, B_2, \ldots, B_n\}$ is normalized at every finite singularity of L_{inp}. We want to normalize it at $x = \infty$. The process of normalizing an integral basis in an algebraic function field at infinity was introduced in [39] as one of the steps in the integration of algebraic functions. We can normalize a global integral basis for L_{inp} at infinity as follows:

Algorithm 4.3.1 ([25]). General Outline of `normalize_at_infinity`.

Inputs:

- $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ = A regular singular differential operator,
- $\{B_1, B_2, \ldots, B_n\}$ = An integral basis for L_{inp}.

Output:

- A normalized global integral basis for L_{inp} and a list of integers (degrees of the basis elements at infinity).

1. Compute a local integral basis $\{b_1, b_2, \ldots, b_n\}$ for L_{inp} at $x = \infty$ with Remark 4.2.5 and Algorithm 4.2.1

2. Write

$$B_i = \sum_{j=1}^{n} r_{i,j} b_j$$

where $r_{i,j} \in \mathbb{Q}(x)$.

3. Let $D \in \mathbb{Q}[x]$ be a non-zero polynomial for which

$$a_{i,j} := D r_{i,j} \in \mathbb{Q}[x]$$

for all i, j.

4. (a) For each $i \in \{1, 2, \ldots, n\}$ let m_i be the maximum of the degrees of $a_{i,1}, a_{i,2}, \ldots, a_{i,n}$. Let $V_i \in \mathbb{Q}^n$ be the vector whose j-th entry is the x^{m_i}-coefficient of $a_{i,j}$. Let

$$d_i := m_i - \deg(D), \qquad \text{“}d_i = \text{degree of } B_i \text{ at infinity”}. \tag{4.12}$$

   (b) If $V_1, \ldots, V_n$ are linearly independent, then compute

$$B_i = \frac{1}{D} \sum_{j=1}^{n} a_{i,j} b_j. \tag{4.13}$$

and return

$$[B_1, \ldots, B_n] \text{ and } [d_1, \ldots, d_n]$$

sorted in such a way that $d_1 \leq \cdots \leq d_n$.

Otherwise, take $c_1, \ldots, c_n$ such that $c_1 V_1 + \cdots + c_n V_n = 0$.

(c) Among those $i \in \{1, 2, \ldots, n\}$ for which $c_i \neq 0$, choose one for which d_i is maximal. For this i, update $a_{i,j}$ as

$$a_{i,j} \leftarrow \sum_{k=1}^{n} c_k x^{d_i - d_k} a_{k,j}. \tag{4.14}$$

Then go back to Step 4a. Assignment in (4.14) lowers the degree of the corresponding coefficient $a_{i,j}$ in (4.13). Algorithm updates $a_{i,j}$ (4.14) if degree reduction is possible.

The $\{B_1, B_2, \ldots, B_n\}$ remains an integral basis of L_{inp} throughout Algorithm 4.3.1 because the B_i in (4.13) in Step 6.1 can be written as a non-zero rational number times the old B_i plus a $\mathbb{Q}[x]$-linear combination of the B_j with $j \neq i$. When we go back to Step 4a the non-negative m_i decreases while m_j $(i \neq j)$ stays the same. Hence Algorithm 4.3.1 must terminate.

Definition 4.3.3. Let $L \in \mathbb{Q}(x)[\partial]$ be an operator of order n. Let $\{B_1, \ldots, B_n\}$ be a normalized integral basis for L such that d_i is the degree of B_i at infinity for all $i \in \{1, \ldots, n\}$ and $d_1 \leq \cdots \leq d_n$. We say $d_1 \leq \cdots \leq d_n$ is the *degree sequence* for L.

Lemma 4.3.1. *Let $L \in \mathbb{Q}(x)[\partial]$ be an operator of order n. Let $\{B_1, \ldots, B_n\}$ be a normalized integral basis for L with the degree sequence $d_1 \leq \cdots \leq d_n$.*

- *Let $\mathcal{G} = r_1 B_1 + \cdots + r_n B_n$ where $r_1, \ldots, r_n \in \mathbb{Q}(x)$. The degree of $\mathcal{G}$ at infinity equals the maximum of the degrees of the terms $r_i B_i$ at infinity.*
- *Let $\mathcal{G} = r_1 B_1 + \cdots + r_n B_n$ where $r_1, \ldots, r_n \in \mathbb{Q}[x]$. The degree of $\mathcal{G}$ at infinity equals the maximum of* $\deg(r_i) + d_i$ *where $i = 1, \ldots, n$ and* $\deg(0) = -\infty$.

Proof. Follows from Definition 4.3.2 and Remark 4.3.1. □

Definition 4.3.4. Let $L \in \mathbb{Q}(x)[\partial]$ be an operator of order n. Let $\{B_1, \ldots, B_n\}$ be a normalized integral basis for L with the degree sequence $d_1 \leq \cdots \leq d_n$. For $m \in \mathbb{Z}$, define

$$\begin{aligned} \mathcal{O}_L(m) &= \{\mathcal{G} \,|\, \mathcal{G} \text{ is integral for } L \text{ and the degree of } \mathcal{G} \text{ at infinity is} \leq m\} \\ &= \{r_1 B_1 + \cdots + r_n B_n \,|\, r_i \in \mathbb{Q}[x] \text{ with } \deg(r_i) \leq m - d_j\} \end{aligned}$$

where $\deg(r_i) < 0$ means $r_i = 0$.

Table 4.1: Comparison of timings of Kauers' and Koutschan's integral basis algorithm [31] and our integral basis algorithm (in seconds) on a computer with a 2.5 GHZ Intel Core i5-3210M CPU and 8 GB RAM.

Example	[31]	Our algorithm
1	0.185	0.111
2	0.863	0.156
3	0.233	0.182
4	0.592	0.226
5	12.351	0.294
6	66.537	0.377
7	124.197	0.499
8	151.942	0.515
9	175.580	0.569
10	157.484	0.596
11	145.185	0.602
12	230.897	0.688
13	1609.865	0.699
14	> 1600	0.918
15	> 1600	1.133
16	> 1600	1.156
17	> 1600	1.251

Remark 4.3.2. Let $L_1, L_2 \in \mathbb{Q}(x)[\partial]$. If L_1 and L_2 are gauge equivalent, then the isomorphism between $\mathcal{O}_{L_1}$ and $\mathcal{O}_{L_2}$ gives a bijection between $\mathcal{O}_{L_1}(m)$ and $\mathcal{O}_{L_2}(m)$. This implies that L_1 and L_2 have the same degree sequences, since $\dim(\mathcal{O}_{L_1}(m))$ (for $m \in \mathbb{Z}$) determines (and determined by) the degree sequence.

Remark 4.3.3. The examples used for the comparison in Table 4.1 come from [9, 11, 12] (see the plain text file named `equations17` in [24] for these examples).

CHAPTER 5

APPLICATIONS OF NORMALIZED INTEGRAL BASES

In this chapter, we define *standard forms* of an arbitrary order regular singular differential operator. In the last part of this section, we give a general algorithm (Algorithm 5.2.1) to compute hypergeometric solutions of second order regular singular differential equations. This algorithm is combination of quotient method algorithm (Algorithm 3.2.1), integral basis algorithm (Algorithm 4.2.4), and normalization algorithm (Algorithm 4.3.1).

Remark 5.0.1. Some of the contents of this chapter (apart from some new materials and adoptions) have been published in [26].

5.1 Standard Forms of a Differential Operator

The following algorithm is an answer for the problem stated in Example 4.1.2.

Algorithm 5.1.1 ([25])**.** General Outline of `standard_forms`.

Input:

- $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ = An irreducible regular singular differential operator of order n.

Output:

- A set of n-th order operators which are gauge equivalent to L_{inp} and have few apparent singularities.

1. Compute a normalized integral basis $\{B_1, \ldots, B_n\}$ for L_{inp} using Algorithm 4.2.4 and Algorithm 4.3.1. For all $i \in \{1, \ldots, n\}$, let d_i be the degree of B_i at infinity such that $d_1 \leq \cdots \leq d_n$.
2. If $d_1 < d_2$ then, find L_1 such that $L_{\text{inp}} \xrightarrow{B_1}_G L_1$. Let $S = \{L_1\}$ and return S.
3. If $d_1 = d_2 = \cdots = d_k$ (with $d_k < d_{k+1}$ or $k = n$), then:
   (a) Let $S = \{\}$.

(b) Make the ansatz

$$\mathfrak{B} = c_1 B_1 + \cdots + c_k B_k. \tag{5.1}$$

with unknowns $c_1, \ldots, c_k$.

(c) For $j = 0, \ldots, n-1$, compute the remainders $\mathfrak{R}_j$ when $\partial^j \cdot \mathfrak{B}$ is right-divided by L_{inp}. In Maple this means,

$$\mathfrak{R}_j := \texttt{rightdivision}\left(\texttt{mult}\left(\partial^j, \mathfrak{B}\right), L_{\text{inp}}\right)[2].$$

(d) Let

$$\mathfrak{R}_j = r_{j,n-1}\partial^{n-1} + \cdots + r_{j,0}$$

and let $\mathfrak{E}$ be the primitive part[1] of the determinant of the matrix

$$\begin{bmatrix} r_{0,0} & \cdots & r_{0,n-1} \\ \vdots & \ddots & \vdots \\ r_{n-1,0} & \cdots & r_{n-1,n-1} \end{bmatrix}$$

with respect to $c_1, \ldots, c_k$. Here $\mathfrak{E}$ is the location where the apparent singularities would be if we use the gauge transformation defined by $\mathfrak{B}$ in (5.1).

i. Let $\text{TrueSing}(L_{\text{inp}})$ be the set of non-removable singularities of L_{inp}. For each $s \in \text{TrueSing}(L_{\text{inp}})$:

A. If $s \neq \infty$, then substitute $x = s$ into $\mathfrak{E}$, equate it to 0, and get the equation $\mathfrak{E}_s$. If $s = \infty$, then let $\mathfrak{E}_s$ be the eqaution $\text{lc}(\mathfrak{E}) = 0$ where $\text{lc}(\mathfrak{E})$ is the leading coefficient of $\mathfrak{E}$.

ii. Let $\{\mathfrak{E}_s \,|\, s \in \text{TrueSing}(L_{\text{inp}})\}$ be the set of all equations obtained in the previous step (Step 3(d)i). For each k-element subset $\mathfrak{S}$ of $\{\mathfrak{E}_s \,|\, s \in \text{TrueSing}(L_{\text{inp}})\}$:

A. Try to solve the system $\mathfrak{S}$ for unknowns $c_1, \ldots, c_k$. If there is a solution, then find it. Then update $\mathfrak{B}$ in (5.1) accordingly and find $\mathfrak{L}$ such that $L_{\text{inp}} \xrightarrow{\mathfrak{B}}_G \mathfrak{L}$. Then update S as $S \cup \{\mathfrak{L}\}$.

(e) Return S.

Remark 5.1.1. If the above method does not produce sufficiently many equations, then in Step 3(d)i we add more equations (e.g., $\mathfrak{E}'|_s = 0$ when $s \neq \infty$, and $\text{lc}(\mathfrak{E} - \text{lc}(\mathfrak{E})) = 0$ when $s = \infty$).

Definition 5.1.1. If L is an element of the output of Algorithm 5.1.1 for an input $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$, then we say that L is a *standard form* of L_{inp}.

[1]What is computed here is the factor of the Wronskian [29] that depends on $c_1, \ldots, c_k$.

Example 5.1.1. Consider the third order differential operator (15) in [5], which is

$$N_3 = \partial^3 + \frac{p_2}{(4\,x+1)\,x\,q_2}\partial^2 + \frac{p_1}{(4\,x+1)\,x^2\,t_0\,q_0}\partial + \frac{p_0}{(4\,x+1)\,x^3\,{t_0}^2\,q_0}.$$

Here polynomials p_0, p_1, p_2, q_0 , q_2, and t_0 are given at Appendix A. Maple's `dsolve` command can not find solutions of N_3. Algorithm 5.1.1 gives us a standard form of N_3, which is

$$\begin{aligned}\tilde{N}_3 = \partial^3 &+ \frac{12\left(32\,x^2+3\right)x}{(4\,x-1)\,(4\,x^2+1)\,(4\,x+1)}\partial^2 \\ &+ \frac{432\,x^4+16\,x^2-1}{(4\,x-1)\,x^2\,(4\,x^2+1)\,(4\,x+1)}\partial + \frac{48\,x^4+1}{(4\,x-1)\,x^3\,(4\,x^2+1)\,(4\,x+1)}.\end{aligned}$$

This operator $\tilde{N}_3$ which is gauge equivalent to N_3, is easier to solve and Maple's `dsolve` command finds its solutions in less than 1 second. We obtain solutions of N_3 from solutions of $\tilde{N}_3$.

5.2 Computing All Hypergeometric Solutions

Let $\partial = \frac{d}{dx}$. In this chapter we introduce a heuristic algorithm to compute hypergeometric solutions of a second order regular singular differential operators $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ in the form of

$$\exp\left(\int r\,dx\right)\left(r_0 \cdot {}_2\mathrm{F}_1(a_1,a_2;b_1;f) + r_1 \cdot {}_2\mathrm{F}_1'(a_1,a_2;b_1;f)\right) \tag{5.2}$$

where $f, r, r_0, r_1 \in \overline{\mathbb{Q}(x)}$ and $a_1, a_2, b_1 \in \mathbb{Q}$. Our algorithm tries to transform L_{inp} to a simpler operator $\tilde{L}_{\text{inp}}$ (which hopefully has a solution in form (3.1)). The key idea is to follow the strategy of the `POLRED` algorithm in [15].

Example 5.2.1 (Finding Solutions in the form of (5.2) using a Normalized Integral Basis)**.** Consider the differential operator[2]

$$\begin{aligned}L_{\text{inp}} = \partial^2 &- \frac{512\,x^5+384\,x^4-64\,x^3-88\,x^2-10\,x-1}{x\,(4\,x-1)\,(4\,x+1)\,(16\,x^3+24\,x^2+5\,x+1)}\partial \\ &+ \frac{512\,x^5+64\,x^4-128\,x^3-60\,x^2-8\,x-1}{x^2\,(4\,x-1)\,(4\,x+1)\,(16\,x^3+24\,x^2+5\,x+1)}.\end{aligned}$$

Algorithm 3.2.1 can not solve L_{inp}. We try to transform L_{inp} to simpler operator $\tilde{L}_{\text{inp}}$. First we compute an integral basis. Then we normalize the basis at infinity and obtain $\{B_1, B_2\}$ where

$$\begin{aligned}B_1 = &\frac{16\,x^4-x^2}{(16\,x^3+24\,x^2+5\,x+1)\,x}\partial \\ &+ \frac{-34359738400\,x^3-51539607556\,x^2-10737418241\,x-2147483648}{(16\,x^3+24\,x^2+5\,x+1)\,x}\end{aligned}$$

[2] Prof. Jean-Marie Maillard sent us this differential operator [5].

and

$$B_2 = \frac{16\,x^3 - x}{(16\,x^3 + 24\,x^2 + 5\,x + 1)\,x}\partial + \frac{-32\,x^2 - 4\,x - 1}{(16\,x^3 + 24\,x^2 + 5\,x + 1)\,x}.$$

with $d_1 = -2$ and $d_2 = 0$. We try to find a suitable $\mathcal{G} \in \mathbb{Q}(x)[\partial]/\mathbb{Q}(x)[\partial]L_{\text{inp}}$. For this example, we take $\mathcal{G} = B_1$ because it is the unique (up to constant factors) integral element of minimal degree at infinity, leading to a unique standard form. This $\mathcal{G}$ gives us a gauge transformation which maps solutions of L_{inp} to solutions of

$$\tilde{L}_{\text{inp}} = \partial^2 + \frac{48\,x^2 - 1}{x\,(16\,x^2 - 1)}\partial + \frac{16}{16\,x^2 - 1}.$$

$\tilde{L}_{\text{inp}}$ is a standard form of L_{inp} and it has a solution in the form of (5.2), which is

$$y(x) = {}_2\mathrm{F}_1\left(\frac{1}{2}, \frac{1}{2}; 1; 16x^2\right)$$

and easy to find with Algorithm 3.2.1. Then we apply the inverse gauge transformation (inverse of $\mathcal{G} = B_1$) and obtain a solution of L_{inp} in the form of (5.2), which is

$$Y(x) = \left(4x^3 + x^2 + \frac{x}{2}\right) \cdot {}_2\mathrm{F}_1\left(\frac{1}{2}, \frac{1}{2}; 1; 16x^2\right) + \left(32x^5 - 2x^3\right) \cdot {}_2\mathrm{F}_1\left(\frac{3}{2}, \frac{3}{2}; 2; 16x^2\right).$$

Algorithm 5.2.1 ([24])**.** General Outline of `hypergeometricsols`.

Inputs:

- L_{inp} = A second order regular singular irreducible operator,
- $a_f max$ = A bound for the algebraic degree a_f. If omitted, then $a_f max = 2$ which means our implementation tries $a_f = 1$ and $a_f = 2$.

Output:

- Solutions of L_{inp} in the form of (3.1) or (5.2), or an empty list.

1. Try to find solutions of L_{inp} in the form of (3.1) by using Algorithm 3.2.1 in Section 3. If none are found go to Step 2.

2. (a) Use Algorithm 5.1.1 to compute standard forms of L_{inp} and store the gauge transformations that give the standrad forms of L_{inp}.

   (b) Let $\tilde{L}_{\text{inp}}$ be a standard form of L_{inp} and $\mathcal{G}$ be the gauge transformation that transforms L_{inp} to $\tilde{L}_{\text{inp}}$. For each $(\tilde{L}_{\text{inp}}, \mathcal{G})$:

      i. Try to find solutions of $\tilde{L}_{\text{inp}}$ in the form of (3.1) by using Algorithm 3.2.1.

ii. If Step 2(b)i succeeds, then apply the inverse of the gauge transformation $\mathcal{G}$ to the solutions of $\tilde{L}_{\text{inp}}$. This will give us the solutions L_{inp} in the form of (5.2). Then, return these solutions. Otherwise, proceed with the next candidate (if any). If no candidates remain, return an empty list.

5.2.1 Examples from Physics

Example 5.2.2 (The Feynman Diagrams)**.** Consider the equation (2.14) in [2], which is

$$\begin{aligned} 0 = {} & \frac{d^2}{dx^2}Y(x) + \frac{\left(5\,x^4 - 30\,x^2 + 9\right)}{x\,(x^2-1)\,(-x^2+9)}\,\frac{d}{dx}Y(x) + \frac{-8\,(x^2-3)}{(-x^2+9)\,(x^2-1)}Y(x) \\ & + \frac{-32\,x^2\,(\ln(x))^3}{(-x^2+9)\,(x^2-1)} + \frac{12\,\left(2\,x^4+13\,x^2-9\right)(\ln(x))^2}{(-x^2+9)\,(x^2-1)} \\ & - \frac{6\,\left(x^6+x^4+62\,x^2-54\right)\ln(x)}{(-x^2+9)\,(x^2-1)} + \frac{9\,x^6+61\,x^4+251\,x^2-1161}{2\,(-x^2+9)\,(x^2-1)}. \end{aligned} \tag{5.3}$$

The operator which corresponds to the homogeneous part of the equation (5.3) is

$$L_{\text{inp}} = \partial^2 + \frac{\left(5\,x^4-30\,x^2+9\right)}{x\,(x^2-1)\,(-x^2+9)}\,\partial + \frac{-8\,(x^2-3)}{(-x^2+9)\,(x^2-1)}. \tag{5.4}$$

Algorithm 3.2.1 can not find a hypergeometric solution of (5.4). Algorithm 5.2.1 finds an hypergeometric solution of (5.4) and it reads

$$\begin{aligned} Y(x) \; = {} & \frac{x^2\,(x-1)\,(x+3)^2}{-2\sqrt{x+1}\,(x-3)^{3/2}} \cdot {}_2\mathrm{F}_1\left(\frac{1}{2},\frac{1}{2};\,1;\,\frac{-16x^3}{(x+1)\,(x-3)^3}\right) \\ & + \frac{x^3\,(x-1)\left(x^2+3\right)(x+3)^3}{2\,(x-3)^{9/2}\,(x+1)^{3/2}} \cdot {}_2\mathrm{F}_1\left(\frac{3}{2},\frac{3}{2};\,2;\,\frac{-16x^3}{(x+1)\,(x-3)^3}\right). \end{aligned}$$

Solutions of inhomogeneous equations (5.3) can be derived from solutions of its homogeneous part (5.4) [2]. Algorithm 5.2.1 plays an important role to find solutions of (5.3).

5.2.2 Examples from Combinatorics

We tested Algorithm 3.2.1 on many examples, including from the On-line Encyclopedia of Integer Sequences [1]. Another source of examples comes from [9, 11, 12] (see the plain text file named `equations17` in [24] for these operators). Four of them have solutions in the form of (3.1) and Algorithm 3.2.1 finds these solutions. However, Algorithm 3.2.1 does not solve the other thirteen operators from that list. We know from [11, 12] that these operators do have solutions in the form of (5.2). It means that these operators must be gauge equivalent to operators with solutions in the form of (3.1). As mentioned in the beginning of this section the key idea is to follow [15] `POLRED`'s strategy; compute an integral basis and then normalize it at infinity. Algorithm 5.2.1 computes their normalized integral bases, finds their standard forms, and finds hypergeometric solutions in the form of (5.2) of the remaining thirteen equations.

Example 5.2.3. Consider the differential equation `Linp[9]` in the file `equations17` in [24], which is

$$\partial^2 + \frac{A}{B}\partial + \frac{C}{B} \tag{5.5}$$

where

$A = 2752512\,x^{12} - 2473984\,x^{11} + 1345536\,x^{10} + 497216\,x^9 - 2492736\,x^8 + 997104\,x^7 - 136176\,x^6 + 56533\,x^5 - 3285\,x^4 - 2741\,x^3 - 27\,x^2 - 228\,x + 60$,

$B = 172032\,x^{13} - 155648\,x^{12} + 20480\,x^{11} + 78016\,x^{10} - 188224\,x^9 + 78144\,x^8 - 6528\,x^7 + 701\,x^6 + 2131\,x^5 - 1183\,x^4 + 97\,x^3 - 30\,x^2 + 12\,x$,

$C = 9633792\,x^{11} - 8601600\,x^{10} + 7953408\,x^9 - 552960\,x^8 - 6703296\,x^7 + 2567040\,x^6 - 414624\,x^5 + 231579\,x^4 - 60420\,x^3 + 12561\,x^2 - 2184\,x - 300$.

The unique standard form (up to constant factors) of (5.5) is

$$\partial^2 + \frac{320\,x^4 - 1}{64\,x^5 - x}\partial + \frac{192x^2}{64\,x^4 - 1}. \tag{5.6}$$

Algorithm 5.2.1 obtains hypergeometric solutions of (5.5) from hypergeometric solutions of (5.6).

APPENDIX A

OPERATOR POLYNOMIAL DEFINITIONS

$$
\begin{aligned}
p_0 = {} & 273690768629642986782720\,x^{37} - 211359767643880613904384\,x^{36} \\
& + 783972504748826595672064\,x^{35} - 798202280307771391610880\,x^{34} \\
& + 529828395288499148986368\,x^{33} - 404444091888046259051520\,x^{32} \\
& + 228209038204604432213504\,x^{31} - 87336151574531227459584\,x^{30} \\
& + 50153229940291807271936\,x^{29} - 39982094900964694180032\,x^{28} \\
& + 24384549315393061440864\,x^{27} - 11788010919590396492272\,x^{26} \\
& + 4902330084007966551208\,x^{25} - 1589963053121052166560\,x^{24} \\
& + 304744587512802289232\,x^{23} + 16506284162934483376\,x^{22} \\
& - 41000039398409867868\,x^{21} + 20742149600984969600\,x^{20} \\
& - 7212005436663336980\,x^{19} + 1852871276944889952\,x^{18} - 333772782940373404\,x^{17} \\
& + 28423554873971452\,x^{16} + 7532729171676060\,x^{15} - 4293988070154760\,x^{14} \\
& + 986843365675252\,x^{13} - 89915103220404\,x^{12} - 9436839756874\,x^{11} \\
& + 3809320554856\,x^{10} - 564751409700\,x^{9} + 49607139664\,x^{8} + 356966776\,x^{7} \\
& - 632334612\,x^{6} + 31450724\,x^{5} + 1326944\,x^{4} + 272064\,x^{3} - 18408\,x^{2} - 2696\,x + 72
\end{aligned} \tag{A.1}
$$

$$
\begin{aligned}
p_1 = {} & 853186235804614656\,00\,x^{31} - 591275542562745384\,96\,x^{30} + 2184472622425578168\,32\,x^{29} \\
& - 2085015781058460078\,08\,x^{28} + 1126111984234432849\,92\,x^{27} - 4926454914095980697\,6\,x^{26} \\
& + 5363044383950807040\,x^{25} + 9908091314931060224\,x^{24} - 6004424721650285504\,x^{23} \\
& + 1626875693444819872\,x^{22} - 222594258778946672\,x^{21} - 105820382901860152\,x^{20} \\
& + 109213685415135432\,x^{19} - 41945444103056288\,x^{18} + 8021551526743336\,x^{17} \\
& - 113658269629740\,x^{16} - 679274249487472\,x^{15} + 368537977098520\,x^{14} \\
& - 107672107809992\,x^{13} + 16030172387980\,x^{12} - 130330811308\,x^{11} - 456469811472\,x^{10} \\
& + 121723656358\,x^{9} - 19315052396\,x^{8} + 696949584\,x^{7} + 343267454\,x^{6} \\
& - 40979606\,x^{5} - 1504616\,x^{4} + 240988\,x^{3} + 10236\,x^{2} + 392\,x - 92
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
p_2 = {} & 1103269952225280\,x^{24} - 228305326678016\,x^{23} + 2331485726244864\,x^{22} \\
& - 1249271454269440\,x^{21} + 549381083516928\,x^{20} - 225290952722816\,x^{19} \\
& - 39003496295360\,x^{18} + 46746500840896\,x^{17} - 10249554621312\,x^{16} \\
& + 1973847887848\,x^{15} + 157900491180\,x^{14} - 637108022672\,x^{13} \\
& + 233984558200\,x^{12} - 24645390372\,x^{11} - 4177273140\,x^{10} \\
& + 2621821288\,x^{9} - 942904492\,x^{8} + 195411966\,x^{7} + 1609130\,x^{6} \\
& - 6956791\,x^{5} + 515168\,x^{4} + 60240\,x^{3} - 2676\,x^{2} - 256\,x - 20
\end{aligned} \tag{A.3}
$$

$$
\begin{aligned}
q_0 = {} & 91939162685440\,x^{24} - 79628753641472\,x^{23} + 202946718879744\,x^{22} \\
& - 207932599859712\,x^{21} + 135671205331456\,x^{20} - 75320543530112\,x^{19} \\
& + 30199474895552\,x^{18} - 7567681417120\,x^{17} + 1089997410032\,x^{16} \\
& + 167043589576\,x^{15} - 213500102028\,x^{14} + 56207492972\,x^{13} + 6076506627\,x^{12} \\
& - 8171378448\,x^{11} + 3438773202\,x^{10} - 1089652708\,x^{9} + 215327593\,x^{8} \\
& - 8698490\,x^{7} - 6071267\,x^{6} + 1203089\,x^{5} - 37292\,x^{4} - 8220\,x^{3} + 314\,x^{2} + 20\,x + 2
\end{aligned} \tag{A.4}
$$

$$
\begin{aligned}
q_2 = {} & 22984790671360\,x^{23} - 14160990742528\,x^{22} + 47196432034304\,x^{21} \\
& - 40184041956352\,x^{20} + 23871790843776\,x^{19} - 12862188171584\,x^{18} \\
& + 4334321680992\,x^{17} - 808339934032\,x^{16} + 70414369000\,x^{15} \\
& + 59364489644\,x^{14} - 38533903096\,x^{13} + 4418397469\,x^{12} \\
& + 2623726024\,x^{11} - 1386913106\,x^{10} + 512965024\,x^{9} - 144171921\,x^{8} \\
& + 17788918\,x^{7} + 2272607\,x^{6} - 949665\,x^{5} + 63356\,x^{4} \\
& + 6516\,x^{3} - 426\,x^{2} - 28\,x - 2
\end{aligned} \tag{A.5}
$$

$$
t_0 = 6874\,x^6 - 2913\,x^5 + 660\,x^4 - 230\,x^3 + 60\,x^2 + 6\,x - 2 \tag{A.6}
$$

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