Pretty Mathematics

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A good deal of my research work in physics has consisted in, not setting out to solve some particular problem, but simply examining mathematical quantities of a kind that physicists use and trying to fit them together in an interesting way, regardless of any application that the work may have. It is simply a search for pretty mathematics. It may turn out later that the work does have an application. Then one has had good luck.

I can give a good example of this procedure. At one time, in 1927, I was playing around with those $2 \times 2$ matrices whose squares are equal to unity and which anticommute with one another. Calling them $\sigma_1, \sigma_2, \sigma_3$, I noticed that if one multiplied them into the three components of a momentum, so as to form $\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3$, one obtained a quantity whose square was just $p_1^2 + p_2^2 + p_3^2$. This was an exciting result, but what use could one make of it?

One could use $\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3$ as the Hamiltonian in a Schrödinger wave equation, giving the wave function two components so that the $\sigma$ matrices can be applied to it. One then had a relativistic wave equation. But it applied only to a particle of zero rest mass — useless for the electron, which was what I was mainly interested in.

To get a theory for a particle of non-zero rest mass one would need four $\sigma$ matrices, anticommuting with one another, and they did not exist. So my work I therefore had to abandon it.

It was not until some weeks later that I realized there was no need to restrict oneself to $2 \times 2$ matrices. One could go on to $4 \times 4$ matrices, and the problem is easily soluble. In retrospect, it seems strange that one can be so much held up over such an elementary point.
The resulting wave equation for the electron turned out to be very successful.

It led to correct values for the spin and the magnetic moment. This was quite unexpected. The work all followed from a study of pretty mathematics, without any thought being given to these physical properties of the electron.

Another example of pretty mathematics that followed led to the idea of the magnetic monopole. When I did this work I was hoping to find some explanation of the fine-structure constant $\frac{e^2}{\hbar c}$. But this failed. The mathematics led inexorably to the monopole.

"From the theoretical point of view we would think that monopoles should exist, because of the prettiness of the mathematics. Many attempts to find them have been made, but all have been unsuccessful. One should conclude that pretty mathematics by itself is not an adequate reason for Nature to have made use of a theory. We still have much to learn in seeking for the basic principles of Nature."
I would like to discuss a further aspect of the development of quantum mechanics that is provided by a relativistic theory of a particle that I put forward in 1970. The wave function has only one component, instead of the usual four, but the particle has internal degrees of freedom, consisting of two harmonic oscillations, each the coordinates of the oscillations $q_1$, $q_2$, and the conjugate momenta $p_1$, $p_2$, so that the $q_i$ satisfy the commutation relations (with $\hbar = 1$)

$$\{q_i, q_j\} = i \hbar \epsilon_{ij} \quad i, j = 1, 2, 3, 4,$$

where $\epsilon$ is the matrix

$$\epsilon = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Note that $\epsilon$ is skew and $\epsilon^2 = -1$.

We then consider the four $q_i$'s as a column matrix, so that $q_i\psi$ appears as a column matrix with four elements. We set up the ward equation

$$\frac{\partial^2 \psi}{\partial t^2} + \partial \frac{\partial \psi}{\partial x} + \epsilon \psi = 0 \quad (1)$$

where the $\epsilon$'s are $4 \times 4$ matrices whose squares are unity and which anticommute with one another and with $B$. Also the $\epsilon$'s must be chosen to have only real elements.

This wave equation is very similar to the usual electron wave equation, except that we have the column matrix $q_i\psi$, with just one $\psi$, replacing the four components of the usual $\psi$.

There are four equations in (1), corresponding to the four components of $q_i\psi$. Of these, it turns out that only three are independent. For one function $\psi$ to satisfy three independent equations, it is necessary that certain consistency conditions shall be fulfilled. One finds that they are fulfilled, provided $\psi$ satisfies the de Broglie equation (with $m = 1$) for all values of the internal coordinates. One finds also that the theory is relativistic, and that the mass has to be finite. These are pretty results, and lead one to wonder if the theory can describe any particle in nature.
One cannot assign a change to the particle and let it interact with the electromagnetic field in the usual way, by making
\[ i \frac{\partial}{\partial x} \rightarrow i \frac{\partial}{\partial x} + e A^v \]  
besides this the equations cease to be consistent. The consistency conditions are so restrictive that there is very little one can do with the theory, and I did not find anyway of developing the theory so as to lend to a hope of applying it.

Recently, an extension of the theory has been made by Sundaram and his co-workers. They extend the internal degrees of freedom of the particle by replacing
\[ q^v \rightarrow \tilde{q}^v \tilde{q}^v \]
for each of the four values of \( v \). For each \( \tilde{q}^v \) the \( \tilde{q}^v \) are like the original \( q^v \) in describing two harmonic oscillators, but the \( \tilde{q}^v \) for one values of \( \tilde{q}^v \) intercommute with those for another value of \( \tilde{q}^v \). Under these conditions the equations are still consistent and we still have a reasonable relativistic theory.

This is a truly remarkable result. It makes the theory much more flexible. It becomes possible to introduce interaction with the electromagnetic field in the usual way, according to the procedure (2). It reveals hope that the theory will have an application to some particle in nature, but a great deal more work will first have to be done on it.

With an approach like this to a new theory, there is a chance of new features appearing which we could never find by just making straight-forward development of the old theories.