2007

Analysis and Approximation of a Two-Band Ginzburg-Landau Model of Superconductivity

Wan-Kan Chan

THE FLORIDA STATE UNIVERSITY

COLLEGE OF ARTS AND SCIENCES

ANALYSIS AND APPROXIMATION OF A TWO-BAND GINZBURG-LANDAU MODEL OF SUPERCONDUCTIVITY

By

WAN-KAN CHAN

A Dissertation submitted to the
Department of Mathematics
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Degree Awarded:
Fall Semester, 2007

The members of the Committee approve the Dissertation of Wan-Kan Chan defended on Sep 14, 2007.

Max Gunzburger
Professor Directing Dissertation

Janet Peterson
Professor Co-Directing Dissertation

Efstratios Manousakis
Outside Committee Member

Xiaoming Wang
Committee Member

The Office of Graduate Studies has verified and approved the above named committee members.

To my parents, my wife and my daughters.

ACKNOWLEDGEMENTS

I wish to express my deep and sincere gratitude to my advisor, Prof. Max Gunzburger. His kindness, endless patience and support have allowed me to finish my doctoral study at FSU. Over these years, I have learned a lot and received a great deal of inspirations through his teachings, talks and mentoring. I always admire his insightful understanding and breadth in mathematics and science, in particularly, in superconductivity. And his creative ways of tackling mathematical and computational problems have always surprised me. Most importantly, he taught me how to do and enjoy doing good research. It is my pleasure and honor to have the opportunity to work with and learn from this outstanding applied mathematician.

I also wish to express my warm and sincere thanks to my co-advisor, Prof. Janet Peterson, for her excellent teachings and constant concern of my life. My interest in computational mathematics began with her wonderful finite element methods lecture at Iowa State University.

I am grateful and indebted to my committee member, Prof. Xiaoming Wang, for assisting me in many ways and for bringing many excellent advanced courses to FSU. I believe that the advanced PDE courses and the fluid dynamics course were few of the most important and useful courses that I have ever taken. He always passionately and patiently showed us the mathematical skills and insights of functional analysis and Navier-Stokes equations. His office door was always opened to me and he always assisted me in my mathematical problems, and gave me encouragement and advice. His teachings have laid a solid foundation for my research.

I am also grateful and indebted to my committee member, Prof. Efstratios Manousaki, for his inspirations and enlightening discussions on electromagnetism and superconductivity, and for his insightful commentary on my work.

I owe my sincere gratitude to Dr. Alex Gurevich from the Applied Superconductivity

Center at the National High Magnetic Field Laboratory. He was always willing to answer my questions about dual-band superconductivity in detail and provide valuable help and suggestions through emails and meetings. My simulations of dual-band superconductivity would not be completed without his assistance.

I must also express my gratitude to Prof. Qiang Du from Penn State University, for his valuable discussions and his kind words of encouragement. His papers on superconductivity models have provided me an excellent learning source about mathematical analysis of superconductivity models. The existence proofs of solutions in this present work is mainly based on the framework established in his paper on the existence and uniqueness of solutions for the TDGL equations.

I am forever indebted to my great parents. Ever since I left my beautiful hometown and began my pursuit of my dreams overseas, first in Taiwan and now in the US, you both has always showed me your endless love and faith in me, and given me the greatest encouragement and support. Your honesty, sympathy and kindness to all people and living things, and your diligence in work have always reminded me to be a better person. Father, I share your passions for woodwork, mechanics, cars and invention. These hobbies have made my life full of excitements.

My special gratitude to my sisters, my brother and their families, for taking the best care of our parents, and giving every support I needed.

My special thanks to my friends and colleagues, Zheng Chen, Hoaming Lin and Yuki Saka, for being my friends, and willing to share their knowledge with me.

Lastly, I am most indebted to my beloved wife Chin-Chu Chang. Without your unconditional love, understanding and support, I wouldn't be able to finish my study. And to my two lovely daughters Yu-Chin and Shuan-Yu, you both are always my main source of happiness and my greatest reason to stand up strong. Thank you all for accompanying and helping me through the tough times in these few years.

TABLE OF CONTENTS

LIST OF FIGURES

ABSTRACT

In 2001, the discovery of the intermetallic compound superconductor MgB_2 having a critical temperature of 39K stirred up great interest in using a generalization of the Ginzburg-Landau model, namely the two-band time-dependent Ginzburg-Landau (2B-TDGL) equations, to model the phenomena of two-band superconductivity. In this work, various mathematical and numerical aspects of the two-dimensional, isothermal, isotropic 2B-TDGL equations in the presence of a time-dependent applied magnetic field and a time-dependent applied current are investigated. A new gauge is proposed to facilitate the inclusion of a time-dependent current into the model. There are three parts in this work. First, the 2B-TDGL model which includes a time-dependent applied current is derived. Then, assuming sufficient smoothness of the boundary of the domain, the applied magnetic field, and the applied current, the global existence, uniqueness and boundedness of weak solutions of the 2B-TDGL equations are proved. Second, the existence, uniqueness, and stability of finite element approximations of the solutions are shown and error estimates are derived. Third, numerical experiments are presented and compared to some known results which are related to MgB_2 or general two-band superconductivity. Some novel behaviors are also identified.

CHAPTER 1

Introduction

The discovery of the intermetallic compound superconductor MgB_2 stirred up intense research to investigate the novel properties of this material. The compound MgB_2 differs from conventional low critical temperature ($\mathcal{T}_c$) superconductors and cuprate- based high $\mathcal{T}_c$ superconductor compounds mainly in its possession of two distinct energy gaps; the other superconductors are known to only have one energy gap. It is its two-band structure that gives MgB_2 many novel properties unseen in any other superconductors; for example, interband phase soliton textures occur in a two-band superconductor [27]. Because of the existence of multiple distinct energy gaps in a multiband superconductor, there exists multiple distinct order parameters which interact with each other through a Josephson tunneling like mechanism. The conventional isotropic or anisotropic time-dependent Ginzburg-Landau (TDGL) model [31] which has been widely accepted as a successfully phenomenological model for a single-band superconductor sample at temperatures near its critical temperature does not include any appropriate coupling terms to account for the coupling interactions that are shown to be significant factors in determining the novel properties of a multiband superconductor such as MgB_2. Therefore, the TDGL model is not a correct model for multiband superconductivity. [32] investigates the breakdown of the anisotropic GL model in modeling MgB_2. The 2B-TDGL model generalizes the TDGL model by adding coupling terms to model the interband interaction of the two distinct order parameters corresponding to the two distinct energy bands. The 2B-TDGL model has now been widely used by the physics community as a phenomenological model to investigate the properties of multiband superconductor such as MgB_2.

1.1 Previous Work and Outline of Present Work

Recently, there have been many research papers which investigated new properties of MgB_2 and of other multiband superconductors. Many of the papers that investigated the phenomenological properties of two-band superconductivity used the 2B-TDGL model and its variants. However, to our knowledge, none of the published papers using the 2B-TDGL model have done a full investigation of the analytical properties of the 2B-TDGL model such as the existence, uniqueness and boundedness of weak solutions; or the properties of the finite element approximations such as the existence, uniqueness, stability and error estimates for approximate solutions. The purpose of the present work is to examine these issues associated with the 2B-TDGL model and then present some two-dimensional numerical results.

It should be mentioned that analogous analytical and approximation issues for the TDGL model and its variations have been addressed in part or in whole by many authors. For example, to show existence and uniqueness of solutions of the TDGL equations, Du in [2], [3] used Galerkin finite dimensional approximation and compactness methods similar to the methods used by Temam for the Navier-Stokes equations [45]. Du used the zero-electric-potential (ZEP) gauge on a modified TDGL equations with a regularization term $\epsilon\nabla(\nabla \cdot \mathbf{A})$ added to the equations. Existence and uniqueness of the original TDGL equations is then proved by passage to the limit $\epsilon \to 0$. Chen, Hoffmann and Liang in [8] adopted the Lorentz guage $\phi = -\nabla \cdot \mathbf{A}$ and used the Leray-Schauder fixed point theorem. Tang and Wang in [7] used the London gauge $\nabla \cdot \mathbf{A} = 0$, and the same methods as in [45]. However, all of these papers did not include an applied current into their analyses and only Chen *et al.* in [8] included an applied magnetic field in their analysis. Pelle, Kaper and Takac in [9] and Zaouch in [10] used semigroup methods to show the existence, uniqueness and long-time asymptotic behavior of the TDGL equations under a generalized gauge $\phi = -\omega(\nabla \cdot \mathbf{A})$, $\omega > 0$, in the presence of a time-dependent applied magnetic field.

In this work, we follow the methods used by Du in [2] to prove the existence and uniqueness of the 2B-TDGL in the presence of a time-dependent applied magnetic field and a time-dependent applied current. We choose Du's approach mainly because we want to generalize the zero-electric-potential gauge to a "current gauge" which allows us to include time-dependent current into the 2B-TDGL model. Basically the "current gauge" replaces the electric potential ϕ in the 2B-TDGL equations with a predefined auxiliary function ϕ_a.

As a result, unlike all the other gauges mentioned before other than the ZEP gauge, this "current gauge" does not change the coercivity of the non-coercive 2B-TDGL equations. In order to facilitate the proofs of the existence and regularity theorems, we also add a regularization term $-\epsilon\nabla(\mathrm{div}\mathbf{A})$ to the 2B-TDGL equation for the magnetic potential $\mathbf{A}$ to make the equation coercive. Du's method is designed specifically to handle the convergence of this modified problem. We want to point out that as in the zero-electric-potential gauge case, our "current gauge" is well-suited for numerical computation.

We organize this work as follows. In section 1.2, we introduce some general concepts and properties of superconductivity, the conventional TDGL model, some properties of MgB_2 and the 2B-TDGL model. In chapter 2, We first present our model- the isotropic, isothermal 2B-TDGL equations. After we nondimensionalize the 2B-TDGL equations, we then discuss some issues of adding current to TDGL models. The "current gauge" which takes the time-dependent applied current into account is introduced into the nondimensionalzed 2B-TDGL equations. In chapter 3, we start our analytical studies. The weak form of the gauged equations are presented first, then based on a modified weak form, the existence, uniqueness and boundedness of two-dimensional solutions are proved. After the analytical studies, we move to the two-dimensional Galerkin approximations in chapter 4. We first present a fully discretized problem using the backward Euler scheme for the time discretization and conforming finite element methods for the space discretization. Then, the existence and uniqueness of the approximate solutions, stability and error estimates of the approximation scheme are examined. In chapter 5, numerical results based on a two-dimensional backward Euler finite element approximation are presented. Finally, conclusions and future research are discussed in chapter 6.

1.2 Models and Phenomena of Superconductivity

Superconductivity is a very fascinating and remarkable physical phenomenon which was first discovered in 1911 by H.K Onnes. The two hallmark properties of superconductivity are perfect conductivity and perfect diamagnetism. When a superconductor is cooled to a temperature below its critical temperature $\mathcal{T}_c$, which is one of the material characteristic parameters of the superconductor, its electric resistivity is reduced to a negligible, if not zero, value. In other words, in the superconducting state, the superconductor behaves like a perfect conductor. In addition to this striking characteristic, a superconductor cooled below its $\mathcal{T}_c$ also exhibits a novel perfect diamagnetism phenomenon called the Meissner effect. When a sufficiently small magnetic field is applied to a superconductor cooled below its $\mathcal{T}_c$, no magnetic field can penetrate into the superconductor. Moreover, if a superconductor is cooled through its critical temperature in the present of a magnetic field, it expels the field from inside. The conductivity of a superconductor can be destroyed by a large enough magnetic field strength or applied current. Thus in addition to the critical temperature $\mathcal{T}_c$, there are also critical magnetic field H_c and critical current j_c associated with a particular superconductor.

The first sucessful microscopic description of superconductivity was proposed by Bardeen, Cooper, and Schrieffer in their seminal BCS theory in 1957. Before the BCS theory, various theories were proposed, including the London equations in 1935, and the Ginzburg-Landau macroscopic theory in 1950. The Ginzburg-Landau (GL) theory was not appreciated until, in 1959, Gorkov proved that the GL theory is actually a limiting case of the BCS theory. The GL theory is now commonly accepted as a successful phenomenological model for superconductivity.

1.2.1 Ginzburg-Landau phenomenological model of superconductivity

The Ginzburg-Landau (GL) theory generalizes the London theory of second-order phase transitions. In the Ginzburg-Landau model, a complex-valued order parameter $\psi = |\psi|e^{\theta}$ is introduced with $|\psi|^2$ representing the local density of superconducting electrons. Under the assumption that the temperature is close to the transition temperature below $\mathcal{T}_c$, and ψ varies slowly spatially, the free energy of the superconductor can be expanded and approximated in terms of the order parameter and its gradients. Then the total free energy is:

$$\begin{aligned} \mathcal{G} &= \int_\Omega \left[f_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m^*}\left|(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A})\psi\right|^2 + \frac{|\mathbf{h}|^2}{8\pi}\right] d\Omega \qquad (1.1) \\ &:= \mathcal{F}_s + \int_\Omega \frac{|\mathbf{h}|^2}{8\pi} d\Omega, \end{aligned}$$

where α and β are two temperature dependent material constants, c is the speed of light, $\hbar$ is the Planck's constant m^* is the effective mass and e^* is the effective electron charge, $\mathbf{A}$ is the magnetic vector potential, $\mathbf{h} = \text{curl}\mathbf{A}$ is the magnetic field. The first term f_n is the normal state energy density without a magnetic field and the last term is the magnetic field energy density.

The gradient term in equation (1.1) is the momentum operator form of kinetic energy of the supercurrent in the presence of a magnetic field, and is gauge invariant as required by the fact that the physically observable free energy, superconducting electron density $|\psi|^2$ and magnetic field $\mathbf{h}$ do not change when a mathematical gauge transformation G_χ is performed simultaneous on ψ and $\mathbf{A}$ to give $(\psi', \mathbf{A}')$:

$$G_\chi : (\psi, \mathbf{A}) \mapsto (\psi', \mathbf{A}') = (\psi e^{i\chi}, \mathbf{A} + \frac{\hbar c}{e^*} \nabla \chi), \tag{1.2}$$

where χ is an arbitrary but sufficiently smooth single-valued scalar function of the spatial coordinates. Both $(\psi, \mathbf{A})$ and $(\psi', \mathbf{A}')$ represent the same physical state of the system and this physical state determines the same physically observable variables mentioned above, namely, $\mathcal{F}$, $|\psi|^2$ and $\mathbf{h}$.

In the presence of an external applied magnetic field $\mathbf{H}_e$, the superconductor acquires an energy density $-\mathbf{h} \cdot \mathbf{H}_e / 4\pi$. The total energy then becomes

$$\mathcal{E} \quad = \quad \mathcal{F}_s + \int_\Omega \left[\frac{|\mathbf{h}|^2}{8\pi} - \frac{\mathbf{h} \cdot \mathbf{H}_e}{4\pi} \right] d\Omega. \tag{1.3}$$

For mathematical convenience, the last magnetic field integrands in (1.3) are replaced by $|\mathrm{curl}\mathbf{A} - \mathbf{H}_e|^2/8\pi$ and the new energy functional becomes

$$\mathcal{F} \quad = \quad \mathcal{F}_s + \int_\Omega \frac{|\mathrm{curl}\mathbf{A} - \mathbf{H}_e|^2}{8\pi} d\Omega. \tag{1.4}$$

The energy functional $\mathcal{F}$ is nonnegative and it gives the same minimizer as $\mathcal{E}$ when they are minimized. This is the free energy functional we will consider later.

Thermodynamic principles require that when in equilibrium, the free energy of the superconductor is minimized. The material parameters α and β are temperature dependent. α is positive when the superconductor is in the normal state ($\mathcal{T} > \mathcal{T}_c$) and is negative when in the superconducting state ($\mathcal{T} < \mathcal{T}_c$). β is positive or else $\mathcal{F}$ has no minimizer. For $\mathcal{T}$ close to $\mathcal{T}_c$, α and β are approximated as

$$\alpha(\mathcal{T}) \approx \alpha(0) \left[\frac{\mathcal{T}}{\mathcal{T}_c} - 1 \right], \qquad \beta \approx \beta(0),$$

where $\alpha(0)$ and $\beta(0)$ are the corresponding parameters at absolute zero ($\mathcal{T} = 0$).

Taking the first variation of the free energy functional $\mathcal{F}$ given in (1.4) with respect to ψ, ψ^* and $\mathbf{A}$ gives

$$\begin{aligned}\delta_{total}\mathcal{F} &= \int_\Omega \left[\left(\alpha\psi + \beta|\psi|^2 + \frac{1}{2m^*}(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A})^2\psi\right)\delta\psi^* + c.c.\right] d\Omega \\ &+ \int_\Omega \left[\frac{\text{curl}(\text{curl}\mathbf{A} - \mathbf{H}_e)}{8\pi} - \frac{e^*}{2m^*c}\left(\psi^*(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A})\psi + c.c.\right)\right]\delta\mathbf{A}\, d\Omega \\ &+ \int_{\partial\Omega}\left[\delta\psi^*(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A})\psi\cdot\mathbf{n} + c.c. + \frac{1}{8\pi}(\text{curl}\mathbf{A} - \mathbf{H}_e)\times\mathbf{n}\cdot\delta\mathbf{A}\right] d\partial\Omega, \quad (1.5)\end{aligned}$$

where $c.c.$ denotes the complex conjugate and $\delta\psi$, $\delta\psi^*$ and $\delta\mathbf{A}$ are the variation variables.

We minimize the energy functional $\mathcal{F}$ by taking $\delta_{total}\mathcal{F} = 0$. Setting $\delta_{total}\mathcal{F}/\delta\psi^* = 0$ gives the first stationary GL equation

$$\alpha\psi + \beta|\psi|^2 + \frac{1}{2m^*}(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A})^2\psi = \frac{\delta\mathcal{F}_s}{\delta\psi^*} = 0 \qquad \text{in } \Omega. \tag{1.6}$$

Setting $\delta_{total}\mathcal{F}/\delta\mathbf{A} = \mathbf{0}$ gives the second stationary GL equation

$$\begin{aligned}\frac{\mathbf{j}_s}{c} = \frac{\text{curl}^2\mathbf{A} - \text{curl}\mathbf{H}_e}{4\pi} &= -\frac{\delta\mathcal{F}_s}{\delta\mathbf{A}} && (1.7)\\ &= \frac{e^*}{2m^*c}\left(\psi^*(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A})\psi + c.c.\right) \\ &= -i\frac{e^*\hbar}{2m^*c}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{e^{*2}}{m^*c^2}|\psi|^2\mathbf{A} \\ &= \frac{e^*}{m^*}|\psi|^2\left(\hbar\nabla\theta - \frac{e^*}{c}\mathbf{A}\right) \qquad \text{in } \Omega, && (1.8)\end{aligned}$$

where $\mathbf{j_s}$ is the supercurrent density and θ is the phase of the order parameter ψ. The natural boundary conditions resulting from the minimization process are

$$(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A})\psi\cdot\mathbf{n} = 0 \qquad \text{on } \partial\Omega \tag{1.9}$$

and

$$(\text{curl}\mathbf{A} - \mathbf{H}_e)\times\mathbf{n} = 0 \qquad \text{on } \partial\Omega. \tag{1.10}$$

If the domain Ω is a bounded region in $\mathbb{R}^3$, we need to consider the effect of the Maxwell system in $\mathbb{R}^3 \setminus \Omega$. But when we are considering Ω as a simply connected bounded region with smooth boundary in $\mathbb{R}^2$, it can be shown that we can omit the Maxwell system outside Ω [59]. For simplicity, we will only work on domains in $\mathbb{R}^2$. We can then assume that the magnetic field outside of Ω is equal to the applied magnetic field $\mathbf{H}_e$ and use only the natural boundary conditions (1.9)-(1.10) as the boundary conditions (B.C.s) for the partial differential equations (PDE) system.

The second boundary condition (1.10) ensures that the magnetic field is continuous in the tangential direction along the boundary. The first boundary condition (1.9) ensures that no supercurrent passes through the boundary. De Gennes [61] showed from a microscopic theory that this boundary condition is correct for an insulator-superconductor interface but for a superconductor-normal metal (S-N) interface in which the proximity effects occur, a more general boundary condition

$$(-i\hbar\,\nabla - \frac{e^*}{c}\mathbf{A})\psi\cdot\mathbf{n} = i\hbar\gamma\psi \qquad \text{on } \partial\Omega \tag{1.11}$$

must be used, also see [1], [62], where γ is an real-valued constant and is equal to zero for insulator. Note that (1.11) still implied that $\mathbf{j}_s\cdot\mathbf{n} = 0$ on $\partial\Omega$, i.e. no supercurrent passes through the boundary. The boundary condition (1.11) can be derived from the energy functional $\mathcal{F} + \int_{\partial\Omega}\tilde{\gamma}|\psi|^2 dS$, for some constant $\tilde{\gamma}$. However, Chapman, *et al.* in [5] showed that this B.C. is still not general enough and thus proposed a modified GL model to include the free energy of the normal material in the form of GL free energy with positive parameter α, as if the normal material is in its normal state above its critical temperature. This model allows the existence of superelectrons in the normal material and thus takes into account the effect of magnetic field in the normal material on the superconductor through the coupling of the GL equations in the bulk and exterior of the superconductor. The magnetic field in the normal material is created by the supercurrent diffused from the superconductor through the proximity effects, and will in return affect the solution of the superconductor. In our present work, the S-N type B.C. (1.11) will be considered.

The stationary GL equations (1.6)-(1.7) and (1.9)-(1.10) has two particular solutions. In case α is positive, $\psi = 0$ and curl$\mathbf{A} = \mathbf{H}_e$ minimizes the functional $\mathcal{F}$, this corresponds to the normal state. In case of $\alpha < 0$ with zero field and homogeneous B.C. (with zero gradient), equation (1.6) has one solution $|\psi|^2 = -\alpha/\beta$, this corresponds to the superconducting state at $\mathcal{T} < \mathcal{T}_c$.

1.2.2 The Time-Dependent Ginzburg-Landau Model

The solution of the stationary GL equations with external field absent puts the superconductor in equilibrium state. The time-dependent Ginzburg-Landau (TDGL) model (see [22] and

[60]) generalizes the stationary Ginzburg-Landau model by including relaxation processes driven by deviations from equilibrium state. The validity of the TDGL model needs $\mathcal{T}$ to be close to $\mathcal{T}_c$ and that deviations from the equilibrium state are small. When a superconductor is deviated from equilibrium, the relaxation rate of ψ back to the equilibrium state depends on the deviation from equilibrium, expressed mathematically as

$$\Gamma \frac{\partial \psi}{\partial t} = -\frac{\delta \mathcal{F}}{\delta \psi^*}, \tag{1.12}$$

where Γ is a positive damping constant. The equation (1.12) should be gauge invariant under the gauge transformation

$$G_\chi : (\psi, \mathbf{A}, \phi) \mapsto (\psi e^{i\chi}, \mathbf{A} + \frac{\hbar c}{e^*} \nabla \chi, \phi - \frac{\hbar}{e^*} \frac{\partial \chi}{\partial t}), \tag{1.13}$$

where χ is an arbitrary but sufficiently smooth single-valued scalar function of time and spatial coordinates, and ϕ is a scalar electric potential. We change equation (1.12) into a gauge invariant form by adding the electric potential ϕ to the equation, which gives

$$\Gamma \left(\frac{\partial \psi}{\partial t} + \frac{ie^*}{\hbar} \phi \psi \right) = -\frac{\delta \mathcal{F}}{\delta \psi^*}.$$

Maxwell's equations give the normal current density as

$$\mathbf{j}_n = \sigma_n \mathbf{E} = \sigma_n \left(-\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \right), \tag{1.14}$$

where σ_n is the conductivity in the normal state and $\mathbf{E}$ is the electric field that induces the normal current.

The total current $\mathbf{j} = (c/4\pi)\mathrm{curl}^2 \mathbf{A}$ in the superconductor can be contributed from a supercurrent $\mathbf{j}_s$, from a normal current $\mathbf{j}_n$ induced by an electric field by the relation (1.14), and from a current $\mathbf{j}_m = (c/4\pi)\mathrm{curl}\, \mathbf{H}_e$ induced by the external field $\mathbf{H}_e$. Therefore, the total current becomes

$$\begin{aligned} \mathbf{j} = \frac{c}{4\pi} \mathrm{curl}^2 \mathbf{A} &= \mathbf{j}_n + \mathbf{j}_s + \mathbf{j}_m \\ &= \mathbf{j}_n - c \frac{\partial \mathcal{F}_s}{\partial \mathbf{A}} + \frac{c}{4\pi} \mathrm{curl}\, \mathbf{H}_e. \end{aligned} \tag{1.15}$$

The complete TDGL model is

$$\Gamma \left(\frac{\partial \psi}{\partial t} + \frac{ie^*}{\hbar} \phi \psi \right) + \alpha \psi + \beta |\psi|^2 + \frac{1}{2m^*} (-i\hbar \nabla - \frac{e^*}{c} \mathbf{A})^2 \psi = 0 \qquad \text{in } \Omega,$$

$$
\begin{aligned}
&\frac{c}{4\pi}(\mathrm{curl}^2\mathbf{A} - \mathrm{curl}\,\mathbf{H}_e) \\
&\quad = \sigma_n\left(-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi\right) - i\frac{e^*\hbar}{2m^*}\left(\psi^*\,\nabla\psi - \psi\,\nabla\psi^*\right) - \frac{e^{*2}}{m^*c}|\psi|^2\mathbf{A} \qquad \text{in } \Omega.
\end{aligned}
$$

In addition to the boundary conditions (1.11) and (1.10), repeated below, we also have two initial conditions. Together they are

$$
\begin{array}{ll}
(-i\hbar\,\nabla - \frac{e^*}{c}\mathbf{A})\psi\cdot\mathbf{n} = i\gamma\psi & \text{on } \partial\Omega, \\
(\mathrm{curl}\mathbf{A} - \mathbf{H}_e)\times\mathbf{n} = 0 & \text{on } \partial\Omega, \\
\psi(x,0) = \psi_0(x) & \text{in } \Omega, \\
\mathbf{A}(x,0) = \mathbf{A}_0(x) & \text{in } \Omega,
\end{array}
$$

where ψ_0 and $\mathbf{A_0}$ are given functions.

As mentioned before, the TDGL equations are gauge invariant and any two solutions that are gauge equivalent through the gauge transformation G_χ (1.13) produce physically indistinguishable observables such as current, magnetic field and density of superconducting electrons $|\psi|^2$, and thus represent the same physical solution. Because of this gauge invariant equivalent relation, the TDGL initial-boundary-value problem (IBVP) is not mathematically well-posed since it has infinite number of solutions. Therefore, we need to choose a fixed gauge function χ in our TDGL equations in order to obtain a unique solution in our analytical and numerical studies of the TDGL IBVP. The good point is that we can choose any gauge function χ to fit our convenient need and to simplify our problem. There are many possible choices of the gauge; for example, one can choose the zero electric potential gauge to eliminate the electric potential ϕ in the TDGL equations and this gauge is well-suited for computation, see e.g., [2]; and [13] for other choices of gauge.

1.2.3 Characteristic Lengths

There are two important temperature dependent material parameters which arise from the GL model that characterize the phenomenological properties of a superconductor. First is the coherence length ξ defined as:

$$
\xi(\mathcal{T}) = \left(\frac{\hbar^2}{2m^*|\alpha(\mathcal{T})|}\right)^{\frac{1}{2}}. \tag{1.16}
$$

This parameter specifies the spatial width of the transition layer of the order parameter ψ in the neighborhood of the boundary between a normal region and a superconducting region. Within the transition layer, the magnitude of the order parameter which represents the density of the superconducting electrons can rise from zero at the boundary to a value $|\psi_\infty|^2$ inside the superconductor. As a result of this layer, the order parameter on the normal-superconducting boundary can not change abruptly. We will see in the section 1.2.5 concerning the Josephson effects that the proximity effect occurs near the boundary of a normal-superconductor interface causes the continuity of the order parameter on the boundary.

The second characteristic parameter is the temperature dependent spatial penetration depth λ defined as:

$$\lambda(\mathcal{T}) = \left(\frac{m^* c^2 \beta(\mathcal{T})}{4\pi e^{*2} |\alpha(\mathcal{T})|} \right)^{\frac{1}{2}}. \tag{1.17}$$

An external applied magnetic field can penetrate the superconductor in the neighborhood of the boundary of a normal-superconducting region to a distance specified by the penetration depth. As a result, the magnetic field does not drop abruptly to zero inside the superconductor but decays exponentially within the penetration depth where the screening supercurrent flows. We will see that the parameters ξ and λ form the fundamental length scales among others for the nondimensionalization of the TDGL equations.

1.2.4 Type-I and Type-II Superconductors and Vortices

The ratio between the two charateristic lengths ξ and λ is called the Ginzburg-Landau parameter

$$\kappa = \frac{\lambda}{\xi} = \left(\frac{m^{*2} c^2 \beta}{2\pi e^{*2} \hbar^2} \right)^{\frac{1}{2}}. \tag{1.18}$$

The significance of this parameter is that when $\kappa > 1/\sqrt{2}$, negative surface energy is formed on any normal-superconducting surface in the presence of magnetic field with magnitude in the range between H_{c1} and H_{c2} [61], where H_{c1} and H_{c2} are the temperature-dependent lower critical field and upper critical field, respectively. Thus if the decrease in surface energy surpasses the increase in energy in the normal region, it is energetically more favorable to form normal regions inside the superconductor than to maintain the Meissner state. A mixed state is reached when the normal regions are formed in such a way to give maximum surface area relative to the volume of the normal regions. It is shown that each such normal region formed can only allow magnetic flux in an integer multiple of quantum unit $\Phi_0 = 2\pi\hbar c/e^*$ called fluxion to pass. In the mixed state, the magnetic flux is allowed to penetrate the superconductor in the form of flux tubes or vortices with each isolated vortex carrying a fluxion. We call any such superconductor with $\kappa > \frac{1}{\sqrt{2}}$ a type-II superconductor. On the contrary, a normal region inside a superconductor with $\kappa < \frac{1}{\sqrt{2}}$ gives positive surface energy and is thus energetically unfavorable, and we call such superconductors as type-I superconductors. Type-I superconductors have no mixed state, when the magnitude of an applied magnetic field, denoted as H_e, is lower than the characteristic thermodynamic critical field H_c of a type-I superconductor, the phase of the type-I superconductor changes into the Meissner state, and when H_a is higher than H_c, the phase of the superconductor changes to the normal state. As we have just mentioned at the beginning of this section, the mixed state in a type-II superconductor happens when $H_{c1} < H_e < H_{c2}$. These three temperature-dependent fields are defined on Φ_0, ξ, λ and κ as

$$
\begin{aligned}
H_(\mathcal{T}) &= \frac{\Phi_0}{2\sqrt{2}\pi\xi(\mathcal{T})\lambda(\mathcal{T})}, \\
H_{c1}(\mathcal{T}) &= \frac{\Phi_0 \ln\kappa}{4\pi\lambda^2(\mathcal{T})} = \frac{H_c(\mathcal{T})\ln\kappa}{\sqrt{2}\kappa}, \\
H_{c2}(\mathcal{T}) &= \frac{\Phi_0}{2\pi\xi^2(\mathcal{T})} = \sqrt{2}\kappa H_c(\mathcal{T}),
\end{aligned}
$$

and they satisfy the relation

$$H_{c1}(\mathcal{T}) < H_c(\mathcal{T}) < H_{c2}(\mathcal{T}).$$

When $H_e < H_{c1}$, the type-II superconductor is in the Meissner state, and when $H_e > H_{c2}$, the superconductor is in the normal state. Together with the thermodynamic critical field H_c, the upper and lower critical fields form three characteristic field strengths intrinsic to the type-II superconductor.

Since vortices carry magnetic flux, they can be moved by many means, for example, by the Lorentz force $\mathbf{J} \times \Phi_0$ induced by an applied current of density $\mathbf{J}$ acting on each vortex, by mutual vortex repulsion, by Magnus "lift" force which exerts on spinning moving object in a fluid medium [63], or by thermal fluctuations. Suppose a vortex moves at a velocity of v_ϕ, then an electric field $\mathbf{E} = v_\Phi \times \mu_0\mathbf{J}$, where μ_0 is the permeability of the superconductor, is generated which in turn generates an ohmic loss $\mathbf{J} \cdot \mathbf{E}$ as heat dissipation which may drive the superconductor into the normal state. Thus, it is desirable to trap or pin down the vortices from moving. Intensive research is being done to create sufficiently strong pinning forces and optimal pinning locations to reduce heat dissipation and allow the maximal current flow. The maximal current that can flow through the superconductor before the pinned vortices start to move or before it is pushed to the normal state is called the critical current $\mathbf{J}_c$, and it is another characteristic parameter of the superconductor.

1.2.5 Josephson Effects

Electron tunneling is a common quantum mechanical phenomenon in which electrons confined in a space by an energy barrier have a probability to cross the barrier. Tunneling involving quasiparticles, which are the energetic excited superconducting electrons (superelectrons), occurs in superconductive "weak link" structures composed of a barrier made of a thin insulator sandwiched between two superconductors (S-I-S) or between a superconductor and a normal material (S-I-N). Such tunneling requires an application of a biased voltage across the link to shift the energy level in one superconductor on one side of the barrier to facilitate the current flow. In 1962, B.D. Josephson made a prediction based on microscopic theory that a remarkable tunneling effect involving only superelectrons can occur across a S-N-S weak link without the need of voltage bias. This non-voltage tunneling effect was confirmed experimentally shortly later in 1963. It is now understood that this effect also occurs, in addition to the S-N-S link, in more general links such as a narrow constriction which joints two pieces of superconductors of the same kind continuously, and a superconductor-normal metal direct contact whereby the superelectrons from the supercondcutor diffuse into the normal metal by a proximity effect. The supercurrent tunneling effect is called the Josephson effect and all these links are collectively called Josephson junctions.

A variety of behaviors of the Josephson effects can be described by or derived from the Josephson relations:

$$\mathbf{j}_s = \mathbf{j}_c \sin\Delta\theta, \tag{1.19}$$

and

$$\frac{d(\Delta\theta)}{dt} = \frac{2e}{\hbar}V, \tag{1.20}$$

where $\Delta\theta = \theta_1 - \theta_2$, and θ_i is the phase of the order parameter ψ_i. (Note: In the presence of a magnetic field, the phase difference $\Delta\theta$ must be generalized to the gauge invariant phase difference $\Theta = \Delta\theta - (2\pi/\Phi_0)\int \mathbf{A}\cdot ds$, as now Θ enters the gauge invariant gradient term in (1.8).)

The first equation (1.19) says that the tunneling current is a function of the phase difference between the two distinct order parameters of the two materials across the junction. The maximum current density flows across the junction is bounded by the critical current density $\mathbf{j}_c$ that the junction can sustain. The current flow can occur even in the absence of an applied voltage across the junction and in this case, by (1.20) $\Delta\theta$ is a constant over time across the junction and thus by (1.19) the current is stationary. This effect is called the direct current (dc) Josephson effect.

The second equation (1.20) says that the rate of change of the phase difference $\Delta\theta$ in time is proportional to the applied voltage across the junction. Suppose the voltage V is independent of time, then we have:

$$\Delta\theta(t) = \theta_0 + \frac{2e}{\hbar}Vt, \tag{1.21}$$

and thus

$$\mathbf{j}_s = \mathbf{j}_c \sin(\nu t + \theta_0), \tag{1.22}$$

where $\nu = 2eV/\hbar$ is called the Josephson frequency. Equation (1.22) shows that an applied voltage across the junction causes the the current flow to alternate at frequency ν with amplitude $\mathbf{j}_c$. This effect is called the alternating current (ac) Josephson effect. More complicated Josephson effects can be derived based on the two Josephson relations; following are some examples (see, e.g., [62], [63] and [64]):

1. Inverse ac Josephson effect occurs when an radio frequency ac or electromagnetic field excitation such as a microwave is imposed into the junction, a dc voltage is generated across the unbiased junction. A "staircase" like I versus V (current-voltage) characteristic pattern is the hallmark of this type of effect (see e.g., chapter 13 in [63]).

2. In the presence of a magnetic field, the behaviors of the Josephson effects, depending on the structural configuration of the junction and the orientation of the magnetic field, are rich and complicated. Consider the case when a magnetic field is applied parallel to a "short" S-N-S junction which we mean the size of the junction is small enough that the magnetic field (screening field) generated by the junction current is negligible as compared to the applied field. The net tunneling current across the junction in this case is attenuated

from the zero field current by the factor:

$$\frac{i_{max}}{i_c(0)} = \left| \frac{\sin(\pi\Phi/\Phi_0)}{\pi\Phi/\Phi_0} \right|, \tag{1.23}$$

where $i_c(0)$ is the critical current at zero applied magnetic field, Φ is the applied flux and Φ_0 is the quantum flux unit described in section 1.2.4. Equation (1.23) is called the Josephson diffraction equation and it shows that the net current vanishes when the magnetic flux Φ has values equal to $n\Phi_0$, where n is an integer, and the net current has (decaying) local maximum when $\Phi = (n + 1/2)\Phi_0$. This accounts for the fact that Θ (the gauge invariant phase difference) changes as a spatial function along the entire cross section of the junction parallel to the direction of the magnetic field. Suppose the phase difference is $2n\pi$ across the entire length of the junction perpendicular to the field, then according to the first Josephson relation (1.19), the current changes sign $2n$ times cancelling itself and thus produces a zero net current; but then it forms n current loops along the entire length. Each current loop encircles magnetic flux of a quantum flux unit Φ_0, and is called Josephson vortex.

For a "long" Josephson junction in which its screening field produced by the tunneling current is not negligible as compared to the applied field, the resulting behaviors are even more complicated. In general, its behaviors can be described by a Sine-Gordon equation:

$$\left(\triangle - \frac{1}{c^2} \frac{\partial^2}{\partial t} \right) \Theta = \frac{1}{\lambda_j^2} \sin\Theta, \tag{1.24}$$

where c is a constant and λ_j is the Josephson penetration depth which measure how significant is the screening field compared to the applied field and thus is used to distinguish what we called the short and long junctions. If the size of the junction is very small compared to λ_j, the screening effect is insignificant and we call this junction a short junction. On the opposite, we call the junction a long junction. The solutions of the Sine-Gondon equation are solitary waves (solitons) and represent the dynamics of the vortices in the Josephson effect context (see, e.g., [62] and [64]).

1.2.6 The Two-Band Superconductivity

Discovery of MgB_2

Superconductors with multiple superconducting energy gaps (multiband superconductors) have been an interest of theoretical study based on the microscopic BCS theory since the paper of H. Suhl, *et al.* [21] was published in 1959. However materials containing two or more distinct energy gaps had never been observed in an experiment. In 2001, Japanese physicists discovered experimentally the novel superconductor magnesium diboride MgB_2 which has a remarkably high critical temperature of 39 K, twice the highest transition temperature ever recorded in any conventional metallic superconductors [23]. Theis class of superconductors is classified as low $\mathcal{T}_c$ superconductors, as opposed to the high $\mathcal{T}_c$ superconductors which are commonly fabricated in the form of copper-oxide compounds. In 2002 theorists predicted that MgB_2 contains two superconducting energy gaps on the Fermi surface which is the highest occupied electron energy state [24]. In 2003 experiments conducted by Japanese and US physicists confirmed that MgB_2 contains two energy gaps [25].

Electronic Structure of MgB_2

MgB_2 is an intermetallic compound and is known now as an electron-phonon mediated superconductor with two distinct energy gaps and therefore two order parameters. Its crystal structure consists of hexagonal honeycombed layers of boron atoms separated by planes of magnesium atoms. It has four distinctive Fermi surface sheets. Electrons form superconducting pairs on different sheets with different binding or gap energies. The superconducting energy gaps on the two σ bands which form in two nested Fermi cylindrical sheets confined in the Boron planes have values around 7 meV. The energy gaps on the two π bands which form in two "webbed tunnels" Fermi sheets have values around 2 meV. The strong intraband electron pairing in the σ bands is considered to be the main contribution for the high temperature superconductivity , see [24],[25],[26] and its references. MgB_2 is a type-II superconductor with estimated London penetration depth $\lambda(0K)$ = 125-140 nm, coherence length $\xi(0K) = 5.2$ nm, and GL parameter $\kappa(0K) = 26$ [28].

The Isotropic Two-Band Time-Dependent Ginzburg-Landau (2B-TDGL) Model

Due to its two-band nature with two distinct energy gaps associated to two separate group of Fermi surfaces, MgB_2 has many peculiar properties not explainable by standard one-band BCS and one-band TDGL models, and their variants such as the anisotropic TDGL model [31], [32]. Generalization of the microscopic BCS models such as the two-band BCS theory and the two-band Eliashberg theory are used by many authors to predict or explain the quantitative characteristics of two-band superconductors, for example, energy gaps, specific heat and critical fields [24], [41], [26]. The TDGL model is also generalized into the two-band TDGL model from the two-band BCS theory [41] and is used to predict and explain the phenomenological behaviors of two-band superconductors such as vortices [41], [36], penetration depth [28], critical fields [29] and interband phase texture [27].

CHAPTER 2

The Isothermal, Isotropic 2B-TDGL Equations

The two-band time-dependent Ginzburg-Landau model generalizes the conventional one-band Ginzburg-Landau model essentially by coupling the two order parameters with coupling terms analogous to that of a Josephson junction. As in the one-band TDGL equations, the derivation of the 2B-TDGL equations are also based on the first variations of a free energy, but now the free energy of a two-band superconductivity model consists of two distinct energy contributions, each from a distinct order parameter. There are also interband coupling energy terms to account for the interband interactions. The 2B-TDGL equations are given by

$$\Gamma_\mu \left(\frac{\partial \psi_\mu}{\partial t} + \frac{ie^*}{\hbar} \phi \psi_\mu \right) = -\frac{\delta \mathcal{F}_{12}}{\delta \psi_\mu^*} \qquad \mu = 1, 2, \tag{2.1}$$

$$\sigma_n \left(\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \nabla \phi \right) = -c \frac{\partial \mathcal{F}_{12}}{\partial \mathbf{A}}, \tag{2.2}$$

where

$$\mathcal{F}_{12}(\psi_1,\, \psi_2) = \int_\Omega (\, f_1 + f_2 + f_{12} + f_m)\, d\Omega, \tag{2.3}$$

with

$$f_\mu = \alpha_\mu |\psi_\mu|^2 + \frac{\beta_\mu}{2} |\psi_\mu|^4 + \frac{1}{2m_\mu^*} \left| (-i\hbar \nabla - \frac{e^*}{c} \mathbf{A}) \psi_\mu \right|^2 \qquad \mu = 1, 2, \tag{2.4}$$

$$f_{12} = \epsilon [\psi_1^* \psi_2 + c.c.] + \epsilon_1 \left[(i\hbar \nabla - \frac{e^*}{c} \mathbf{A}) \psi_1^* (-i\hbar \nabla - \frac{e^*}{c} \mathbf{A}) \psi_2 + c.c. \right], \tag{2.5}$$

$$f_m = \frac{1}{8\pi} \left| \mathrm{curl} \mathbf{A} - \mathbf{H}_e \right|^2, \tag{2.6}$$

where *c.c.* denotes the complex conjugate, and the real numbers ϵ and ϵ_1 are coupling parameters describing the interband interactions of the two order parameters and their gradients, respectively. Here f_1 and f_2 are the conventional one-band free energy densities for the order parameters ψ_1 and ψ_2, respectively. All the other parameters are the same as those found in the TDGL equations.

In general, the critical temperature for each order parameter in a multiband superconductor is distinct from each other. To take into account this situation, we retain the explicit temperature dependences of $\alpha_\mu = \alpha_\mu(\mathcal{T})$ and $\beta_\mu = \beta_\mu(\mathcal{T})$ in the 2B-TDGL equations and approximate them as

$$\alpha_\mu = \alpha_\mu(0)\left[\frac{\mathcal{T}}{\mathcal{T}_{c\mu}} - 1\right], \qquad \beta_\mu = \beta_\mu(0), \tag{2.7}$$

where $\alpha_\mu(0) > 0$ and $\beta_\mu(0) > 0$ are the corresponding parameters at absolute zero ($\mathcal{T} = 0$), and $\mathcal{T}_{c\mu}$ is the critical temperature of the band μ. From the one-band Ginzburg-Landau theory, the above approximations are valid for small variation of $\mathcal{T}$ near the critical temperature $\mathcal{T}_c$ of each individual band. However, the $\mathcal{T}_c$ of one band may be several times larger or smaller than the $\mathcal{T}_c$ of another band. In order to correctly model a phenomenon, we may need to set the operating temperature $\mathcal{T}$ below both $\mathcal{T}_{c1}$ and $\mathcal{T}_{c2}$. As a result, the temperature $\mathcal{T}$ may be slightly below the lowest critical temperature, say $\mathcal{T}_{c1}$, but becomes deviated largely below the highest critical temperature, $\mathcal{T}_{c2}$. In this case, whether this 2B-TDGL model remains qualitatively valid needs further investigation.

We want to emphasize that the temperature $\mathcal{T}$ in our model is a given fixed value. In other words, we only consider an isothermal 2B-TDGL model and neglect the effect of Joule heating produced by the interchanging of thermal energy and electro-magnetic energy loss which can be generated by a time-varying current or a time-varying magnetic field. As a limitation to this isothermal simplification in simulations of real world situations, we may only consider a time-dependent applied current or a magnetic field that varies slowly in time. For non-isothermal TDGL model, we refer the reader to papers such as [14] and [15].

From equations (2.1) and (2.2), we obtain the following dimensional isothermal isotropic 2B-TDGL equations:

$$\begin{aligned}\Gamma_1\left(\frac{\partial\psi_1}{\partial t} + \frac{ie^*}{\hbar}\phi\psi_1\right) &+ \alpha_1\psi_1 + \beta_1|\psi_1|^2\psi_1 + \frac{1}{2m_1^*}\left(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A}\right)^2\psi_1 \\ &+ \epsilon\,\psi_2 + \epsilon_1\left(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A}\right)^2\psi_2 = 0 \qquad \text{in } \Omega\times(0,\mathrm{T}),\end{aligned}$$

$$\Gamma_2\left(\frac{\partial\psi_2}{\partial t}+\frac{ie^*}{\hbar}\,\phi\psi_2\right)+\,\alpha_2\psi_2+\,\beta_2|\psi_2|^2\psi_2+\frac{1}{2m_2^*}\left(-i\hbar\nabla-\frac{e^*}{c}\mathbf{A}\right)^2\psi_2$$
$$+\,\epsilon\,\psi_1+\epsilon_1\left(-i\hbar\nabla-\frac{e^*}{c}\mathbf{A}\right)^2\psi_1=0 \qquad \text{in } \Omega\times(0,\mathrm{T}),$$

$$\begin{aligned}
\frac{c}{4\pi}\mathrm{curl}^2\mathbf{A} &= \frac{c}{4\pi}\mathrm{curl}\,\mathbf{H}_e+\sigma_n\left(-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}-\nabla\phi\right)\\
&+ \frac{e^*}{m_1^*}\left[\frac{i\hbar}{2}\left(\psi_1\nabla\psi_1^*-\psi_1^*\nabla\psi_1\right)-\frac{e^*}{c}|\psi_1|^2\mathbf{A}\right]\\
&+ \frac{e^*}{m_2^*}\left[\frac{i\hbar}{2}\left(\psi_2\nabla\psi_2^*-\psi_2^*\nabla\psi_2\right)-\frac{e^*}{c}|\psi_2|^2\mathbf{A}\right]\\
&- \epsilon_1 i\hbar e^*(\psi_1^*\nabla\psi_2-\psi_1\nabla\psi_2^*+\psi_2^*\nabla\psi_1-\psi_2\nabla\psi_1^*)\\
&- \epsilon_1\frac{2e^{*2}}{c}\mathbf{A}(\psi_1\psi_2^*+\psi_2\psi_1^*) \qquad \text{in } \Omega\times(0,\mathrm{T}),
\end{aligned}$$

or equivalently,

$$\begin{aligned}
\frac{c}{4\pi}\mathrm{curl}^2\mathbf{A} &= \frac{c}{4\pi}\mathrm{curl}\,\mathbf{H}_e+\sigma_n\left(-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}-\nabla\phi\right)\\
&+ e^*\psi_1^*\left[\frac{1}{2}\frac{1}{m_1^*}\left(-i\hbar\nabla\psi_1-\frac{e^*}{c}\mathbf{A}\psi_1\right)+\epsilon_1\left(-i\hbar\nabla\psi_2-\frac{e^*}{c}\mathbf{A}\psi_2\right)\right]\\
&+ e^*\psi_1\left[\frac{1}{2}\frac{1}{m_1^*}\left(i\hbar\nabla\psi_1^*-\frac{e^*}{c}\mathbf{A}\psi_1^*\right)+\epsilon_1\left(i\hbar\nabla\psi_2^*-\frac{e^*}{c}\mathbf{A}\psi_2^*\right)\right]\\
&+ e^*\psi_2^*\left[\frac{1}{2}\frac{1}{m_2^*}\left(-i\hbar\nabla\psi_2-\frac{e^*}{c}\mathbf{A}\psi_2\right)+\epsilon_1\left(-i\hbar\nabla\psi_1-\frac{e^*}{c}\mathbf{A}\psi_1\right)\right]\\
&+ e^*\psi_2\left[\frac{1}{2}\frac{1}{m_2^*}\left(i\hbar\nabla\psi_2^*-\frac{e^*}{c}\mathbf{A}\psi_2^*\right)+\epsilon_1\left(i\hbar\nabla\psi_1^*-\frac{e^*}{c}\mathbf{A}\psi_1^*\right)\right]\\
&\qquad \text{in } \Omega\times(0,\mathrm{T}). \qquad (2.8)
\end{aligned}$$

The first variations of the energy functional (2.3)-(2.6) give the natural dimensional boundary conditions, together with the initial conditions; they are:

$$\begin{aligned}
\left[\frac{1}{2}\frac{1}{m_1^*}\left(-i\hbar\nabla\psi_1-\frac{e^*}{c}\mathbf{A}\psi_1\right)+\epsilon_1\left(-i\hbar\nabla\psi_2-\frac{e^*}{c}\mathbf{A}\psi_2\right)\right]\cdot\mathbf{n} &= 0 \quad \text{on } \partial\Omega\times(0,\mathrm{T}), \quad (2.9)\\
\left[\frac{1}{2}\frac{1}{m_2^*}\left(-i\hbar\nabla\psi_2-\frac{e^*}{c}\mathbf{A}\psi_2\right)+\epsilon_1\left(-i\hbar\nabla\psi_1-\frac{e^*}{c}\mathbf{A}\psi_1\right)\right]\cdot\mathbf{n} &= 0 \quad \text{on } \partial\Omega\times(0,\mathrm{T}), \quad (2.10)\\
\mathrm{curl}\mathbf{A}\times\mathbf{n} &= \mathbf{H}_e\times\mathbf{n} \quad \text{on } \partial\Omega\times(0,\mathrm{T}),\\
\psi_1(x,0) &= \psi_{10}(x) \qquad \text{in } \Omega,\\
\psi_2(x,0) &= \psi_{20}(x) \qquad \text{in } \Omega,\\
\mathbf{A}(x,0) &= \mathbf{A}_0(x) \qquad \text{in } \Omega.
\end{aligned}$$

Note that the above PDEs and boundary conditions are gauge invariant, i.e., given a solution $(\psi_1, \psi_2, \mathbf{A}, \phi)$, let G_χ be a gauge transformation defined as

$$G_\chi : (\psi_1, \psi_2, \mathbf{A}, \phi) \mapsto (\psi_1', \psi_2', \mathbf{A}', \phi') = (\psi_1 e^{i\chi},\ \psi_2 e^{i\chi},\ \mathbf{A} + \frac{\hbar c}{e^*}\nabla\chi,\ \phi - \frac{\hbar}{e^*}\frac{\partial \chi}{\partial t}), \qquad (2.11)$$

where χ is an arbitrary but sufficiently smooth single-valued scalar function of time and spatial coordinates, then the transformed solution $(\psi_1', \psi_2', \mathbf{A}', \phi')$ is also a solution for the 2B-TDGL system.

For a superconductor-normal metal (S-N) interface, the boundary conditions (2.9) and (2.10) should be modified to the De Gennes's S-N type boundary conditions (see (1.11)) which are implied by the requirement that the normal component of the supercurrent does not cross the superconductor sample, i.e., $\mathbf{j}_s \cdot \mathbf{n} = 0$ on $\partial\Omega$, see [1] and [61]. Imposing this S-N interface requirement on equation (2.8), we get:

$$\begin{aligned}
\left[\frac{1}{2}\frac{1}{m_1^*}\left(-i\hbar\nabla\psi_1 - \frac{e^*}{c}\mathbf{A}\psi_1\right) + \epsilon_1\left(-i\hbar\nabla\psi_2 - \frac{e^*}{c}\mathbf{A}\psi_2\right)\right]\cdot\mathbf{n} &= i\frac{\hbar}{2m_1^*}\gamma_1\psi_1 + \epsilon_1 i\hbar\gamma_2\psi_2 \\
&\quad \text{on } \partial\Omega\times(0,\mathrm{T}), \qquad (2.12)\\
\left[\frac{1}{2}\frac{1}{m_2^*}\left(-i\hbar\nabla\psi_2 - \frac{e^*}{c}\mathbf{A}\psi_2\right) + \epsilon_1\left(-i\hbar\nabla\psi_1 - \frac{e^*}{c}\mathbf{A}\psi_1\right)\right]\cdot\mathbf{n} &= i\frac{\hbar}{2m_2^*}\gamma_2\psi_2 + \epsilon_1 i\hbar\gamma_1\psi_1 \\
&\quad \text{on } \partial\Omega\times(0,\mathrm{T}). \qquad (2.13)
\end{aligned}$$

Here γ_1 and γ_2 are assumed to be nonnegative real valued functions satisfing $\gamma_i(\mathbf{x}) \geq 0$, for $\mathbf{x} \in \partial\Omega$ and $\gamma_i \in L^\infty(\partial\Omega)$. For the above set of boundary conditions, we need $\gamma_1 = \gamma_2$ to be satisfied in order to satisfy the S-N interface requirement.

By adding the additional term

$$\int_{\partial\Omega}\left(\frac{\hbar^2}{2m_1^*}\gamma_1|\psi_1|^2 + \frac{\hbar^2}{2m_2^*}\gamma_2|\psi_2|^2 + \epsilon_1(\hbar^2\gamma_1\psi_1\psi_2^* + \hbar^2\gamma_1\psi_1^*\psi_2 + \hbar^2\gamma_2\psi_2\psi_1^* + \hbar^2\gamma_2\psi_2^*\psi_1)\right)dS$$

to the energy functional (2.3), boundary conditions (2.12) and (2.13) become the natural boundary conditions of this new functional's Euler-Lagrange equations.

Other possible boundary conditions which also satisfy the S-N interface requirement and replace the boundary conditions (2.9) and (2.10) are

$$\begin{aligned}
\left[\frac{1}{2}\frac{1}{m_1^*}\left(-i\hbar\nabla\psi_1 - \frac{e^*}{c}\mathbf{A}\psi_1\right) + \epsilon_1\left(-i\hbar\nabla\psi_2 - \frac{e^*}{c}\mathbf{A}\psi_2\right)\right]\cdot\mathbf{n} &= i\frac{\hbar}{2m_1^*}\gamma_1\psi_1 \\
&\quad \text{on } \partial\Omega\times(0,\mathrm{T}), (2.14)\\
\left[\frac{1}{2}\frac{1}{m_2^*}\left(-i\hbar\nabla\psi_2 - \frac{e^*}{c}\mathbf{A}\psi_2\right) + \epsilon_1\left(-i\hbar\nabla\psi_1 - \frac{e^*}{c}\mathbf{A}\psi_1\right)\right]\cdot\mathbf{n} &= i\frac{\hbar}{2m_2^*}\gamma_2\psi_2 \\
&\quad \text{on } \partial\Omega\times(0,\mathrm{T}). (2.15)
\end{aligned}$$

By adding the additional term

$$\int_{\partial\Omega} \left(\frac{\hbar^2}{2m_1^*}\gamma_1|\psi_1|^2 + \frac{\hbar^2}{2m_2^*}\gamma_2|\psi_2|^2 \right) dS$$

to the energy functional (2.3), boundary conditions (2.14) and (2.15) become the natural boundary conditions of this new functional's Euler-Lagrange equations.

Note that when $\epsilon_1 = 0$, we recover the boundary conditions proposed by De Gennes in both cases:

$$(-i\hbar\,\nabla - \frac{e^*}{c}\mathbf{A})\psi_i \cdot \mathbf{n} = i\hbar\gamma\psi_i \qquad \text{on } \partial\Omega.$$

2.1 Nondimensionalization of the 2B-TDGL Equations

We non-dimensionalize the 2B-TDGL equations with the following non-dimensional variables (those with $'$) and parameters

For $\mu = 1, 2$,

$$\left.\begin{aligned}
&x = x_0 x', && t = \frac{\Gamma_1}{|\alpha_1(0)|}t', \\
&\phi = \frac{\hbar|\alpha_1(0)|}{2\Gamma_1 e^*}\phi', && \psi_\mu = \left(\frac{|\alpha_\mu(0)|}{\beta_\mu(0)}\right)^{\frac{1}{2}}\psi'_\mu, \\
&\mathbf{A} = \left(\frac{8\pi|\alpha_1(0)|^2}{\beta_1(0)}\right)^{\frac{1}{2}} x_0\mathbf{A}', && \mathbf{H}_e = \sqrt{2}\left(\frac{4\pi|\alpha_1(0)|^2}{\beta_1(0)}\right)^{\frac{1}{2}}\mathbf{H}'_e = \sqrt{2}H_c^1(0)\mathbf{H}'_e, \\
&\xi_\mu = \left(\frac{\hbar^2}{2m_\mu^*|\alpha_\mu(0)|}\right)^{\frac{1}{2}}, && \lambda_\mu = \left(\frac{c^2 m_\mu^*\beta_\mu(0)}{4\pi e^{*2}|\alpha_\mu(0)|}\right)^{\frac{1}{2}}, \\
&\kappa_\mu = \left(\frac{c^2 m_\mu^{*2}\beta_\mu(0)}{2\pi e^{*2}\hbar^2}\right)^{\frac{1}{2}}, && \nu = \left(\frac{|\alpha_1(0)|^2\beta_2(0)}{|\alpha_2(0)|^2\beta_1(0)}\right)^{\frac{1}{2}} = \left(\frac{\lambda_2(0)\xi_2(0)}{\lambda_1(0)\xi_1(0)}\right), \\
&\eta = \epsilon\left(\frac{\beta_1(0)|\alpha_2(0)|}{\beta_2(0)|\alpha_1(0)|}\right)^{\frac{1}{2}}\frac{1}{|\alpha_1(0)|}, && \eta_1 = \epsilon_1 2(m_1^* m_2^*)^{\frac{1}{2}}, \\
&\Gamma = \frac{\Gamma_2|\alpha_1(0)|}{\Gamma_1|\alpha_2(0)|}, && \sigma = \frac{\sigma_n m_1^*\beta_1(0)}{\Gamma_1 e^{*2}},
\end{aligned}\right\} \quad (2.16)$$

where

$$H_c^1(0) = \left(\frac{4\pi|\alpha_1(0)|^2}{\beta_1(0)}\right)^{\frac{1}{2}}$$

is the thermodynamic critical field of the first band at $\mathcal{T} = 0$.

Define

$$\mathcal{T}_1 = \left[1 - \frac{\mathcal{T}}{\mathcal{T}_{c1}}\right], \qquad \mathcal{T}_2 = \left[1 - \frac{\mathcal{T}}{\mathcal{T}_{c2}}\right]. \tag{2.17}$$

Then the non-dimensionalized 2B-TDGL equations (with $'$ dropped) are:

$$\begin{aligned}&\left(\frac{\partial\psi_1}{\partial t} + i\,\phi\psi_1\right) + \left(|\psi_1|^2 - \mathcal{T}_1\right)\psi_1 + \left(-i\frac{\xi_1}{x_0}\nabla - \frac{x_0}{\lambda_1}\mathbf{A}\right)^2\psi_1 \\ &\quad + \eta\psi_2 + \eta_1\frac{\xi_1}{\xi_2}\frac{1}{\nu}\left(-i\frac{\xi_2}{x_0}\nabla - \nu\frac{x_0}{\lambda_2}\mathbf{A}\right)^2\psi_2 = 0 \qquad \text{in}\,\Omega\times(0,\mathrm{T}),\end{aligned} \tag{2.18}$$

$$\begin{aligned}&\Gamma\left(\frac{\partial\psi_2}{\partial t} + i\,\phi\psi_2\right) + \left(|\psi_2|^2 - \mathcal{T}_2\right)\psi_2 + \left(-i\frac{\xi_2}{x_0}\nabla - \nu\frac{x_0}{\lambda_2}\mathbf{A}\right)^2\psi_2 \\ &\quad + \eta\nu^2\psi_1 + \eta_1\frac{\xi_2}{\xi_1}\nu\left(-i\frac{\xi_1}{x_0}\nabla - \frac{x_0}{\lambda_1}\mathbf{A}\right)^2\psi_1 = 0 \qquad \text{in}\,\Omega\times(0,\mathrm{T}),\end{aligned} \tag{2.19}$$

$$\begin{aligned}\mathrm{curl}^2\mathbf{A} &= \mathrm{curl}\,\mathbf{H}_e + \sigma\left(-\frac{x_0^2}{\lambda_1^2}\frac{\partial\mathbf{A}}{\partial t} - \frac{1}{\kappa_1}\nabla\phi\right) \\ &+ i\frac{1}{2}\frac{1}{\kappa_1}\left(\psi_1\nabla\psi_1^* - \psi_1^*\nabla\psi_1\right) - \frac{x_0^2}{\lambda_1^2}|\psi_1|^2\mathbf{A} \\ &+ i\frac{1}{2}\frac{1}{\nu}\frac{1}{\kappa_2}\left(\psi_2\nabla\psi_2^* - \psi_2^*\nabla\psi_2\right) - \frac{x_0^2}{\lambda_2^2}|\psi_2|^2\mathbf{A} \\ &- \eta_1 i\frac{1}{2}\frac{\xi_1}{\lambda_2}(\psi_1^*\nabla\psi_2 - \psi_1\nabla\psi_2^* + \psi_2^*\nabla\psi_1 - \psi_2\nabla\psi_1^*) \\ &- \eta_1\frac{x_0^2}{\lambda_1\lambda_2}\mathbf{A}(\psi_1\psi_2^* + \psi_2\psi_1^*) \qquad \text{in}\,\Omega\times(0,\mathrm{T}).\end{aligned} \tag{2.20}$$

As in the dimensional case, the first possible set of non-dimensional S-N interface boundary conditions together with the initial conditions are:

$$\begin{aligned}\left[\left(-i\frac{\xi_1}{x_0}\nabla\psi_1 - \frac{x_0}{\lambda_1}\mathbf{A}\psi_1\right) + \eta_1\frac{1}{\nu}\left(-i\frac{\xi_2}{x_0}\nabla\psi_2 - \nu\frac{x_0}{\lambda_2}\mathbf{A}\psi_2\right)\right]\cdot\mathbf{n} &= i\gamma_1\frac{\xi_1}{x_0}\psi_1 + i\eta_1\frac{1}{\nu}\gamma_2\frac{\xi_2}{x_0}\psi_2 \\ &\quad \text{on}\,\partial\Omega\times(0,\mathrm{T}),\end{aligned} \tag{2.21}$$

$$\begin{aligned}\left[\left(-i\frac{\xi_2}{x_0}\nabla\psi_2 - \nu\frac{x_0}{\lambda_2}\mathbf{A}\psi_2\right) + \eta_1\nu\left(-i\frac{\xi_1}{x_0}\nabla\psi_1 - \frac{x_0}{\lambda_1}\mathbf{A}\psi_1\right)\right]\cdot\mathbf{n} &= i\gamma_2\frac{\xi_2}{x_0}\psi_2 + i\eta_1\nu\gamma_1\frac{\xi_1}{x_0}\psi_1 \\ &\quad \text{on}\,\partial\Omega\times(0,\mathrm{T}),\end{aligned} \tag{2.22}$$

$$\begin{aligned}\mathrm{curl}\mathbf{A}\times\mathbf{n} &= \mathbf{H}_e\times\mathbf{n} \quad \text{on}\,\partial\Omega\times(0,\mathrm{T}), \\ \psi_1(x,0) &= \psi_{10}(x) \quad \text{in}\,\Omega, \\ \psi_2(x,0) &= \psi_{20}(x) \quad \text{in}\,\Omega, \\ \mathbf{A}(x,0) &= \mathbf{A}_0(x) \quad \text{in}\,\Omega.\end{aligned}$$

Here for $i = 1, 2$, $\gamma_i(\mathbf{x}) \geq 0$, for $\mathbf{x} \in \partial\Omega$ and $\gamma_i \in L^\infty(\partial\Omega)$. Boundary conditions (2.21) and (2.22) together require $\gamma_1 = \gamma_2$ to be satisfied in order to satisfy the S-N interface requirement.

As in the dimensional case, other possible S-N interface boundary conditions which replace the boundary conditions (2.21)-(2.22) are

$$\begin{aligned}
\left[\left(-i\frac{\xi_1}{x_0}\nabla\psi_1 - \frac{x_0}{\lambda_1}\mathbf{A}\psi_1\right) + \eta_1\frac{1}{\nu}\left(-i\frac{\xi_2}{x_0}\nabla\psi_2 - \nu\frac{x_0}{\lambda_2}\mathbf{A}\psi_2\right)\right]\cdot\mathbf{n} &= i\gamma_1\frac{\xi_1}{x_0}\psi_1 \\
&\qquad \text{on } \partial\Omega\times(0,\mathrm{T}), \\
\left[\left(-i\frac{\xi_2}{x_0}\nabla\psi_2 - \nu\frac{x_0}{\lambda_2}\mathbf{A}\psi_2\right) + \eta_1\nu\left(-i\frac{\xi_1}{x_0}\nabla\psi_1 - \frac{x_0}{\lambda_1}\mathbf{A}\psi_1\right)\right]\cdot\mathbf{n} &= i\gamma_2\frac{\xi_2}{x_0}\psi_2 \\
&\qquad \text{on } \partial\Omega\times(0,\mathrm{T}),
\end{aligned}$$

The physical meanings associated to each of these two sets of boundary conditions have yet to be investigated.

2.2 2B-TDGL Equations with Time-Dependent Applied Current and Magnetic Field

For the rest of this work, we assume that the domain Ω is an open, bounded, simply connected 2-D domain with Lipschitz boundary $\partial\Omega$; in particular, we are interested in convex polygonal domains. Following the notations used in [2] and [3], let $H^s(\Omega)$ be a real-valued Sobolev space of order integer s in the domain Ω; let $\mathcal{H}^s$ be the corresponding space of complex-valued functions, with real and complex parts belong to $H^s(\Omega)$; and let $\mathbf{H}^s(\Omega)$ be the corresponding space of vector-valued functions, with each component belonging to $H^s(\Omega)$, i.e., $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^n$. We will use the same notation $\|.\|_s$ to denote the norms of $H^s(\Omega)$, $\mathcal{H}^s(\Omega)$ and $\mathbf{H}^s(\Omega)$ without any ambiguity. Similarly, Let $L^q(\Omega)$, $\mathcal{L}^q(\Omega)$ and $\mathbf{L}^q(\Omega)$ be the real-valued Lebesgue space, its corresponding complex-valued and vector-valued spaces respectively. We will use $\|.\|_0$ to denote the norms of any of these L^2 Lebesgue spaces, and $\|.\|_{0,q}$ for any of these L^q spaces, $1 \leq q \leq \infty$, $q \neq 2$. We will also denote by $\mathbf{n}$ the unit outward normal vector on the boundary $\partial\Omega$.

We first define some notations for vector-valued spaces:

$$
\begin{aligned}
&\mathbf{H}(\mathrm{curl};\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) |\, \mathrm{curl}\,\mathbf{v} \in L^2(\Omega) \text{ in } \Omega\},\\
&\mathbf{H}_0(\mathrm{curl};\Omega) = \{\mathbf{v} \in \mathbf{H}(\mathrm{curl};\Omega) |\ \mathbf{v} \times \mathbf{n}|_{\partial\Omega} = 0\},\\
&\mathbf{H}(\mathrm{div};\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) |\, \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\},\\
&\mathbf{H}(\mathrm{div}_0;\Omega) = \{\mathbf{v} \in \mathbf{H}(\mathrm{div};\Omega) |\ \mathrm{div}\,\mathbf{v} = 0 \text{ in } \Omega\},\\
&\mathbf{H}_n(\mathrm{div}_0;\Omega) = \{\mathbf{v} \in \mathbf{H}(\mathrm{div};\Omega) |\ \mathrm{div}\,\mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0\},\\
&\mathbf{H}^1_0(\mathrm{div}_0;\Omega) = \Big\{\mathbf{v} \in \mathbf{H}^1_0(\Omega) |\, \mathrm{div}\,\mathbf{v} = 0 \text{ in } \Omega\Big\},\\
&\mathbf{H}^1_n(\Omega) = \Big\{\mathbf{v} \in \mathbf{H}^1(\Omega) |\, \mathbf{v}\cdot \mathbf{n} = 0 \text{ on } \partial\Omega\Big\},\\
&\mathbf{H}^1_n(\mathrm{div};\Omega) = \Big\{\mathbf{v} \in \mathbf{H}^1_{\mathrm{n}}(\Omega) |\, \mathrm{div}\,\mathbf{v} = 0 \text{ in } \Omega\Big\}.
\end{aligned}
$$

We quote a result from Girault and Raviart [44]: If Ω is an open, bounded and simply connected domain in $\mathbb{R}^2$ with a boundary $\partial\Omega$ of class $C^{1,1}$ or with a piecewise smooth $\partial\Omega$ with no reentrant corners, such as a bounded convex polygon, then for any $\mathbf{A} \in \mathbf{H}^1_n(\Omega)$, we have

$$
C||\mathbf{A}||_{\mathbf{H}^1(\Omega)} \leq ||\mathbf{A}||_{\mathbf{H}^1_n(\Omega)} := ||\mathrm{div}\mathbf{A}||_{L^2(\Omega)} + ||\mathrm{curl}\mathbf{A}||_{\mathbf{L}^2(\Omega)}. \tag{2.23}
$$

Next we define some time dependent Sobolev spaces. For any given $T > 0$, and Banach space X, define the spaces

$$
\begin{aligned}
&L^p(0,T;X) = \left\{ u \Big| u(\cdot,t) \in X,\ t \in (0,T)\ a.e.;\ \left[\int_0^T \|u(\cdot,t)\|_X^p dt\right]^{\frac{1}{p}} < \infty \right\},\\
&L^\infty(0,T;X) = \Big\{ u \big| u(\cdot,t) \in X,\ t \in (0,T)\ a.e.;\ \ \mathrm{ess\ sup}_{0\leq t\leq T}\|u(\cdot,t)\|_X < \infty\Big\},\\
&H^1(0,T;X) = \left\{ u \Big| u(\cdot,t) \in X,\ t \in (0,T)\ a.e.;\ \left[\int_0^T \left(\|u(\cdot,t)\|_X^2 + \|u'(\cdot,t)\|_X^2\right) dt\right]^{\frac{1}{2}} < \infty \right\}.
\end{aligned}
$$

The corresponding complex-valued and vector-valued spaces are defined in an analogous way. In 2-D, we know that $\mathrm{curl}\phi = (\partial\phi/\partial x_2, -\partial\phi/\partial x_1)$ is a vector, while $\mathrm{curl}\,\mathbf{v} = \partial v_2/\partial x_1 - \partial v_1/\partial x_2$ is a scalar. We also understand that in 2-D, $\mathrm{curl}\,\phi \times \mathbf{n}|_{\partial\Omega} = \mathrm{curl}\,\phi\cdot\boldsymbol{\tau}|_{\partial\Omega} = -\nabla\phi\cdot\mathbf{n}|_{\partial\Omega}$ in the trace sense, where $\mathbf{n}$ and $\boldsymbol{\tau}$ are the unit vectors outward normal to and tangential to the boundary $\partial\Omega$, respectively.

Now we want to generalize the 2B-TDGL to include the effects of an applied current. First let us discuss how other authors include an applied current into a one-band time-dependent

Ginzburg-Landau model with various gauge adoptions. Assume a superconductor occupying the domain Ω in the normal state possesses some material properties which will be stated later. Then the total current inside and outside the superconductor is $\mathbf{j} = \mathrm{curl}^2\mathbf{A}$. Since it is divergence free, by using a standard method used in electromagnetics (see for example [65], sec. 2.8), we can derive the following boundary condition that must be satisfied on $\partial\Omega$

$$[\mathbf{j} \cdot \mathbf{n}] = [\mathrm{curl}^2\mathbf{A} \cdot \mathbf{n}] = 0 \qquad \text{on } \partial\Omega,$$

where $[\cdot]$ denotes the jump of the enclosed quantity across the boundary $\partial\Omega$. In other words, if an applied current $\mathbf{j}_a$ is applied to the superconductor either through direct metal lead contacts or by indirect means such as an external electrogmagnetic field, we must have

$$\mathbf{j} \cdot \mathbf{n} = \mathrm{curl}^2\mathbf{A} \cdot \mathbf{n} = \mathbf{j}_a \cdot \mathbf{n}_o \qquad \text{on } \partial\Omega, \tag{2.24}$$

where $\mathbf{n}$ and $\mathbf{n}_o$ are the unit outward normal vector and unit inward normal vector on the boundary $\partial\Omega$, respectively. We will called a current which is applied to the superconductor through metal leads a Type-A current, and a current which is applied to the conductor by means of an external electromegnetic field induction a Type-B current. In the case where there is only a Type-B current source, $\mathbf{j}_a$ is equal to $\mathrm{curl}\mathbf{H}_e$, with $\mathbf{H}_e$ being the external electromagnetic field which induces the current and is assumed throughout this work to be the same inside and outside of the superconductor, and unaffected by the magnetic field or demagnetization created by the superconductor.

Consider a case where there exists a Type-A current $\mathbf{j}_a$ and a Type-B current induced by $\mathbf{H}_e$. Then we have (with some parameters dropped)

$$\begin{aligned} \mathbf{j} = \mathrm{curl}^2\mathbf{A} = -\sigma(\frac{\partial \mathbf{A}}{\partial t} + \nabla\phi) + \mathrm{curl}\mathbf{H}_e + \mathbf{j}_s \qquad &\text{in } \Omega, \\ \mathrm{curl}^2\mathbf{A} = \mathbf{j}_a + \mathrm{curl}\mathbf{H}_e \qquad &\text{outside } \Omega, \end{aligned}$$

where $\mathbf{j}_s$ is the superconcurrent. Now since the normal component of the supercurrent vanishes on the boundary, i.e., $\mathbf{j}_s \cdot \mathbf{n}|_{\partial\Omega} = 0$, then by the continuity boundary condition (2.24), the normal current must satisfy

$$-\sigma(\frac{\partial \mathbf{A}}{\partial t} + \nabla\phi) \cdot \mathbf{n} \;=\; \mathbf{j}_a \cdot \mathbf{n} \qquad \text{on } \partial\Omega, \tag{2.25}$$

where in the above equation, the term $\mathrm{curl}\mathbf{H}_e \cdot \mathbf{n}|_{\partial\Omega}$ has been cancelled out on both sides since by our assumption on $\mathbf{H}_e$, it is the same throughout the region inside and outside of the superconductor sample.

We now investigate how currents are added to the TDGL equations under different gauge choices.

(I) Suppose the divergence-free Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$ and $\mathbf{A} \cdot \mathbf{n}|_{\partial\Omega} = 0$) is applied to the TDGL model, then the continuity boundary condition (2.25) becomes

$$-\sigma \nabla \phi \cdot \mathbf{n} = \mathbf{j}_a \cdot \mathbf{n} \qquad \text{on } \partial\Omega. \tag{2.26}$$

This means that we have to solve for the electric potential ϕ which has to satisfy the TDGL equations and its standard boundary conditions in addition to the boundary condition (2.26) at every computational step solving for ψ and $\mathbf{A}$. Because of this and the divergence free requirement, the Coulomb gauge adoption is not suitable for computation.

(II) Suppose the Lorenz gauge ($\phi = -\nabla \cdot \mathbf{A}$ and $\mathbf{A} \cdot \mathbf{n}|_{\partial\Omega} = 0$) is used, then the electric potential ϕ is eliminated from the TDGL equations and the continuity B.C. (2.25) becomes

$$\nabla(\nabla \cdot \mathbf{A}) \cdot \mathbf{n} = \mathbf{j}_a \cdot \mathbf{n} \qquad \text{on } \partial\Omega. \tag{2.27}$$

(III) Suppose the zero electric potential (ZEP) gauge ($\phi = 0$ and $\mathbf{A} \cdot \mathbf{n}|_{\partial\Omega} = 0$) is used, the right hand side of the continuity B.C. (2.25) vanishes, this means the ZEP gauge can only be used when $\mathbf{j}_a|_{\partial\Omega} = 0$.

The above discussion shows that the Coulomb gauge, Lorentz gauge and the ZEP gauge are not suitable for the modeling of Type-A current explicitly. One viable way to add applied current into the TDGL model is to add Type-B current through external electromagnetic field $\mathbf{H}_e$ by the relation $\mathbf{j}_e = \text{curl}^2\mathbf{A}_e = \text{curl}\mathbf{H}_e$. But there is a limitation to this approach- it can not be used to model a Type-A current because such a normal (stationary or slowly varying) current vanishes inside a superconducting superconductor away from the boundary (see [60], sec. 1.2 and sec. 11.4) and the $\text{curl}\mathbf{H}_e$ term can not model this phenomenon from the outside and without knowledge of the dynamics of the superconductor.

(IV) One way to add a Type-A current explicitly to the TDGL model is to use an extension of the ZEP gauge. For instance, in [4], a gauge which satisfies $\phi = \phi_c$ in Ω and $\mathbf{A} \cdot \mathbf{n} = 0$ on $\partial\Omega$ is adopted to the TDGL equations. Here ϕ_c satisfies the relation $\nabla\phi_c = \mathbf{j}_a$ in $\overline{\Omega}$. By utilizing this gauge, it is easy to see that the continuity B.C. (2.25) is satisfied. Moreover, unlike (2.26), now $\phi = \phi_c$ is a fixed function that is independent of the TDGL equations and thus may be calculated ahead of each computational step solving for ψ and $\mathbf{A}$ of the TDGL equations. The point here is that the role of ϕ_c now only serves as an auxiliary variable and does not represent the true electric potential occurring in the superconductor. However, this gauge choice can only model Type-A applied current of constant value either in space or in time. The reason will be stated in the remark below.

We want to generalize the above gauge choice (IV) to one which can also model time-dependent Type-A applied current of spatial function. For simplicity, throughout the rest of this work we will assume that the superconductor when in the normal state is a linear, isotropic and nondispersive material. We also assume that the superconductor in the normal state is a good conductor with a large but finite conductivity σ, i.e., a metallic superconductor such as the MgB_2 (but it is not an isotropic material). Assume that a time-dependent current $\mathbf{j}_c$ exists in the superconductor in the normal state, i.e. as a conductor, then the electric field $\mathbf{E}_c$, electric displacement $\mathbf{D}_c$, magnetic field $\mathbf{H}_c$, magnetic potential $\mathbf{A}_c$ and electric potential ϕ_c associated with the applied current must satisfy the following Maxwell equations and some constitutive relations

$$
\begin{aligned}
\mathbf{E}_c &= -\left(\frac{x_0^2}{\lambda_1^2}\frac{\partial \mathbf{A}_c}{\partial t} + \frac{1}{\kappa_1}\nabla\phi_c\right), && (2.28)\\
\mathbf{j}_c &= \sigma \mathbf{E}_c, && (2.29)\\
\nabla\cdot\mathbf{j}_c &= -\frac{\partial \rho_v}{\partial t}, && (2.30)\\
\mathbf{D}_c &= \epsilon \mathbf{E}_c, && (2.31)\\
\nabla\cdot\mathbf{D}_c &= \rho_v, && (2.32)\\
\mathrm{curl}\mathbf{H}_c &= \mathrm{curl}^2\mathbf{A}_c = \mathbf{j}_c + \frac{\partial \mathbf{D}_c}{\partial t}, && (2.33)\\
\nabla\cdot\mathbf{H}_c &= 0, && (2.34)\\
\mathbf{B}_c &= \mu \mathbf{H}, && (2.35)
\end{aligned}
$$

where ρ_v, ϵ and μ are nondimensionalized parameters corresponding to the charge density, the absolute permittivity and permeability of the superconductor, respectively. Since we assumed that the superconductor is a linear and isotropic material, the constitutive relations (2.31) and (2.35) hold with constant permittivity and permeability, respectively. In addition, we assumed that the superconductor is also a nondispersive material, so the constitutive relation (2.29) is stationary and σ is a constant. We will also assume that the applied current varies slowly in time (e.g., length of circuit $<<$ wavelength of current or $\sigma/\omega\epsilon >> 1$, where ω is the frequency of the current) and satisfies a smoothness regularity in time which will be stated later. With the above assumptions, the displacement current $\partial\mathbf{D}_c/\partial t$ can be ignored and equation (2.33) can be approximated as

$$
\begin{aligned}
\mathrm{curl}^2\mathbf{A}_c &= \mathbf{j}_c \quad \text{in } \Omega, && (2.36)\\
\mathrm{curl}\mathbf{A}_c \times \mathbf{n} &= 0 \quad \text{on } \partial\Omega. && (2.37)
\end{aligned}
$$

Here for the purpose of exploring the relationship of $\mathbf{A}_c$ and later of ϕ_c to $\mathbf{j}_c$, we need not consider other external field and so we assumed that there is no external magnetic field applied to the sample. Other boundary conditions will be determined by proper gauge choice.

From the above Maxwell equations, the consequences of the approximation that $\partial\mathbf{D}_c/\partial t = 0$ are $\partial\rho_v/\partial t = 0$, $\rho_v = 0$ and $\nabla\cdot\mathbf{j}_c = 0$. This implies that the Kirchoff's law holds across the sample, i.e., $\int_S \mathbf{j}_c\cdot\mathbf{n}dS = 0$, for any closed surface S. As discussed before, the divergence free condition of the current $\mathbf{j}_c$ gives a continuity condition across the boundary of the domain in the normal direction (see for example [65], sec. 2.8 and sec.3.2.2), namely

$$
[\mathrm{curl}^2\mathbf{A}\cdot\mathbf{n}] = [\mathbf{j}_c\cdot\mathbf{n}] = 0 \qquad \text{on } \partial\Omega, \tag{2.38}
$$

where [.] denotes the jump of the enclosed quantity across $\partial\Omega$.

When we are considering the case that an applied current $\mathbf{j}_a$ is supplied to the boundary of the superconductor sample in the normal state through current leads, we can only specify the applied current on the sample boundary as $\mathbf{j}_a|_{\partial\Omega}$. We find $\tilde{\mathbf{j}}_c(t)$ as an extension of $\mathbf{j}_a|_{\partial\Omega}$ to Ω, for $t\geq 0$, by solving

$$
\begin{aligned}
\nabla\cdot\tilde{\mathbf{j}}_c(t) &= 0 \qquad \text{in } \Omega, && (2.39)\\
\tilde{\mathbf{j}}_c(t) &= \mathbf{j}_a(t) \quad \text{on } \partial\Omega, && (2.40)
\end{aligned}
$$

where $\mathbf{j}_a$ satisfies $\int_{\partial\Omega}\mathbf{j}_a\cdot\mathbf{n}dS = 0$.

For each $t\geq 0$, given a prescribed $\mathbf{j}_a(t)|_{\partial\Omega}\in\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ with $\int_{\partial\Omega}\mathbf{j}_a\cdot\mathbf{n}dS = 0$, the solution of the above problem exists and is unique up to an additive function in $\mathbf{V} := \mathbf{H}_0^1(\mathrm{div}_0;\Omega)$, see [44]. Moreover, we have

$$
||\tilde{\mathbf{j}}_c(t)||_{\mathbf{H}^1(\Omega)/\mathbf{V}} \leq C||\mathbf{j}_a(t)||_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)}, \tag{2.41}
$$

where the constant C is independent of the time t. The following lemma shows that we can find a particular solution.

Lemma 2.2.1 *For $t \geq 0$, there exists a solution $\mathbf{j}_c(t) \in \mathbf{H}^1(\Omega)$ of the BVP (2.39)-(2.40) that satisfies*

$$||\mathbf{j}_c(t)||_{\mathbf{H}^1(\Omega)} = ||\tilde{\mathbf{j}}_c(t)||_{\mathbf{H}^1(\Omega)/\mathbf{V}} := \inf_{\mathbf{v}\in\mathbf{V}} ||(\tilde{\mathbf{j}}_c + \mathbf{v})(t)||_{\mathbf{H}^1(\Omega)} \leq C||\mathbf{j}_a(t)||_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)}, \tag{2.42}$$

where $\tilde{\mathbf{j}}_c$ is obtained from the BVP (2.39)-(2.40).

Proof Let $m = \inf_{\mathbf{v}\in\mathbf{V}} ||\tilde{\mathbf{j}}_c + \mathbf{v}||_{\mathbf{H}^1(\Omega)}$. Let $\{\mathbf{v}_n\} \subset \mathbf{V}$ be a sequence such that $\{\tilde{\mathbf{j}}_c + \mathbf{v}_n\}$ is a minimizing sequence of m, i.e.,

$$||\tilde{\mathbf{j}}_c + \mathbf{v}_n||_{\mathbf{H}^1(\Omega)} \to m \qquad \text{as } n \to \infty.$$

Then by the inequality (2.41) m is bounded, so $\{\tilde{\mathbf{j}}_c + \mathbf{v}_n\}$ is uniformly bounded in $\mathbf{H}^1(\Omega)$. Therefore, there exists a bounded subsequence, again denoted as $\{\tilde{\mathbf{j}}_c + \mathbf{v}_n\}$, converges weakly in $\mathbf{H}^1(\Omega)$ to a limit $\mathbf{j}_c \in \mathbf{H}^1(\Omega)$, i.e.,

$$\{\tilde{\mathbf{j}}_c + \mathbf{v}_n\} \rightharpoonup \mathbf{j}_c \quad \text{in } \mathbf{H}^1(\Omega) \qquad \text{as } n \to \infty. \tag{2.43}$$

Since the linear, bounded mapping $\mathbf{u} \in \mathbf{H}^1(\Omega) \mapsto ||\mathbf{u}||_{\mathbf{H}^1(\Omega)}$ is compact, then by passing to yet another subsequence if necessary, we obtain

$$||\tilde{\mathbf{j}}_c + \mathbf{v}_n||_{\mathbf{H}^1(\Omega)} \to ||\mathbf{j}_c||_{\mathbf{H}^1(\Omega)} = m \qquad \text{as } n \to \infty. \tag{2.44}$$

Since the Hilbert space $\mathbf{H}^1(\Omega)$ is locally uniformly convex, the convergences in (2.43) and (2.44) together imply a strong convergence (see [53])

$$\{\tilde{\mathbf{j}}_c + \mathbf{v}_n\} \to \mathbf{j}_c \quad \text{in } \mathbf{H}^1(\Omega) \qquad \text{as } n \to \infty. \tag{2.45}$$

Now since the Sobolev trace operator $\gamma_0 : \mathbf{H}^1(\Omega) \mapsto \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ defined as $\gamma_0(\mathbf{v}) = \mathbf{v}|_{\partial\Omega}$ in the trace sense, is linear and continuous, the strong convergence (2.45) implies $\gamma_0(\tilde{\mathbf{j}}_c + \mathbf{v}_n) \to \gamma_0(\mathbf{j}_c)$ as $n \to \infty$. A passage to the limit $n \to \infty$ of the sequence $\{\tilde{\mathbf{j}}_c + \mathbf{v}_n\}$ in the BVP (2.39)-(2.40) shows that $\mathbf{j}_c$ also satisfies the BVP (2.39)-(2.40). ∎

By the above lemma, we assume without loss of generality that given a prescribed boundary current $\mathbf{j}_a|_{\partial\Omega}$ we can always extend it to a current $\mathbf{j}_c \in \mathbf{H}^1(\Omega)$ defined in the domian Ω. We can decompose $\mathbf{j}_a|_{\partial\Omega}$ as $\mathbf{j}_a|_{\partial\Omega} = \mathbf{j}_a \cdot \boldsymbol{\tau} + \mathbf{j}_a \cdot \mathbf{n}$, where $\boldsymbol{\tau}$ is the unit tangential vector on $\partial\Omega$. We will see later that we actually only concern about $\mathbf{j}_a \cdot \mathbf{n}|_{\partial\Omega}$. The result of lemma 2.2.1 also implies $\mathbf{j}_c \in \mathbf{H}(\text{div};\Omega)$ and thus $\mathbf{j}_c \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{j}_a \cdot \mathbf{n}|_{\partial\Omega} \in H^{-\frac{1}{2}}(\Omega)$. By Sobolev's extension theorem on trace, there is a lifting function $\Phi \in \mathbf{H}(\text{div};\Omega)$ such that $\Phi \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{j}_c \cdot \mathbf{n}|_{\partial\Omega}$, see [44]. Clearly, $\mathbf{j}_c$ can be such a lifting function. Therefore, from the extension theorem, we obtain that for $t \geq 0$,

$$||\mathbf{j}_c(t)||_{\mathbf{H}(\text{div};\Omega)} \leq ||\mathbf{j}_a(t) \cdot \mathbf{n}||_{H^{-\frac{1}{2}}(\partial\Omega)}. \tag{2.46}$$

By the same reasoning, we can find a divergence free lifting function $\partial\mathbf{j}_c/\partial t \in \mathbf{H}(\text{div}_0;\Omega)$ of $\partial\mathbf{j}_c/\partial t \cdot \mathbf{n}|_{\partial\Omega} = \partial\mathbf{j}_a/\partial t \cdot \mathbf{n}|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$. Moreover, for $t \geq 0$, we have

$$||\frac{\partial\mathbf{j}_c}{\partial t}(t)||_{\mathbf{H}(\text{div};\Omega)} \leq ||\frac{\partial\mathbf{j}_a}{\partial t}(t) \cdot \mathbf{n}||_{H^{-\frac{1}{2}}(\partial\Omega)}. \tag{2.47}$$

Therefore, from the norm estimates (2.46) and (2.47), if $\mathbf{j}_a \cdot \mathbf{n}$, $\partial\mathbf{j}_a/\partial t \cdot \mathbf{n} \in L^q(0,T;H^{-\frac{1}{2}}(\partial\Omega))$, we have $\mathbf{j}_c$, $\partial\mathbf{j}_c/\partial t \in \mathbf{L}^q(0,T;\mathbf{H}(\text{div}_0;\Omega)$, for $q \in [1,\infty]$.

In view of the results obtained above, we may now prescribe the applied current either inside or on the boundary of the superconductor sample. We want to point out that $\mathbf{j}_c$ in Ω only represent a current flowing in a conductor occupying the region Ω, it does not represent the true normal current occurring inside the superconductor in the superconducting state. As we stated before, inside a superconducting superconductor there is zero normal current away from the boundary. In other words, any $\mathbf{j}_c$ as a divergence free lifting function will serve our purpose, albeit it is not unique. Without loss of generality, we assume $\mathbf{j}_c$, $\partial\mathbf{j}_c/\partial t \in \mathbf{L}^q(0,T;\mathbf{H}(\text{div}_0;\Omega))$. From (2.28) and (2.29), we have for almost all $t \in [0,T]$,

$$\mathbf{j}_c(t) \quad = \quad -\sigma\left(\frac{x_0^2}{\lambda_1^2}\frac{\partial\mathbf{A}_c(t)}{\partial t} + \frac{1}{\kappa_1}\nabla\phi_c(t)\right), \tag{2.48}$$

where $\mathbf{A}$ satisfies the BVP (2.36)-(2.37). Since $\mathbf{j}_c \in \mathbf{L}^2(\Omega)$, by applying the Helmholtz decomposition (see [44], section 3 of chapter I) to $\mathbf{j}_c$, we know for a.e. $t \in [0,T]$, $\phi_c(t) \in H^1(\Omega)/\mathbb{R}$ is the only solution of

$$(\nabla\phi_c(t),\, \nabla v) \quad = \quad (-\frac{\kappa_1}{\sigma}\mathbf{j}_c(t), \nabla v) \qquad \forall v \in H^1(\Omega), \tag{2.49}$$

which has the following distributional interpretation

$$\nabla^2\phi_c(t) \quad = \quad -\frac{\kappa_1}{\sigma}\nabla \cdot \mathbf{j}_c(t) = 0 \qquad \text{in } \Omega, \tag{2.50}$$

$$\frac{\partial\phi_c(t)}{\partial n} \quad = \quad -\frac{\kappa_1}{\sigma}\mathbf{j}_c(t) \cdot \mathbf{n} \qquad \text{on } \partial\Omega. \tag{2.51}$$

Here since $\mathbf{j}_c(t) \in \mathbf{H}(\mathrm{div};\Omega)$, $\mathbf{j}_c(t)\cdot\mathbf{n} \in H^{-\frac{1}{2}}(\partial\Omega)$ is well-defined. From the above elliptic BVP, we have for a.e. $t \in [0,T]$, $\phi_c(t) \in H^1(\Omega)$, unique up to an additive function of time only and

$$||\phi_c(t)||_{H^1(\Omega)} \leq C||\mathbf{j}_c(t)\cdot\mathbf{n}||_{H^{-\frac{1}{2}}(\partial\Omega)}, \tag{2.52}$$

where the constant C is independent of the time t. Here in the above estimate, we have used the fact that the semi-norm $|\cdot|_{H^1(\Omega)}$ is always smaller or equal to the quotient norm $||\cdot||_{H^1(\Omega)/\mathbb{R}}$, and that $H^1(\Omega)/\mathbb{R}$ is isomorphic to the space $\{v \in H^1(\Omega);\ \int_\Omega v\, d\Omega = 0\}$ in which by the Poincaré-Friedrichs type inequality we have $||\cdot||_{H^1(\Omega)} \leq C|\cdot|_{H^1(\Omega)}$. From the above estimate, we get $\phi_c \in L^q(0,T;H^1(\Omega))$ if $\mathbf{j}_c \in \mathbf{L}^q(0,T;\mathbf{H}(\mathrm{div}_0;\Omega))$, for $q \in [1,\infty]$. Note that the condition $\int_\Omega v\, d\Omega = 0$ allows us to find a unique solution in $\phi_c \in L^q(0,T;H^1(\Omega))$. Indeed, suppose $p_1 := \phi_c(t)+C_1(t)$ and $p_2 := \phi_c(t)+C_2(t)$ are two solutions in $L^q(0,T;H^1(\Omega))$ which differ in an additive time function, then $\int_\Omega (p_1-p_2)d\Omega = \int_\Omega (C_1(t)-C_2(t))d\Omega = 0$, this gives $C_1(t)-C_2(t) = 0$ for almost all $t \in [0,T]$. Hereafter, we will set $H^1(\Omega)/\mathbb{R} = \{v \in H^1(\Omega);\ \int_\Omega v\, d\Omega = 0\}$.

Analogous to the finding of ϕ_c, we can find, for almost all $t \in [0,T]$, a unique $\partial\phi_c/\partial t(t) \in H^1(\Omega)/\mathbb{R} = \{v \in H^1(\Omega);\ \int_\Omega v\, d\Omega = 0\}$, by solving $\zeta \in H^1(\Omega)/\mathbb{R}$ in

$$\nabla^2\zeta(t) = -\frac{\kappa_1}{\sigma}\nabla\cdot\frac{\partial\mathbf{j}_c(t)}{\partial t} = 0 \qquad \text{in } \Omega, \tag{2.53}$$

$$\frac{\partial\zeta(t)}{\partial n} = -\frac{\kappa_1}{\sigma}\frac{\partial\mathbf{j}_c(t)}{\partial t}\cdot\mathbf{n} \qquad \text{on } \partial\Omega. \tag{2.54}$$

Here since $\partial\mathbf{j}_c(t)/\partial t \in \mathbf{H}(\mathrm{div};\Omega)$, $\partial\mathbf{j}_c(t)/\partial t\cdot\mathbf{n} \in H^{-\frac{1}{2}}(\partial\Omega)$ is well-defined. From the above elliptic BVP, we have for almost all $t \in [0,T]$,

$$||\zeta(t)||_{H^1(\Omega)} \leq C||\frac{\partial\mathbf{j}_c(t)}{\partial t}\cdot\mathbf{n}||_{H^{-\frac{1}{2}}(\partial\Omega)}, \tag{2.55}$$

where the constant C is independent of the time t. Clearly, $\zeta = \partial\phi_c/\partial t$. From the above estimate, we get $\partial\phi_c/\partial t \in L^q(0,T;H^1(\Omega))$ if $\partial\mathbf{j}_c(t)/\partial t \in \mathbf{L}^q(0,T;\mathbf{H}(\mathrm{div}_0;\Omega))$, for $q \in [1,\infty]$.

Summarizing the above regularity information, we have that if $\mathbf{j}_c \in \mathbf{H}^1(0,T;\mathbf{H}(\mathrm{div}_0;\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{H}(\mathrm{div}_0;\Omega))$, then $\phi_c \in H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$. This further implies $\phi_c \in C([0,T];H^1(\Omega))$.

By the uniqueness of the Helmholtz orthogonal decomposition and equation (2.48), for almost all $t \in [0,T]$, $\partial\mathbf{A}_c(t)/\partial t$ is the only solution defined as

$$\frac{x_0^2}{\lambda_1^2}\frac{\partial\mathbf{A}_c(t)}{\partial t} = -\mathbf{j}_c(t) - \frac{\sigma}{\kappa_1}\nabla\phi_c(t). \tag{2.56}$$

Moreover, we have $\partial \mathbf{A}_c(t)/\partial t \in \mathbf{H}_n(\mathrm{div}_0;\Omega)$. However, we will find $\mathbf{A}_c$ and $\partial \mathbf{A}_c/\partial t$ by solving the BVPs (2.36)-(2.37) which will be repeated below to give us some norm estimates of higher regularity.

Remark In general $\partial \mathbf{A}_c(\mathbf{x}, t)/\partial t$ is not identically zero in (2.56) for almost all $t \in [0,T]$, this is because if it was indeed equal to $\mathbf{0}$, then we would have $\mathbf{j}_c(t) = -\frac{\sigma}{\kappa_1}\nabla\phi_c(t)$ for almost all $t \in [0,T]$. This would imply that we must have $\mathrm{curl}\,\mathbf{j}_c(t) = 0$ for almost all $t \in [0,T]$. But in general $\mathrm{curl}\,\mathbf{j}_c(t) \neq 0$ for almost all $t \in [0,T]$. In other words, if we want to keep $\mathbf{j}_c(t) = -\frac{\sigma}{\kappa_1}\nabla\phi_c(t)$ as is needed in the gauge choice discussed in case (IV), then we are bound to consider only a limited set of $\mathbf{j}_a$ which makes $\mathbf{j}_c$ in Ω satisfying $\mathrm{div}\mathbf{j}_c = \mathrm{curl}\,\mathbf{j}_c = 0$ (see [44], Theorem 2.9 which holds when Ω is simply connected) such as a constant $\mathbf{j}_a$ in space. Or we can consider only stationary $\mathbf{j}_a$, in this case $\partial \mathbf{A}_c(\mathbf{x}, t)/\partial t$ simply vanishes in (2.56).

As in the gauge choice case (IV) discussed before, both ϕ_c and $\mathbf{A}_c$, and even the $\mathbf{j}_c$ which is defined inside of Ω, need not have accurate physical meanings or actually reflect a physical situation. What we need is to ensure that they together satisfy the equation (2.36) and (2.48). Our plan is to use these functions as auxiliary variables to fix a gauge from which the continuity boundary condition (2.25) on $\partial\Omega$ can be established and thus the applied current on the boundary is included into the 2B-TDGL model implicitly.

The space $\mathbf{H}_n(\mathrm{div}_0;\Omega)$ prompts us to apply the Coulomb gauge with $\nabla\cdot\mathbf{A}_c = 0$ in Ω and $\mathbf{A}_c\cdot\mathbf{n}|_{\partial\Omega} = 0$ to equation (2.48) and (2.36)-(2.37). Adopting the Coulomb gauge, we can find, for almost all $t \in [0,T]$, a unique $\mathbf{A}_c(t) \in \mathbf{H}_n^1(\mathrm{div};\Omega)$ by solving

$$(\mathrm{curl}\mathbf{A}_c\,, \mathrm{curl}\mathbf{v}) = (\mathbf{j}_c\,,\,\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{H}_n^1(\mathrm{div};\Omega). \tag{2.57}$$

This is equivalent to the following stongly elliptic problem

$$\begin{aligned}
\mathrm{curl}^2\mathbf{A}_c(t) &= \mathbf{j}_c(t) \quad \text{in } \Omega, && (2.58)\\
\nabla\cdot\mathbf{A}_c(t) &= 0 \qquad \text{in } \Omega, && (2.59)\\
\mathbf{A}_c(t)\cdot\mathbf{n} &= 0 \qquad \text{on } \partial\Omega, && (2.60)\\
\mathrm{curl}\mathbf{A}_c(t)\times\mathbf{n} &= 0 \qquad \text{on } \partial\Omega. && (2.61)
\end{aligned}$$

By utilizing the equivalence of norms (2.23), the existence and uniqueness of $\mathbf{A}_c(t)$ as a weak solution is guaranteed by the Lax-Milgram theorem. Moreover, we have the following estimate

$$||\mathbf{A}_c(t)||_{\mathbf{H}^1(\Omega)} \quad \leq \quad C||\mathbf{j}_c(t)||_{\mathbf{L}^2(\Omega)}, \tag{2.62}$$

where the constant C is independent of the time t. From the above estimate, we get $\mathbf{A}_c \in \mathbf{L}^q(0, T; \mathbf{H}^1_n(\mathrm{div}; \Omega))$ if $\mathbf{j}_c \in \mathbf{L}^q(0, T; \mathbf{L}^2(\Omega))$, for $q \in [1, \infty]$.

Analogous to the finding of $\mathbf{A}_c$, for almost all $t \in [0, T]$, we can also find the unique $\partial \mathbf{A}_c(t)/\partial t \in \mathbf{H}^1_n(\mathrm{div}; \Omega)$ by solving $\mathbf{B} \in \mathbf{H}^1_n(\mathrm{div}; \Omega)$ in

$$(\mathrm{curl}\, \mathbf{B}\, , \mathrm{curl} \mathbf{v}) = (\frac{\partial \mathbf{j}_c(t)}{\partial t}\, , \, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{H}^1_n(\mathrm{div}; \Omega). \tag{2.63}$$

which gives the estimate

$$||\mathbf{B}(t)||_{\mathbf{H}^1(\Omega)} \quad \leq \quad C||\frac{\partial \mathbf{j}_c(t)}{\partial t}||_{\mathbf{L}^2(\Omega)}, \tag{2.64}$$

where the constant C is independent of the time t. Clearly, $\mathbf{B} = \partial \mathbf{A}_c/\partial t$. From the above estimate, we get $\partial \mathbf{A}_c/\partial t \in \mathbf{L}^q(0, T; \mathbf{H}^1_n(\mathrm{div}; \Omega))$ if $\partial \mathbf{j}_c/\partial t \in \mathbf{L}^q(0, T; \mathbf{L}^2(\Omega))$, for $q \in [1, \infty]$.

Summarizing up the spaces we see that if $\mathbf{j}_c \in \mathbf{H}^1(0, T; \mathbf{L}^2(\Omega)) \cap \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))$ then $\mathbf{A}_c \in \mathbf{H}^1(0, T; \mathbf{H}^1_n(\mathrm{div}; \Omega)) \cap \mathbf{L}^\infty(0, \mathrm{T}; \mathbf{H}^1_\mathrm{n}(\mathrm{div}; \Omega))$, and this further implies $\mathbf{A}_c \in \mathbf{C}([0, T]; \mathbf{H}^1_n(\mathrm{div}; \Omega))$. Thus we have $\mathbf{A}_c(0) \in \mathbf{H}^1_n(\mathrm{div}; \Omega)$ and so $\nabla \cdot \mathbf{A}_c(0) = 0$.

We will apply the "current gauge" defined by the following gauge transformation to the 2B-TDGL equations. Given $(\tilde{\psi}_1, \tilde{\psi}_2, \mathbf{B}, \Phi)$,

$$(\psi_1, \psi_2, \overline{\mathbf{A}}, \phi) = G_\chi(\tilde{\psi}_1, \tilde{\psi}_2, \mathbf{B}, \Phi),$$

where $\psi_1 = \tilde{\psi}_1 e^{i\kappa_1 \chi}$, $\psi_2 = \tilde{\psi}_2 e^{i\kappa_1 \chi}$, $\overline{\mathbf{A}} = \mathbf{B} + (\lambda_1^2/x_0^2)\nabla\chi$ and $\phi = \Phi - \kappa_1(\partial\chi/\partial t)$, and χ solve the following problem:

$$\begin{aligned}
\kappa_1 \frac{\partial \chi}{\partial t} \quad &= \quad \Phi - \phi_c \qquad \text{in } \Omega, && (2.65)\\
\frac{\lambda_1^2}{x_0^2}\nabla\chi \cdot \mathbf{n} \quad &= \quad -\mathbf{B} \cdot \mathbf{n} \qquad \text{on } \partial\Omega \quad \text{for } t \geq 0, && (2.66)\\
-\frac{\lambda_1^2}{x_0^2}\triangle\chi \quad &= \quad \mathrm{div}\mathbf{B} \qquad \text{in } \Omega \qquad \text{at } t = 0. && (2.67)
\end{aligned}$$

This amounts to saying that under this "current gauge", we have $\phi = \phi_c$ in Ω for $t > 0$,

$\overline{\mathbf{A}}\cdot\mathbf{n} = 0$ on $\partial\Omega$ for $t \geq 0$ and $\mathrm{div}\overline{\mathbf{A}} = 0$ in Ω at $t = 0$. From equation (2.48), we have

$$
\begin{aligned}
-\frac{\sigma}{\kappa_1}\nabla\phi_c &= \mathbf{j}_c + \sigma\frac{x_0^2}{\lambda_1^2}\frac{\partial \mathbf{A}_c}{\partial t} \qquad \text{in } \Omega. && (2.68)\\
&= \mathrm{curl}\mathbf{H}_c + \sigma\frac{x_0^2}{\lambda_1^2}\frac{\partial \mathbf{A}_c}{\partial t} \qquad \text{in } \Omega. && (2.69)\\
&= \mathrm{curl}^2\mathbf{A}_c + \sigma\frac{x_0^2}{\lambda_1^2}\frac{\partial \mathbf{A}_c}{\partial t} \qquad \text{in } \Omega. && (2.70)
\end{aligned}
$$

Under this "current gauge", from the above equations, we can have various forms to formulate the 2B-TDGL equations, depending on what terms are substituted into the equations. One form is to substitute $\nabla\phi_c$ by equation (2.70), which gives

$$
\begin{aligned}
&\left(\frac{\partial\psi_1}{\partial t} + i\,\phi_c\psi_1\right) + \left(|\psi_1|^2 - \mathcal{T}_1\right)\psi_1 + \left(-i\frac{\xi_1}{x_0}\nabla - \frac{x_0}{\lambda_1}(\mathbf{A}+\mathbf{A}_c)\right)^2\psi_1\\
&\quad + \eta\psi_2 + \eta_1\frac{\xi_1}{\xi_2}\frac{1}{\nu}\left(-i\frac{\xi_2}{x_0}\nabla - \nu\frac{x_0}{\lambda_2}(\mathbf{A}+\mathbf{A}_c)\right)^2\psi_2 = 0 \qquad \text{in}\,\Omega\times(0,\mathrm{T}), \quad (2.71)
\end{aligned}
$$

$$
\begin{aligned}
&\Gamma\left(\frac{\partial\psi_2}{\partial t} + i\,\phi_c\psi_2\right) + \left(|\psi_2|^2 - \mathcal{T}_2\right)\psi_2 + \left(-i\frac{\xi_2}{x_0}\nabla - \nu\frac{x_0}{\lambda_2}(\mathbf{A}+\mathbf{A}_c)\right)^2\psi_2\\
&\quad + \eta\nu^2\psi_1 + \eta_1\frac{\xi_2}{\xi_1}\nu\left(-i\frac{\xi_1}{x_0}\nabla - \frac{x_0}{\lambda_1}(\mathbf{A}+\mathbf{A}_c)\right)^2\psi_1 = 0 \qquad \text{in}\,\Omega\times(0,\mathrm{T}), \quad (2.72)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{curl}^2\mathbf{A} &= \mathrm{curl}\,\mathbf{H}_\mathrm{e} - \sigma\frac{\mathrm{x}_0^2}{\lambda_1^2}\frac{\partial\mathbf{A}}{\partial \mathrm{t}}\\
&+ i\frac{1}{2\kappa_1}\left(\psi_1\nabla\psi_1^* - \psi_1^*\nabla\psi_1\right) - \frac{x_0^2}{\lambda_1^2}|\psi_1|^2(\mathbf{A}+\mathbf{A}_c)\\
&+ i\frac{1}{\nu}\frac{1}{2\kappa_2}\left(\psi_2\nabla\psi_2^* - \psi_2^*\nabla\psi_2\right) - \frac{x_0^2}{\lambda_2^2}|\psi_2|^2(\mathbf{A}+\mathbf{A}_c)\\
&- \eta_1 i\frac{1}{2}\frac{\xi_1}{\lambda_2}(\psi_1^*\nabla\psi_2 - \psi_1\nabla\psi_2^* + \psi_2^*\nabla\psi_1 - \psi_2\nabla\psi_1^*)\\
&- \eta_1\frac{x_0^2}{\lambda_1\lambda_2}(\mathbf{A}+\mathbf{A}_c)(\psi_1\psi_2^* + \psi_2\psi_1^*) \qquad \text{in}\,\Omega\times(0,\mathrm{T}), \quad (2.73)
\end{aligned}
$$

here $\mathcal{T}_1$ and $\mathcal{T}_2$ are defined in (2.17), and $\mathbf{A} := \overline{\mathbf{A}} - \mathbf{A}_c$. Since the normal component of the supercurrent $\mathbf{j}_s$ vanishes on the boundary $\partial\Omega$, by trace extension, equation (2.73) yields that for almost all $t \in [0, T]$,

$$
\mathrm{curl}^2\mathbf{A}\cdot\mathbf{n} = (-\sigma\frac{x_0^2}{\lambda_1^2}\frac{\partial\mathbf{A}}{\partial t} + \mathrm{curl}\,\mathbf{H}_e)\cdot\mathbf{n} \qquad \text{on } \partial\Omega.
$$

Note that later in Theorem (3.2.26) we will show that $\partial(\text{div}\mathbf{A})/\partial t \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))$, and we already knew that $\partial \mathbf{A}_c/\partial t \in \mathbf{L}^q(0,T;\mathbf{H}^1_n(\text{div};\Omega))$. As a result $\partial \mathbf{A}/\partial t$, $\partial\overline{\mathbf{A}}/\partial t \in \mathbf{L}^2(0,T;\mathbf{H}(\text{div};\Omega))$ and thus $(\partial \mathbf{A}/\partial t)\cdot\mathbf{n}$, $(\partial\overline{\mathbf{A}}/\partial t)\cdot\mathbf{n} \in L^2(0,T;H^{-\frac{1}{2}}(\partial\Omega))$. By the construction of $\mathbf{j}_c$, we have $\text{curl}^2\mathbf{A}_c = \mathbf{j}_c$ and $\mathbf{j}_c = \mathbf{j}_a$ on $\partial\Omega$ (by (2.40)), this gives

$$
\begin{aligned}
\text{curl}^2\overline{\mathbf{A}}\cdot\mathbf{n} &= (\text{curl}^2\mathbf{A}_c - \sigma\frac{x_0^2}{\lambda_1^2}\frac{\partial\mathbf{A}}{\partial t} + \text{curl}\,\mathbf{H}_e)\cdot\mathbf{n} \qquad \text{on } \partial\Omega \\
&= (\text{curl}^2\mathbf{A}_c + \text{curl}\,\mathbf{H}_e)\cdot\mathbf{n} \qquad \text{on } \partial\Omega, \\
&= (\mathbf{j}_c + \text{curl}\,\mathbf{H}_e)\cdot\mathbf{n} \qquad \text{on } \partial\Omega, \\
&= (\mathbf{j}_a + \text{curl}\,\mathbf{H}_e)\cdot\mathbf{n} \qquad \text{on } \partial\Omega, \qquad (2.74)
\end{aligned}
$$

where in the first equation the time derivative term vanishes because $(\overline{\mathbf{A}}-\mathbf{A}_c)\cdot\mathbf{n} = 0$ on $\partial\Omega$ for $t \geq 0$. Therefore the continuity B.C. (2.24) (and thus (2.25)) is satisfied, since from the outside of Ω we also have $(\mathbf{j}_a + \text{curl}\,\mathbf{H}_e)\cdot\mathbf{n}$ on $\partial\Omega$. Thus the continuity B.C. is automatically satisfied and needed not be included explicitly into the model as a boundary condition.

The complete set of boundary and initial conditions now becomes

$$
\begin{aligned}
\left(-i\frac{\xi_1}{x_0}\nabla\psi_1 - i\eta_1\frac{1}{\nu}\frac{\xi_2}{x_0}\nabla\psi_2\right)\cdot\mathbf{n} &= i\gamma_1\frac{\xi_1}{x_0}\psi_1 \qquad \text{on } \partial\Omega\times(0,\text{T}), && (2.75)\\
\left(-i\frac{\xi_2}{x_0}\nabla\psi_2 - i\eta_1\nu\frac{\xi_1}{x_0}\nabla\psi_1\right)\cdot\mathbf{n} &= i\gamma_2\frac{\xi_2}{x_0}\psi_2 \qquad \text{on } \partial\Omega\times(0,\text{T}), && (2.76)\\
\text{curl}\mathbf{A}\times\mathbf{n} &= \mathbf{H}_e\times\mathbf{n} \quad \text{on } \partial\Omega\times(0,\text{T}), && (2.77)\\
\mathbf{A}\cdot\mathbf{n} &= 0 \quad \text{on } \partial\Omega\times(0,\text{T}), && (2.78)\\
\psi_1(\mathbf{x},0) &= \psi_{10}(\mathbf{x}) \quad \text{in}\,\Omega, && (2.79)\\
\psi_2(\mathbf{x},0) &= \psi_{20}(\mathbf{x}) \quad \text{in}\,\Omega, && (2.80)\\
\mathbf{A}(\mathbf{x},0) &= \overline{\mathbf{A}}(\mathbf{x},0) - \mathbf{A}_c(\mathbf{x},0) \quad \text{in}\,\Omega, && (2.81)\\
\nabla\cdot\mathbf{A}(\mathbf{x},0) &= 0 \qquad \text{in}\,\Omega. && (2.82)
\end{aligned}
$$

Note that by the B.C. (2.61), $\text{curl}\mathbf{A}_c\times\mathbf{n} = 0$ on $\partial\Omega$. The B.C. (2.78) is from the gauge which gives $\overline{\mathbf{A}}\cdot\mathbf{n} = 0$, and from $\mathbf{A}_c\cdot\mathbf{n} = 0$ on $\partial\Omega$. The last I.C. is also from the gauge $\nabla\cdot\overline{\mathbf{A}}(\mathbf{x},0) = 0$ and by $\nabla\cdot\mathbf{A}_c(\mathbf{x},0) = 0$. This condition is required in later proof. Hereafter, we will denote $\overline{\mathbf{A}}(0,\mathbf{x}) = \mathbf{A}_0$. We will also denote the above initial-boundary-value problem (2.71)-(2.73) and (2.75)-(2.82) corresponding to the first approach of adding Type-A current as IBVP1.

We notice that in (2.74) $\partial \mathbf{A}/\partial t$ actually plays no role in satisfying the continuity B.C., since it's normal component always vanishes on the boundary. So it is reasonable to exclude this term in the addition of Type-A current to the model. We propose our second approach below which is equivalent to the first approach.

Consider again that $\mathbf{j}_a|_{\partial\Omega}$ is specified only on the boundary. We find a unique $\phi_a \in H^1(\Omega)/\mathbb{R} = \{v \in H^1(\Omega);\ \int_\Omega v\, d\Omega = 0\}$ by solving the following BVP

$$\begin{aligned} \nabla^2 \phi_a(t) &= 0 \qquad \text{in } \Omega, & (2.83)\\ \frac{\partial \phi_a(t)}{\partial n} &= -\frac{\kappa_1}{\sigma}\, \mathbf{j}_a(t)\cdot \mathbf{n} \qquad \text{on } \partial\Omega. & (2.84) \end{aligned}$$

This BVP is exactly the same as the BVP (2.50)-(2.51) which results from the Helmholtz decomposition (2.48) for $\mathbf{j}_c$, but now we skip the process to find $\mathbf{j}_c$. We also skip the process to find $\partial \mathbf{A}_c/\partial t$. As in (2.65)-(2.67), we use the same "current gauge". However now since we don't have the knowledge of $\mathbf{j}_c$ and $\partial \mathbf{A}_c/\partial t$, we don't have the relationship (2.68)-(2.70). This means that even if we have $\mathbf{j}_c$ ready, $\nabla\phi_a \neq \mathbf{j}_c$ in Ω and thus this "current gauge" is different from the gauge discussed in case (IV). In our second approach, we simply replace ϕ in the 2B-TDGL for ψ_i by ϕ_a and $\nabla\phi$ in the 2B-TDGL equation for $\mathbf{A}$ by $\nabla\phi_a$. The gauged 2B-TDGL equations now become

$$\begin{aligned} &\left(\frac{\partial \psi_1}{\partial t} + i\,\phi_a \psi_1\right) + \left(|\psi_1|^2 - \mathcal{T}_1\right)\psi_1 + \left(-i\frac{\xi_1}{x_0}\nabla - \frac{x_0}{\lambda_1}\overline{\mathbf{A}}\right)^2 \psi_1 \\ &\quad + \eta\psi_2 + \eta_1 \frac{\xi_1}{\xi_2}\frac{1}{\nu}\left(-i\frac{\xi_2}{x_0}\nabla - \nu\frac{x_0}{\lambda_2}\overline{\mathbf{A}}\right)^2 \psi_2 = 0 \qquad \text{in } \Omega \times (0,\mathrm{T}), \quad (2.85) \end{aligned}$$

$$\begin{aligned} &\Gamma\left(\frac{\partial \psi_2}{\partial t} + i\,\phi_a \psi_2\right) + \left(|\psi_2|^2 - \mathcal{T}_2\right)\psi_2 + \left(-i\frac{\xi_2}{x_0}\nabla - \nu\frac{x_0}{\lambda_2}\overline{\mathbf{A}}\right)^2 \psi_2 \\ &\quad + \eta\nu^2\psi_1 + \eta_1 \frac{\xi_2}{\xi_1}\nu\left(-i\frac{\xi_1}{x_0}\nabla - \frac{x_0}{\lambda_1}\overline{\mathbf{A}}\right)^2 \psi_1 = 0 \qquad \text{in } \Omega \times (0,\mathrm{T}), \quad (2.86) \end{aligned}$$

$$
\begin{aligned}
\operatorname{curl}^2\overline{\mathbf{A}} &= \operatorname{curl}\mathbf{H}_{\mathrm{e}} - \sigma\frac{\mathrm{x}_0^2}{\lambda_1^2}\frac{\partial\overline{\mathbf{A}}}{\partial \mathrm{t}} - \frac{\sigma}{\kappa_1}\nabla\phi_{\mathrm{a}} \\
&+ i\frac{1}{2\kappa_1}\left(\psi_1\nabla\psi_1^* - \psi_1^*\nabla\psi_1\right) - \frac{x_0^2}{\lambda_1^2}|\psi_1|^2\overline{\mathbf{A}} \\
&+ i\frac{1}{\nu}\frac{1}{2\kappa_2}\left(\psi_2\nabla\psi_2^* - \psi_2^*\nabla\psi_2\right) - \frac{x_0^2}{\lambda_2^2}|\psi_2|^2\overline{\mathbf{A}} \\
&- \eta_1 i\frac{1}{2}\frac{\xi_1}{\lambda_2}(\psi_1^*\nabla\psi_2 - \psi_1\nabla\psi_2^* + \psi_2^*\nabla\psi_1 - \psi_2\nabla\psi_1^*) \\
&- \eta_1\frac{x_0^2}{\lambda_1\lambda_2}\overline{\mathbf{A}}(\psi_1\psi_2^* + \psi_2\psi_1^*) \qquad \text{in}\,\Omega\times(0,\mathrm{T}). \qquad (2.87)
\end{aligned}
$$

It is easy to check that the continuity B.C. is satisfied, indeed

$$
\begin{aligned}
-\sigma\left(\frac{x_0^2}{\lambda_1^2}\frac{\partial\overline{\mathbf{A}}}{\partial t} + \frac{1}{\kappa_1}\nabla\phi\right)\cdot\mathbf{n} &= -\frac{\sigma}{\kappa_1}\nabla\phi_a\cdot\mathbf{n} \\
&= \mathbf{j}_a\cdot\mathbf{n} \qquad \text{on } \partial\Omega,
\end{aligned}
$$

where the last equality is obtained from (2.84). The benefit of using this approach is that it is simpler than the previous method in which we have to evaluate $\mathbf{A}_c$ in addition to ϕ_c, and to do that we have to find a non-unique $\mathbf{j}_c$ in Ω first. In this approach, we only need ϕ_a which can be found once $\mathbf{j}_a$ is given on the boundary $\partial\Omega$ and is unique. So this is a clean way to add a Type-A current to the model. To add a Type-B current, as we have already done so, we just add curl$\mathbf{H}_e$ to the equation for $\mathbf{A}$ as presented above.

The complete set of boundary and initial conditions now becomes

$$
\begin{aligned}
\left(-i\frac{\xi_1}{x_0}\nabla\psi_1 - i\eta_1\frac{1}{\nu}\frac{\xi_2}{x_0}\nabla\psi_2\right)\cdot\mathbf{n} &= i\gamma_1\frac{\xi_1}{x_0}\psi_1 \qquad \text{on } \partial\Omega\times(0,\mathrm{T}), && (2.88)\\
\left(-i\frac{\xi_2}{x_0}\nabla\psi_2 - i\eta_1\nu\frac{\xi_1}{x_0}\nabla\psi_1\right)\cdot\mathbf{n} &= i\gamma_2\frac{\xi_2}{x_0}\psi_2 \qquad \text{on } \partial\Omega\times(0,\mathrm{T}), && (2.89)\\
\operatorname{curl}\overline{\mathbf{A}}\times\mathbf{n} &= \mathbf{H}_e\times\mathbf{n} \quad \text{on } \partial\Omega\times(0,\mathrm{T}), && (2.90)\\
\overline{\mathbf{A}}\cdot\mathbf{n} &= 0 \quad \text{on } \partial\Omega\times(0,\mathrm{T}), && (2.91)\\
\psi_1(\mathbf{x},0) &= \psi_{10}(\mathbf{x}) \quad \text{in}\,\Omega, && (2.92)\\
\psi_2(\mathbf{x},0) &= \psi_{20}(\mathbf{x}) \quad \text{in}\,\Omega, && (2.93)\\
\overline{\mathbf{A}}(\mathbf{x},0) &= \mathbf{A}_0 \quad \text{in}\,\Omega, && (2.94)\\
\nabla\cdot\overline{\mathbf{A}}(\mathbf{x},0) &= 0 \qquad \text{in}\,\Omega. && (2.95)
\end{aligned}
$$

Hereafter, we will denote the above initial-boundary-value problem (2.85)-(2.87) and (2.88)-(2.95) corresponding to the second approach of adding Type-A current as IBVP2.

Compared to the first approach IBVP1, the second approach IBVP2 is more suitable for numerical computation. However, in the theoretical analysis of the existence of a weak solution to the 2B-TDGL equations with a Type-A current involved, it is more convenient to work directly on IBVP1. As a result, we will use IBVP1 in our existence and uniqueness analysis and in our finite-element analysis, but in the computational section, we will use IBVP2 to obtain and present our computational results.

CHAPTER 3

Analysis

In this chapter, we focus on the mathematical analysis of the solutions of the 2B-TDGL equations gauged with the "current gauge" in the case where we assume that the superconductor in consideration has null gradient coupling effect, i.e., with $\eta_1 = 0$. Future work will investigate the case where $\eta_1 \neq 0$. We will work on the initial-boundary-value problem IBVP1 and prove that there exists a unique weak solution to the 2B-TDGL equations under the "current gauge", by a method used by Du in his paper [2]. We will also proof that the solution is uniformly bounded in $\Omega \times [0, T]$.

We define

$$\mathcal{V} = \mathcal{L}^\infty(0, T;\ \mathcal{H}^1(\Omega)) \cap \mathcal{H}^1(0, T; \mathcal{L}^2(\Omega)), \tag{3.1}$$

$$\mathbf{V} = \mathbf{L}^\infty(0, T; \mathbf{H}^1_n(\Omega)) \cap \mathbf{H}^1(0, T; \mathbf{L}^2(\Omega)), \tag{3.2}$$

and denote the L^2 inner products $(f,\ g) = \int_\Omega f g^* d\Omega$; $(f,\ g)_{\partial\Omega} = \int_{\partial\Omega} f g^* dS$.

We assume that the following regularity assumptions hold throughout our work

RA: Assume that $\mathbf{A}_0 := \overline{\mathbf{A}}(0, \mathbf{x}) \in \mathbf{H}^1_n(\mathrm{div}; \Omega) \cap \mathbf{H}^2(\Omega)$ and for $i = 1, 2$, $\gamma_i = \gamma_i(\mathbf{x}) \geq 0$ for $\mathbf{x} \in \partial\Omega$ and $\gamma_i \in L^\infty(\partial\Omega)$, $\psi_{i0} := \psi_i(\mathbf{x},\ 0) = \psi_i^\epsilon(\mathbf{x},\ 0) \in \mathcal{H}^2(\Omega)$ and $|\psi_{i0}| \leq a$, where a is defined in Theorem 3.2.16. For the external magnetic field, we assume $\mathbf{H}_e \in \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega)) \cap \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega)) \cap \mathbf{H}^1(0, T; \mathbf{H}(\mathrm{curl}; \Omega))$. For the Type-A applied current, we assume $\mathbf{j}_a|_{\partial\Omega} \in H^1(0, T; H^{\frac{1}{2}}(\partial\Omega)) \cap L^\infty(0, T; H^{\frac{1}{2}}(\partial\Omega))$.

Remark From the norm estimates (2.46) and (2.47), we can see that $\mathbf{j}_a \cdot \mathbf{n}|_{\partial\Omega} \in H^1(0, T; H^{-\frac{1}{2}}(\partial\Omega)) \cap L^\infty(0, T; H^{-\frac{1}{2}}(\partial\Omega))$ implies $\mathbf{j}_c \in \mathbf{H}^1(0, T; \mathbf{H}(\mathrm{div}_0; \Omega)) \cap \mathbf{L}^\infty(0, T; \mathbf{H}(\mathrm{div}_0; \Omega)) \cap \mathbf{C}([0, T]; \mathbf{H}(\mathrm{div}_0; \Omega))$. From this, the norm estimates (2.52) and (2.55) imply that $\phi_c \in H^1(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \cap C([0, T]; H^1(\Omega))$; and also the norm estimates (2.62) and (2.64) imply that $\mathbf{A}_c \in \mathbf{H}^1(0,T;\mathbf{H}_n^1(\mathrm{div};\Omega)) \cap \mathbf{L}^\infty(0,\mathrm{T};\mathbf{H}_\mathrm{n}^1(\mathrm{div};\Omega)) \cap \mathbf{C}([0,\mathrm{T}];\mathbf{H}_\mathrm{n}^1(\mathrm{div};\Omega))$. The last three relations implies $\mathbf{j}_\mathrm{c}(\mathbf{x},0) \in \mathbf{H}(\mathrm{div}_0;\Omega)$, $\phi_c(\mathbf{x},0) \in H^1(\Omega)$ and $\mathbf{A}_c(\mathbf{x},0) \in \mathbf{H}_n^1(\mathrm{div};\Omega)$, respectively. The regularity of $\mathbf{H}_e$ implies $\mathbf{H}_e(\mathbf{x},0) \in \mathbf{L}^2(\Omega)$. In the course of the development of our mathematical analysis, we will show why and how we impose these regularities.

3.1 Weak Formulations

We want to find a unique weak solution $(\psi_1,\psi_2,\overline{\mathbf{A}}) = (\psi_1,\psi_2,\mathbf{A}+\mathbf{A}_c) \in \mathcal{V}\times\mathcal{V}\times\mathbf{V}$. However, since $\mathbf{A}_c$ is known, we can instead find $(\psi_1,\psi_2,\mathbf{A})$ in the following weak formulation corresponding to the problem IBVP1 (2.71)-(2.73) and (2.75)-(2.82) with the "current gauge" applied. After we have obtained all the results for the case of $(\psi_1,\psi_2,\mathbf{A})$, we will come back to investigate the case for $(\psi_1,\psi_2,\overline{\mathbf{A}})$.

Problem (WP): Under the regularity assumption **RA**, seek $(\psi_1,\psi_2,\mathbf{A}) = (\psi_1,\psi_2,\overline{\mathbf{A}} - \mathbf{A}_c) \in \mathcal{V}\times\mathcal{V}\times\mathbf{V}$, where $\mathbf{A}_c$ is given (see the space requirement below), by finding $(\psi_1,\psi_2,\mathbf{A}) \in \mathcal{V}\times\mathcal{V}\times\mathbf{V}$ such that for a.e. $t \in [0,T]$,

$$\begin{aligned}
&\frac{d}{dt}(\psi_1,\tilde{\psi}) + (i\phi_c\psi_1,\tilde{\psi}) + \Big((|\psi_1|^2 - \mathcal{T}_1)\psi_1,\tilde{\psi}\Big)\\
&\quad + \left(-i\frac{\xi_1}{x_0}\nabla\psi_1 - \frac{x_0}{\lambda_1}(\mathbf{A}+\mathbf{A}_c)\psi_1,\ -i\frac{\xi_1}{x_0}\nabla\tilde{\psi} - \frac{x_0}{\lambda_1}(\mathbf{A}+\mathbf{A}_\mathbf{c})\tilde{\psi}\right) + \eta(\psi_2,\tilde{\psi})\\
&\quad + \eta_1\frac{\xi_1}{\xi_2}\frac{1}{\nu}\left(-i\frac{\xi_2}{x_0}\nabla\psi_2 - \nu\frac{x_0}{\lambda_2}(\mathbf{A}+\mathbf{A}_c)\psi_2,\ -i\frac{\xi_2}{x_0}\nabla\tilde{\psi} - \nu\frac{x_0}{\lambda_2}(\mathbf{A}+\mathbf{A}_c)\tilde{\psi}\right)\\
&\quad + \gamma_1\frac{\xi_1^2}{x_0^2}\Big(\psi_1,\tilde{\psi}\Big)_{\partial\Omega} = 0 \qquad \forall\,\tilde{\psi} \in \mathcal{H}^1(\Omega),
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&\Gamma\frac{d}{dt}(\psi_2,\tilde{\psi}) + \Gamma(i\phi_c\psi_2,\tilde{\psi}) + \Big((|\psi_2|^2 - \mathcal{T}_2)\psi_2,\tilde{\psi}\Big)\\
&\quad + \left(-i\frac{\xi_2}{x_0}\nabla\psi_2 - \nu\frac{x_0}{\lambda_2}(\mathbf{A}+\mathbf{A}_c)\psi_2,\ -i\frac{\xi_2}{x_0}\nabla\tilde{\psi} - \nu\frac{x_0}{\lambda_2}(\mathbf{A}+\mathbf{A}_c)\tilde{\psi}\right) + \eta\nu^2(\psi_1,\tilde{\psi})\\
&\quad + \eta_1\frac{\xi_2}{\xi_1}\nu\left(-i\frac{\xi_1}{x_0}\nabla\psi_1 - \frac{x_0}{\lambda_1}(\mathbf{A}+\mathbf{A}_c)\psi_1,\ -i\frac{\xi_1}{x_0}\nabla\tilde{\psi} - \frac{x_0}{\lambda_1}(\mathbf{A}+\mathbf{A}_c)\tilde{\psi}\right)\\
&\quad + \gamma_2\frac{\xi_2^2}{x_0^2}\Big(\psi_2,\tilde{\psi}\Big)_{\partial\Omega} = 0 \qquad \forall\,\tilde{\psi} \in \mathcal{H}^1(\Omega),
\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&(\mathrm{curl}\mathbf{A},\, \mathrm{curl}\tilde{\mathbf{A}}) + \sigma \frac{x_0^2}{\lambda_1^2}\frac{d}{dt}\left(\mathbf{A},\, \tilde{\mathbf{A}}\right) \\
&\quad + \Re\left(i\frac{1}{\kappa_1}\nabla\psi_1,\, \psi_1\tilde{\mathbf{A}}\right) + \frac{x_0^2}{\lambda_1^2}\left(|\psi_1|^2(\mathbf{A}+\mathbf{A}_c),\, \tilde{\mathbf{A}}\right) \\
&\quad + \Re\left(i\frac{1}{\nu}\frac{1}{\kappa_2}\nabla\psi_2,\, \psi_2\tilde{\mathbf{A}}\right) + \frac{x_0^2}{\lambda_2^2}\left(|\psi_2|^2(\mathbf{A}+\mathbf{A}_c),\, \tilde{\mathbf{A}}\right) \\
&\quad + \eta_1\left(\Re\left(i\frac{\xi_1}{\lambda_2}\nabla\psi_2,\, \psi_1\tilde{\mathbf{A}}\right) + \Re\left(i\frac{\xi_1}{\lambda_2}\nabla\psi_1,\, \psi_2\tilde{\mathbf{A}}\right)\right) \\
&\quad + \eta_1\frac{x_0^2}{\lambda_1\lambda_2}\left((\psi_1\psi_2^* + \psi_2\psi_1^*)(\mathbf{A}+\mathbf{A}_c),\, \tilde{\mathbf{A}}\right) = (\mathbf{H}_e,\, \mathrm{curl}\tilde{\mathbf{A}}) \quad \forall\, \tilde{\mathbf{A}} \in \mathbf{H}_n^1(\Omega), \qquad (3.5)
\end{aligned}
$$

with the initial conditions

$$
\begin{aligned}
\psi_1(\mathbf{x},0) &= \psi_{10} \in \mathcal{H}^2(\Omega) && (3.6)\\
\psi_2(\mathbf{x},0) &= \psi_{20} \in \mathcal{H}^2(\Omega) && (3.7)\\
\mathbf{A}(\mathbf{x},0) &= \mathbf{A}_0 - \mathbf{A}_c(\mathbf{x},0) \quad \text{with } \mathbf{A}_0 \in \mathbf{H}^2(\Omega), && (3.8)
\end{aligned}
$$

where $\mathbf{A}_0 = \overline{\mathbf{A}}(0,\mathbf{x})$.

The problem (WP) is not coercive in the current gauge because the curl bilinear form in the weak form for $\mathbf{A}^\epsilon$ is not coercive; this theoretical difficulty can be avoided by working on the following modified problem in which a regularization term in bilinear form of the divergence operator is added.

Problem $(\mathbf{WP}^\epsilon)$: Under the regularity assumption **RA** and for arbitrary $0 < \epsilon \leq 1$, seek $(\psi_1^\epsilon, \psi_2^\epsilon, \mathbf{A}^\epsilon) = (\psi_1^\epsilon, \psi_2^\epsilon, \overline{\mathbf{A}}^\epsilon - \mathbf{A}_c) \in \mathcal{V}\times\mathcal{V}\times\mathbf{V}$, where $\mathbf{A}_c$ is given as in Problem (WP), by finding $(\psi_1^\epsilon, \psi_2^\epsilon, \mathbf{A}^\epsilon) \in \mathcal{V}\times\mathcal{V}\times\mathbf{V}$ such that for a.e. $t \in [0,T]$,

$$
\begin{aligned}
&\frac{d}{dt}(\psi_1^\epsilon, \tilde{\psi}) + (i\phi_c\psi_1^\epsilon, \tilde{\psi}) + \left(\left(|\psi_1^\epsilon|^2 - \mathcal{T}_1\right)\psi_1^\epsilon,\, \tilde{\psi}\right) \\
&\quad + \left(-i\frac{\xi_1}{x_0}\nabla\psi_1^\epsilon - \frac{x_0}{\lambda_1}(\mathbf{A}^\epsilon+\mathbf{A}_c)\psi_1^\epsilon,\; -i\frac{\xi_1}{x_0}\nabla\tilde{\psi} - \frac{x_0}{\lambda_1}(\mathbf{A}^\epsilon+\mathbf{A}_c)\tilde{\psi}\right) \\
&\quad + \eta(\psi_2^\epsilon, \tilde{\psi}) + \gamma_1\frac{\xi_1^2}{x_0^2}\left(\psi_1^\epsilon, \tilde{\psi}\right)_{\partial\Omega} = 0 \qquad \forall\, \tilde{\psi} \in \mathcal{H}^1(\Omega), \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
&\Gamma\frac{d}{dt}(\psi_2^\epsilon, \tilde{\psi}) + \Gamma(i\phi_c\psi_2^\epsilon, \tilde{\psi}) + \left(\left(|\psi_2^\epsilon|^2 - \mathcal{T}_2\right)\psi_2^\epsilon,\, \tilde{\psi}\right) \\
&\quad + \left(-i\frac{\xi_2}{x_0}\nabla\psi_2^\epsilon - \nu\frac{x_0}{\lambda_2}(\mathbf{A}^\epsilon+\mathbf{A}_c)\psi_2^\epsilon,\; -i\frac{\xi_2}{x_0}\nabla\tilde{\psi} - \nu\frac{x_0}{\lambda_2}(\mathbf{A}^\epsilon+\mathbf{A}_c)\tilde{\psi}\right) \\
&\quad + \eta\nu^2(\psi_1^\epsilon, \tilde{\psi}) + \gamma_2\frac{\xi_2^2}{x_0^2}\left(\psi_2^\epsilon, \tilde{\psi}\right)_{\partial\Omega} = 0 \qquad \forall\, \tilde{\psi} \in \mathcal{H}^1(\Omega), \qquad (3.10)
\end{aligned}
$$

$$\sigma \frac{x_0^2}{\lambda_1^2}\frac{d}{dt}\left(\mathbf{A}^\epsilon,\ \tilde{\mathbf{A}}\right) + \epsilon(\mathrm{div}\mathbf{A}^\epsilon, \mathrm{div}\tilde{\mathbf{A}}) + (\mathrm{curl}\mathbf{A}^\epsilon,\ \mathrm{curl}\tilde{\mathbf{A}})$$
$$+\ \Re\left(i\frac{1}{\kappa_1}\nabla\psi_1,\ \psi_1\tilde{\mathbf{A}}\right) + \frac{x_0^2}{\lambda_1^2}\left(|\psi_1|^2(\mathbf{A}^\epsilon + \mathbf{A}_c),\ \tilde{\mathbf{A}}\right) \tag{3.11}$$
$$+\ \Re\left(i\frac{1}{\nu}\frac{1}{\kappa_2}\nabla\psi_2,\ \psi_2\tilde{\mathbf{A}}\right) + \frac{x_0^2}{\lambda_2^2}\left(|\psi_2|^2(\mathbf{A}^\epsilon + \mathbf{A}_c),\ \tilde{\mathbf{A}}\right) = (\mathbf{H}_e,\ \mathrm{curl}\tilde{\mathbf{A}})\ \forall\tilde{\mathbf{A}} \in \mathbf{H}_n^1(\Omega),$$

The initial conditions are the same as the original problem (WP), but here for the proof of existence and uniqueness of the solutions to this modified problem, we only need $\psi_{10},\ \psi_{20} \in \mathcal{H}^1(\Omega)$. A H^2 regularities for the initial conditions will be needed when we try to seek for higher regularities for the time derivatives of the solutions. The initial conditions are

$$\psi_1^\epsilon(\mathbf{x}, 0) = \psi_{10} \in \mathcal{H}^1(\Omega) \tag{3.12}$$
$$\psi_2^\epsilon(\mathbf{x}, 0) = \psi_{20} \in \mathcal{H}^1(\Omega) \tag{3.13}$$
$$\mathbf{A}^\epsilon(\mathbf{x}, 0) = \mathbf{A}_0 - \mathbf{A}_c(\mathbf{x}, 0) \in \mathbf{H}^1(\Omega). \tag{3.14}$$

We will also work with $(\psi_1^\epsilon, \psi_2^\epsilon, \mathbf{A}^\epsilon)$ instead of $(\psi_1^\epsilon, \psi_2^\epsilon, \overline{\mathbf{A}}^\epsilon)$, and come back to the later case after all the results have been obtained for the first case.

We will use the standard techniques used by Lions in [46] and Temam in [45], namely, Galerkin finite dimensional approximations and compactness methods to prove the existence and uniqueness of the solutions of the modified semilinear parabolic problem (WP$^\epsilon$). The existence and uniqueness of the original problem (WP) is then proved by passage to the limit $\epsilon \to 0$, a method used in Du's paper [2].

Since $\mathcal{H}^1(\Omega)$ and $\mathbf{H}_n^1(\Omega)$ are separable Hilbert spaces, there exists linearly independent total sets $\{z_1, \ldots, z_m, \ldots\} \in \mathcal{H}^1(\Omega)$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m, \ldots\} \in \mathbf{H}_n^1(\Omega)$ such that $\overline{\mathrm{span}\{z_i\}} = \mathcal{H}^1(\Omega)$ and $\overline{\mathrm{span}\{\mathbf{w}_i\}} = \mathbf{H}_n^1(\Omega)$. Let $\mathcal{Z}_n = \mathrm{span}\{z_1, \ldots, z_n\}$ and $\mathbf{\Lambda}_n = \mathrm{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be the n-dimensional subspaces of $\mathcal{H}^1(\Omega)$ and $\mathbf{H}_n^1(\Omega)$ respectively.

For each $n > 0$, define approximate solutions $\psi_{1n}^\epsilon,\ \psi_{2n}^\epsilon \in \mathcal{Z}_n$ and $\mathbf{A}_n^\epsilon \in \mathbf{\Lambda}_n$ for the problem (WP$^\epsilon$) as

$$\psi_{1n}^\epsilon = \sum_{i=1}^n a_{1ni}(t) z_i(\mathbf{x}),\quad \psi_{2n}^\epsilon = \sum_{i=1}^n a_{2ni}(t) z_i(\mathbf{x}),\quad \text{and } \mathbf{A}_n^\epsilon = \sum_{i=1}^n b_{ni}(t)\mathbf{w}_i(\mathbf{x}), \tag{3.15}$$

such that $\psi_{1n}^\epsilon,\ \psi_{2n}^\epsilon$ and $\mathbf{A}_n^\epsilon$ solve the following finite dimensional problem

Problem ($\mathbf{WP_n^\epsilon}$): For arbitrary ϵ, $0<\epsilon\le 1$, find $(\psi_{1n}^\epsilon,\psi_{2n}^\epsilon,\mathbf{A}_n^\epsilon)=(\psi_{1n}^\epsilon,\psi_{2n}^\epsilon,\overline{\mathbf{A}}_n^\epsilon-\mathbf{A}_c)\in\mathcal{Z}_n\times\mathcal{Z}_n\times\mathbf{\Lambda}_n$ such that for a.e. $t\in[0,T]$,

$$
\begin{aligned}
&\frac{d}{dt}(\psi_{1n}^\epsilon,\tilde{\psi}_n)+(i\phi_c\psi_{1n}^\epsilon,\tilde{\psi}_n)+\Big((|\psi_{1n}^\epsilon|^2-\mathcal{T}_1)\psi_{1n}^\epsilon,\tilde{\psi}_n\Big)\\
&\quad+\left(-i\frac{\xi_1}{x_0}\nabla\psi_{1n}^\epsilon-\frac{x_0}{\lambda_1}(\mathbf{A}_\mathbf{n}^\epsilon+\mathbf{A}_c)\psi_{1n}^\epsilon,\ -i\frac{\xi_1}{x_0}\nabla\tilde{\psi}_n-\frac{x_0}{\lambda_1}(\mathbf{A}_n^\epsilon+\mathbf{A}_c)\tilde{\psi}_n\right)\\
&\quad+\eta(\psi_{2n}^\epsilon,\tilde{\psi}_n)+\gamma_1\frac{\xi_1^2}{x_0^2}\left(\psi_{1n}^\epsilon,\tilde{\psi}_n\right)_{\partial\Omega}=0\qquad\forall\,\tilde{\psi}_n\in\mathcal{Z}_n,
\end{aligned}\tag{3.16}
$$

$$
\begin{aligned}
&\Gamma\frac{d}{dt}(\psi_{2n}^\epsilon,\tilde{\psi}_n)+\Gamma(i\phi_c\psi_{2n}^\epsilon,\tilde{\psi}_n)+\Big((|\psi_{2n}^\epsilon|^2-\mathcal{T}_2)\psi_{2n}^\epsilon,\tilde{\psi}_n\Big)\\
&\quad+\left(-i\frac{\xi_2}{x_0}\nabla\psi_{2n}^\epsilon-\nu\frac{x_0}{\lambda_2}(\mathbf{A}_\mathbf{n}^\epsilon+\mathbf{A}_c)\psi_{2n}^\epsilon,\ -i\frac{\xi_2}{x_0}\nabla\tilde{\psi}_n-\nu\frac{x_0}{\lambda_2}(\mathbf{A}_\mathbf{n}^\epsilon+\mathbf{A}_c)\tilde{\psi}_n\right)\\
&\quad+\eta\nu^2(\psi_{1n}^\epsilon,\tilde{\psi}_n)+\gamma_2\frac{\xi_2^2}{x_0^2}\left(\psi_{2n}^\epsilon,\tilde{\psi}_n\right)_{\partial\Omega}=0\qquad\forall\,\tilde{\psi}_n\in\mathcal{Z}_n,
\end{aligned}\tag{3.17}
$$

$$
\begin{aligned}
&\sigma\frac{x_0^2}{\lambda_1^2}\frac{d}{dt}\left(\mathbf{A}_n^\epsilon,\,\tilde{\mathbf{A}_\mathbf{n}}\right)+\epsilon(\mathrm{div}\mathbf{A}_n^\epsilon,\mathrm{div}\tilde{\mathbf{A}}_n)+(\mathrm{curl}\mathbf{A}_n^\epsilon,\,\mathrm{curl}\tilde{\mathbf{A}}_n)\\
&\quad+\Re\left(i\frac{1}{\kappa_1}\nabla\psi_{1n}^\epsilon,\,\psi_1\tilde{\mathbf{A}}_n\right)+\frac{x_0^2}{\lambda_1^2}\left(|\psi_1|^2(\mathbf{A}_n^\epsilon+\mathbf{A}_c),\,\tilde{\mathbf{A}}_n\right)\\
&\quad+\Re\left(i\frac{1}{\nu}\frac{1}{\kappa_2}\nabla\psi_{2n}^\epsilon,\,\psi_2\tilde{\mathbf{A}}_n\right)+\frac{x_0^2}{\lambda_2^2}\left(|\psi_2|^2(\mathbf{A}_n^\epsilon+\mathbf{A}_c),\,\tilde{\mathbf{A}}_n\right)=(\mathbf{H}_e,\,\mathrm{curl}\tilde{\mathbf{A}}_n)\quad\forall\tilde{\mathbf{A}}_n\in\mathbf{\Lambda}_n.
\end{aligned}\tag{3.18}
$$

The initial conditions are:

$$\psi_{1n}^\epsilon(\mathbf{x},0)=\psi_{10n},\tag{3.19}$$

$$\psi_{2n}^\epsilon(\mathbf{x},0)=\psi_{20n},\tag{3.20}$$

$$\mathbf{A}_n^\epsilon(\mathbf{x},0)=\mathbf{A}_{0n},\tag{3.21}$$

where ψ_{i0n} is the orthogonal projection in $\mathcal{H}^1(\Omega)$ of ψ_{i0} onto $\mathcal{Z}_n$, and $\mathbf{A}_{0n}$ is the orthogonal projection in $\mathbf{H}_n^1(\Omega)$ of $\mathbf{A}_0-\mathbf{A}_c(\mathbf{x},0)$ onto $\mathbf{\Lambda}_n$.

As a result of the projections, we have

$$\psi_{i0n}\to\psi_{i0}\qquad\text{in }\mathcal{H}^1(\Omega)\text{ as n}\to\infty,\tag{3.22}$$

$$||\psi_{i0n}||_1\le||\psi_{i0}||_1,\tag{3.23}$$

$$\mathbf{A}_{0n}\to\mathbf{A}_0-\mathbf{A}_c(\mathbf{x},0)\qquad\text{in }\mathbf{H}_n^1(\Omega)\text{ as n}\to\infty,\tag{3.24}$$

$$||\mathbf{A}_{0n}||_1\le||\mathbf{A}_0||_1.\tag{3.25}$$

3.2 Existence, Uniqueness and Boundedness of Solutions

We begin our mathematical analysis by showing that the nonlinear system of ODEs obtained from the problem ($\mathbf{WP}_n^{\epsilon}$) has a unique solution.

Lemma 3.2.1 *Given any $\epsilon > 0$ and $n > 0$, there exists a unique solution $(\psi_{1n}^{\epsilon}, \psi_{2n}^{\epsilon}, \mathbf{A}_n^{\epsilon})$ satisfying the problem* (WP_n^{ϵ}) *in some time interval $[0, T_n]$, where $0 < T_n \leq T$.*

Proof It is well-known that the nonlinear system of ODEs for a one-band TDGL system without the presence of time-dependent current and magnetic field is autonomous and has a unique solution in $[0, T_n]$. Similarly this is true for a nonlinear ODE system for a 2B-TDGL system without time-dependent current and field. With the presence of the time dependent electric potential ϕ_c, vector potential $\mathbf{A}_c$ and applied magnetic field $\mathbf{H}_e$, the resulting nonlinear system of ODEs now becomes non-autonomous in the form $\frac{d}{dt}\mathbf{U} = \mathbf{F}(\mathbf{U}, t) = \mathbf{F}_{\alpha\beta}(\mathbf{U}, t)$, with $\alpha = 1, \cdots, 3n; \beta = 1, \cdots, m$, where $\mathbf{U}$ represents the coefficient vector $\{a_{1n1}, \cdots, a_{1nn}, a_{2n1}, \cdots, a_{2nn}, b_{n1}, \cdots, b_{nn}\}$, $\mathbf{F}$ is a matrix function and m is the maximum number of terms in each of the individual equations (3.16)-(3.18). But with the regularity assumptions that $\phi_c \in C([0, T]; H^1(\Omega))$, $\mathbf{A}_c \in \mathbf{C}([0, T]; \mathbf{H}_n^1(\mathrm{div}; \Omega))$ and $\mathbf{H}_e \in \mathbf{C}([0, T]; \mathbf{L}^2(\Omega))$, all the functions $\mathbf{F}_{\alpha\beta}$ involving the time-dependent ϕ_c, $\mathbf{A}_c$ $\mathbf{H}_e$ are Lipschitz in $\mathbf{U}$ and Lebesgue integrable in t for fixed $\mathbf{U}$. Thus by the standard ODEs theory (see, e.g., theorem II.3.2 in [58]), the system has a unique solution $\mathbf{U}$ which consists of absolutely continuous functions in $[0, T_n]$. ∎

Remark By the standard theory of ODEs, the solution is defined on a maximal interval of existence $[0, T_n]$. If $T_n < T$ then we must have one of the coefficients $\{a_{1ni}(t)\}_i$, $\{a_{2ni}(t)\}_i$ or $\{b_{ni}(t)\}_i$ with magnitude tending to ∞ as $t \to T_n$. However the *a priori* estimates that we are going to prove later in the following lemmas and corollaries will show that all the norms $||\psi_{1n}^{\epsilon}||_0, ||\psi_{2n}^{\epsilon}||_0$ and $||\mathbf{A}_n^{\epsilon}||_0$ are always bounded in $[0, T]$, for any $T > 0$. This in turn shows that the magnitudes of the coefficients are actually bounded in $[0, T]$, thus we must have $T_n = T$.

Define an energy functional corresponding to the (steady state) PDEs of the modified problem (WP$_n^\epsilon$) as

$$\begin{aligned}
\mathcal{F}_n^\epsilon(\psi_{1n}^\epsilon, \psi_{2n}^\epsilon, \mathbf{A}_n^\epsilon) &= \int_\Omega \left\{ \frac{1}{2}(|\psi_{1n}^\epsilon|^2 - \mathcal{T}_1)^2 + \left| \left(i\frac{\xi_1}{x_0}\nabla + \frac{x_0}{\lambda_1}(\mathbf{A}_n^\epsilon + \mathbf{A}_c) \right) \psi_{1n}^\epsilon \right|^2 \right. \\
&+ \frac{1}{\nu^2} \left[\frac{1}{2}(|\psi_{2n}^\epsilon|^2 - \mathcal{T}_2)^2 + \left| \left(i\frac{\xi_2}{x_0}\nabla + \nu\frac{x_0}{\lambda_2}(\mathbf{A}_n^\epsilon + \mathbf{A}_c) \right) \psi_{2n}^\epsilon \right|^2 \right] \\
&+ \left. \epsilon|\mathrm{div}\mathbf{A}_n^\epsilon|^2 + |\mathrm{curl}\mathbf{A}_n^\epsilon - \mathbf{H}_e|^2 + \eta(\psi_{1n}^\epsilon\psi_{2n}^{\epsilon*} + \psi_{1n}^{\epsilon*}\psi_{2n}^\epsilon) \right\} d\Omega \\
&+ \int_{\partial\Omega} \left\{ \gamma_1 \left| \frac{\xi_1}{x_0}\psi_{1n}^\epsilon \right|^2 + \gamma_2 \left| \frac{1}{\nu}\frac{\xi_2}{x_0}\psi_{2n}^\epsilon \right|^2 \right\} dS. \qquad (3.26)
\end{aligned}$$

Due to the interband coupling term $\eta(\psi_{1n}^\epsilon\psi_{2n}^{\epsilon*} + \psi_{1n}^{\epsilon*}\psi_{2n}^\epsilon)$, $\mathcal{F}_n^\epsilon(\psi_{1n}^\epsilon, \psi_{2n}^\epsilon, \mathbf{A}_n^\epsilon)$ is not necessarily a nonnegative functional. For mathematical convenience, we will work on the nonnegative functional $\mathcal{E}_n^\epsilon$ defined as

$$\begin{aligned}
\mathcal{E}_n^\epsilon(\psi_{1n}^\epsilon, \psi_{2n}^\epsilon, \mathbf{A}_n^\epsilon) &= \int_\Omega \left\{ \frac{1}{2}\big(|\psi_{1n}^\epsilon|^2 - (\mathcal{T}_1 + |\eta|)\big)^2 + \left| \left(i\frac{\xi_1}{x_0}\nabla + \frac{x_0}{\lambda_1}(\mathbf{A}_n^\epsilon + \mathbf{A}_c) \right) \psi_{1n}^\epsilon \right|^2 \right. \\
&+ \frac{1}{\nu^2} \left[\frac{1}{2}\big(|\psi_{2n}^\epsilon|^2 - (\mathcal{T}_2 + \nu^2|\eta|)\big)^2 + \left| \left(i\frac{\xi_2}{x_0}\nabla + \nu\frac{x_0}{\lambda_2}(\mathbf{A}_n^\epsilon + \mathbf{A}_c) \right) \psi_{2n}^\epsilon \right|^2 \right] \\
&+ \left. \epsilon(\mathrm{div}\mathbf{A}_n^\epsilon)^2 + |\mathrm{curl}\mathbf{A}_n^\epsilon - \mathbf{H}_e|^2 + |\eta| \left| \psi_{1n}^\epsilon + \mathrm{sign}(\eta)\, \psi_{2n}^\epsilon \right|^2 \right\} d\Omega \\
&+ \int_{\partial\Omega} \left\{ \gamma_1 \left| \frac{\xi_1}{x_0}\psi_{1n}^\epsilon \right|^2 + \gamma_2 \left| \frac{1}{\nu}\frac{\xi_2}{x_0}\psi_{2n}^\epsilon \right|^2 \right\} dS. \qquad (3.27)
\end{aligned}$$

The relationship between these two functionals is $\mathcal{E}_n^\epsilon = \mathcal{F}_n^\epsilon + (\mathcal{T}_1 + \mathcal{T}_2)|\eta| + (1+\nu^2)|\eta|^2/2$. Clearly, both $\mathcal{E}_n^\epsilon$ and $\mathcal{F}_n^\epsilon$ have the same minimizer and give the same Euler-Lagrange equations, namely the steady state 2-band GL equations.

Lemma 3.2.2 *For any $\epsilon > 0$, $n > 0$ and $T > 0$, and for $t \in (0, T)$, we have*

$$\begin{aligned}
&\frac{d\mathcal{E}_n^\epsilon}{dt} + 2||\frac{\partial\psi_{1n}^\epsilon}{\partial t}||_0^2 + 2\frac{\Gamma}{\nu^2}||\frac{\partial\psi_{2n}^\epsilon}{\partial t}||_0^2 + 2\frac{\sigma x_0^2}{\lambda_1^2}||\frac{\partial\mathbf{A}_n^\epsilon}{\partial t}||_0^2 \\
&\quad = 2\int_\Omega \phi_c \left\{ \Im\left(\psi_{1n}^\epsilon \frac{\partial\psi_{1n}^{\epsilon*}}{\partial t} \right) + \frac{\Gamma}{\nu^2}\Im\left(\psi_{2n}^\epsilon \frac{\partial\psi_{2n}^{\epsilon*}}{\partial t} \right) \right\} d\Omega \\
&\quad - 2\int_\Omega \frac{\partial\mathbf{H}_e}{\partial t} \cdot (\nabla \times \mathbf{A} - \mathbf{H}_e) d\Omega, \qquad (3.28)
\end{aligned}$$

where $\Im$ denotes the imaginary part.

Proof Let the test function $\tilde{\psi}_n = \frac{\partial \psi_{1n}^\epsilon}{\partial t}$ in the weak form (3.16), and $\tilde{\psi}_n = \frac{\partial \psi_{2n}^\epsilon}{\partial t}$ in the weak form (3.17). Also let $\tilde{\mathbf{A}}_n = \frac{\partial \mathbf{A}_n^\epsilon}{\partial t}$ in the weak form (3.18). Then we have

$$
\begin{aligned}
\left(\frac{\partial \psi_{1n}^\epsilon}{\partial t}, \frac{\partial \psi_{1n}^\epsilon}{\partial t}\right) + \left(i\phi_c \psi_{1n}^\epsilon, \frac{\partial \psi_{1n}^\epsilon}{\partial t}\right) &= \left(\frac{\delta \mathcal{E}_n^\epsilon}{\delta \psi_{1n}^\epsilon}, \frac{\partial \psi_{1n}^\epsilon}{\partial t}\right), \\
\frac{\Gamma}{\nu^2}\left(\frac{\partial \psi_{2n}^\epsilon}{\partial t}, \frac{\partial \psi_{2n}^\epsilon}{\partial t}\right) + \frac{\Gamma}{\nu^2}\left(i\phi_c \psi_{2n}^\epsilon, \frac{\partial \psi_{2n}^\epsilon}{\partial t}\right) &= \left(\frac{\delta \mathcal{E}_n^\epsilon}{\delta \psi_{2n}^\epsilon}, \frac{\partial \psi_{2n}^\epsilon}{\partial t}\right), \\
\sigma \frac{x_0^2}{\lambda_1^2}\left(\frac{\partial \mathbf{A}_n^\epsilon}{\partial t}, \frac{\partial \mathbf{A}_n^\epsilon}{\partial t}\right) &= \left(\frac{\delta \mathcal{E}_n^\epsilon}{\delta \mathbf{A}_n^\epsilon}, \frac{\partial \mathbf{A}_n^\epsilon}{\partial t}\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
-\frac{1}{2}\frac{d}{dt}\mathcal{E}_n^\epsilon(\psi_{1n}^\epsilon, \psi_{2n}^\epsilon, \mathbf{A}_n^\epsilon) &= \Re\left(\frac{\delta \mathcal{E}_n^\epsilon}{\delta \psi_{1n}^\epsilon}, \frac{\partial \psi_{1n}^\epsilon}{\partial t}\right) + \Re\left(\frac{\delta \mathcal{E}_n^\epsilon}{\delta \psi_{2n}^\epsilon}, \frac{\partial \psi_{2n}^\epsilon}{\partial t}\right) \\
&+ \left(\frac{\delta \mathcal{E}_n^\epsilon}{\delta \mathbf{A}_n^\epsilon}, \frac{\partial \mathbf{A}_n^\epsilon}{\partial t}\right) + \int_\Omega \frac{\partial \mathbf{H}_e}{\partial t} \cdot (\nabla \times \mathbf{A} - \mathbf{H}_e) d\Omega. \qquad \blacksquare
\end{aligned}
$$

Lemma 3.2.2 says that the energy functional $\mathcal{E}_n^\epsilon$ is not dissipative, i.e., it is not true that $d\mathcal{E}_n^\epsilon/dt \leq 0$, unless $\phi_c = 0$ and $\partial \mathbf{H}_e/\partial t = 0$. In other words, if there is an external current or a non-stationary magnetic field applied to the superconductor, the energy functional $\mathcal{E}_n^\epsilon(t)$, for $t > 0$, is not bounded by its initial value $\mathcal{E}_n^\epsilon(0)$. However, the next lemma tells us that $\mathcal{E}_n^\epsilon(t)$ is nevertheless bounded as long as ϕ_c and $(\mathbf{H}_e)_t$ is bounded in some norms.

Lemma 3.2.3 *For any $\epsilon > 0$, $n > 0$ and $T > 0$, and for $t \in [0, T]$,*

$$
\begin{aligned}
\mathcal{E}_n^\epsilon(t) \leq\ & e^T \big[\mathcal{E}_n^\epsilon(0) + \varepsilon\big(||\psi_{1n}^\epsilon||^4_{\mathcal{L}^4(0,T;\mathcal{L}^4(\Omega))} + ||\psi_{2n}^\epsilon||^4_{\mathcal{L}^4(0,T;\mathcal{L}^4(\Omega))}\big) \\
& + C_\varepsilon ||\phi_c||^4_{L^4(0,T;L^4(\Omega))} + ||(\mathbf{H}_e)_t||^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))}\big], \qquad (3.29)
\end{aligned}
$$

where $\varepsilon > 0$ is some constant, and C_ε is a constant depends on ε. Note that here $\varepsilon \neq \epsilon$.

Proof From lemma 3.2.2, we have

$$\begin{aligned}
\frac{d\mathcal{E}_n^\epsilon}{dt} &= -2||\frac{\partial \psi_{1n}^\epsilon}{\partial t}||_0^2 - 2\frac{\Gamma}{\nu^2}||\frac{\partial \psi_{2n}^\epsilon}{\partial t}||_0^2 - 2\frac{\sigma x_0^2}{\lambda_1^2}||\frac{\partial \mathbf{A}_n^\epsilon}{\partial t}||_0^2 \\
&\quad + 2\int_\Omega \phi_c \Big\{ \Big[\Im\Big(\psi_{1n}^\epsilon \frac{\partial \psi_{1n}^{\epsilon *}}{\partial t}\Big) + \frac{\Gamma}{\nu^2}\Im\Big(\psi_{2n}^\epsilon \frac{\partial \psi_{2n}^{\epsilon *}}{\partial t}\Big)\Big] - \frac{\partial \mathbf{H}_e}{\partial t}\cdot(\nabla\times\mathbf{A}-\mathbf{H}_e)\Big\} d\Omega \\
&\le -2||\frac{\partial \psi_{1n}^\epsilon}{\partial t}||_0^2 - 2\frac{\Gamma}{\nu^2}||\frac{\partial \psi_{2n}^\epsilon}{\partial t}||_0^2 - 2\frac{\sigma x_0^2}{\lambda_1^2}||\frac{\partial \mathbf{A}_n^\epsilon}{\partial t}||_0^2 \\
&\quad + 2||\phi_c||_{0,4}||\psi_{1n}^\epsilon||_{0,4}||\frac{\partial \psi_{1n}^\epsilon}{\partial t}||_0 + 2\frac{\Gamma}{\nu^2}||\phi_c||_{0,4}||\psi_{2n}^\epsilon||_{0,4}||\frac{\partial \psi_{2n}^\epsilon}{\partial t}||_0 \\
&\quad + 2||(\mathbf{H}_e)_t||_0||\nabla\times\mathbf{A}-\mathbf{H}_e||_0 \\
&\le -||\frac{\partial \psi_{1n}^\epsilon}{\partial t}||_0^2 - \frac{\Gamma}{\nu^2}||\frac{\partial \psi_{2n}^\epsilon}{\partial t}||_0^2 - 2\frac{\sigma x_0^2}{\lambda_1^2}||\frac{\partial \mathbf{A}_n^\epsilon}{\partial t}||_0^2 \\
&\quad + ||\phi_c||_{0,4}^2||\psi_{1n}^\epsilon||_{0,4}^2 + \frac{\Gamma}{\nu^2}||\phi_c||_{0,4}^2||\psi_{2n}^\epsilon||_{0,4}^2 \\
&\quad + ||(\mathbf{H}_e)_t||_0^2 + ||\nabla\times\mathbf{A}-\mathbf{H}_e||_0^2 \qquad (3.30) \\
&\le \big(||\psi_{1n}^\epsilon||_{0,4}^2 + \frac{\Gamma}{\nu^2}||\psi_{2n}^\epsilon||_{0,4}^2\big)||\phi_c||_{0,4}^2 + ||(\mathbf{H}_e)_t||_0^2 + \mathcal{E}_n^\epsilon. \qquad (3.31)
\end{aligned}$$

The second to last inequality is obtained by using Young's inequality. By Gronwall's inequality, we get, for some constant $\varepsilon > 0$,

$$\begin{aligned}
\mathcal{E}_n^\epsilon &\le e^T\big[\mathcal{E}_n^\epsilon(0) + \big(||\psi_{1n}^\epsilon||^2_{\mathcal{L}^4(0,T;\mathcal{L}^4(\Omega))} + \frac{\Gamma}{\nu^2}||\psi_{2n}^\epsilon||^2_{\mathcal{L}^4(0,T;\mathcal{L}^4(\Omega))}\big)||\phi_c||^2_{L^4(0,T;L^4(\Omega))} \\
&\quad + ||(\mathbf{H}_e)_t||^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))}\big] \\
&\le e^T\big[\mathcal{E}_n^\epsilon(0) + \frac{\varepsilon}{2}\big(||\psi_{1n}^\epsilon||^4_{\mathcal{L}^4(0,T;\mathcal{L}^4(\Omega))} + ||\psi_{2n}^\epsilon||^4_{\mathcal{L}^4(0,T;\mathcal{L}^4(\Omega))}\big) \\
&\quad + \frac{1}{2\varepsilon}(1+\frac{\Gamma^2}{\nu^4})||\phi_c||^4_{L^4(0,T;L^4(\Omega))} + ||(\mathbf{H}_e)_t||^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))}\big]. \qquad (3.32)
\end{aligned}$$

Again, the last inequality is obtained by using Young's inequality. ∎

Lemma 3.2.4 *Assume $\psi_{i0} \in \mathcal{H}^1(\Omega)$ and $\mathbf{A}_0$, $\mathbf{A}_c(\mathbf{x},0) \in \mathbf{H}_n^1(\Omega)$, $\mathbf{H}_e(\mathbf{x},0) \in \mathbf{L}^2(\Omega)$, and also $\phi_c \in L^4(0,T;L^4)$ and $(\mathbf{H}_e)_t \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))$, then for any $\epsilon > 0$, $n > 0$ and $T > 0$,*

$$\begin{aligned}
&||\psi_{1n}^\epsilon||^4_{\mathcal{L}^\infty(0,T;\mathcal{L}^4(\Omega))} + ||\psi_{2n}^\epsilon||^4_{\mathcal{L}^\infty(0,T;\mathcal{L}^4(\Omega))} \\
&\qquad \le C(\Omega,T,\varepsilon_1)\Big[\mathcal{E}_n^\epsilon(0) + ||\phi_c||^4_{L^4(0,T;L^4(\Omega))} + ||(\mathbf{H}_e)_t||^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))} + 1\Big], \qquad (3.33)
\end{aligned}$$

where ε_1 is some fixed constant. Therefore, the sequences $\{\psi_{1n}^\epsilon\}$ and $\{\psi_{1n}^\epsilon\}$ are uniformly bounded, all independent of ϵ and n, in $\mathcal{L}^\infty(0,T;\mathcal{L}^4(\Omega))$.

Proof From the energy functional (3.27) we have, for any $t \in [0, T]$,

$$\begin{aligned}\mathcal{E}_n^\epsilon &\geq \int_\Omega \left[\frac{1}{2}\big(|\psi_{1n}^\epsilon|^2 - (\mathcal{T}_1 + |\eta|)\big)^2 + \frac{1}{\nu^2}\frac{1}{2}\big(|\psi_{2n}^\epsilon|^2 - (\mathcal{T}_2 + \nu^2|\eta|)\big)^2\right] d\Omega \\ &\geq \frac{1}{4}\big(||\psi_{1n}^\epsilon||_{0,4}^4 + \frac{1}{\nu^2}||\psi_{2n}^\epsilon||_{0,4}^4\big) - C|\Omega|.\end{aligned}$$

Here in the last inequality, we have used the inequality $|a - b|^2 \geq (1/2)|a|^2 - |b|^2$. Combining this inequality with the result of lemma 3.2.3, then for any $t \in [0, T]$,

$$\begin{aligned}\frac{1}{4}\big(||\psi_{1n}^\epsilon||_{0,4}^4 + \frac{1}{\nu^2}||\psi_{2n}^\epsilon||_{0,4}^4\big) - C(\Omega) \quad &\leq \quad e^T\big[\mathcal{E}_n^\epsilon(0) + \varepsilon\big(||\psi_{1n}^\epsilon||_{\mathcal{L}^4(0,T;\mathcal{L}^4(\Omega))}^4 + ||\psi_{2n}^\epsilon||_{\mathcal{L}^4(0,T;\mathcal{L}^4(\Omega))}^4\big) \\ &\quad + C_\varepsilon||\phi_c||_{L^4(0,T;L^4(\Omega))}^4 + ||(\mathbf{H}_e)_t||_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))}^2\big].\end{aligned}$$

By using the embedding $\mathcal{L}^\infty(0, T; \mathcal{L}^4(\Omega)) \hookrightarrow \mathcal{L}^4(0, T; \mathcal{L}^4(\Omega))$ and rearranging the terms in the above inequality, we get

$$\begin{aligned}&(\frac{1}{4} - e^T\varepsilon D)||\psi_{1n}^\epsilon||_{\mathcal{L}^\infty(0,T;\mathcal{L}^4(\Omega))}^4 + (\frac{1}{4}\frac{1}{\nu^2} - e^T\varepsilon D)||\psi_{2n}^\epsilon||_{\mathcal{L}^\infty(0,T;\mathcal{L}^4(\Omega))}^4 \\ &\quad \leq C(\Omega) + e^T\big[\mathcal{E}_n^\epsilon(0) + C_\varepsilon||\phi_c||_{L^4(0,T;L^4(\Omega))}^4 + ||(\mathbf{H}_e)_t||_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))}^2\big],\end{aligned}$$

where the constant D is the Sobolev embedding constant which depends only on Ω. If the first three assumptions in lemma 3.2.4 hold, then by the initial conditions (3.19)-(3.21) and the projection relations (3.22)-(3.25), we have $\psi_{in}^\epsilon(\mathbf{x}, 0) \in \mathcal{H}^1(\Omega)$ and $\mathbf{A}_n^\epsilon(\mathbf{x}, 0) \in \mathbf{H}_n^1(\Omega)$. Also by the trace theorem, we have $\psi_{in}^\epsilon|_{\partial\Omega}(\mathbf{x}, 0) \in \mathcal{L}^2(\partial\Omega)$. Then from the energy functional (3.27), we have $\mathcal{E}_n^\epsilon(0) \leq \mathcal{E}_n^1(0) := \mathcal{E}_n^{\epsilon=1}(0)$ bounded by a constant which depends only on $||\psi_{i0}||_{\mathcal{H}^1(\Omega)}$, $||\mathbf{A}_0||_{\mathbf{H}^1(\Omega)}$ and $||\mathbf{H}_e(\mathbf{x}, 0)||_{\mathbf{L}^2(\Omega)}$, and is thus independent of ϵ and n. Furthermore, if the last two assumptions in lemma 3.2.4 hold, then the right hand side of the above inequality is bounded. Therefore, by choosing a small enough $\varepsilon = \varepsilon_1$ to make the constant on the L.H.S. of the above inequality greater than zero, $\{\psi_{1n}^\epsilon\}$ and $\{\psi_{1n}^\epsilon\}$ are uniformly bounded in $\mathcal{L}^\infty(0, T; \mathcal{L}^4(\Omega))$, all independent of ϵ and n. ∎

Corollary 3.2.5 *Suppose the assumptions in lemma 3.2.4 hold, then for any $\epsilon > 0$, $n > 0$ and $T > 0$, and for $t \in [0, T]$,*

$$0 \leq \mathcal{E}_n^\epsilon(t) \leq e^T E < \infty, \tag{3.34}$$

where $E = C(\Omega, T, \varepsilon_1)\left[\mathcal{E}_n^1(0) + ||\phi_c||_{L^4(0,T;L^4(\Omega))}^4 + ||(\mathbf{H}_e)_t||_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))}^2 + 1\right]$ is a constant independent of ϵ and n. Here ε_1 is the constant picked in the proof of lemma 3.2.4.

Proof This is a consequence of the embedding $\mathcal{L}^\infty(0,T;\mathcal{L}^4(\Omega) \hookrightarrow \mathcal{L}^4(0,T;\mathcal{L}^4(\Omega))$, lemma 3.2.3 and lemma 3.2.4. Recall that $\mathcal{E}_n^\epsilon$ is a nonnegative functional by definition in (3.27). ∎

Lemma 3.2.6 *Suppose the assumptions in lemma 3.2.4 hold, then for any $\epsilon > 0$, $n > 0$ and $T > 0$,*

$$||\frac{\partial \psi_{1n}^\epsilon}{\partial t}||^2_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))} + ||\frac{\partial \psi_{2n}^\epsilon}{\partial t}||^2_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))} + ||\frac{\partial \mathbf{A}_n^\epsilon}{\partial t}||^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))} \le (1 + e^T T)E, \tag{3.35}$$

where the constant E is defined in corollary 3.2.5. Therefore, the sequences $\{\frac{\partial \psi_{1n}^\epsilon}{\partial t}\}$ and $\{\frac{\partial \psi_{2n}^\epsilon}{\partial t}\}$ are uniformly bounded in $\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))$, and $\{\frac{\partial \mathbf{A}_n^\epsilon}{\partial t}\}$ is uniformly bounded in $\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))$, all independent of ϵ and n.

Proof From the inequality (3.30)-(3.31),

$$\begin{aligned}
&\frac{d\mathcal{E}_n^\epsilon}{dt} + ||\frac{\partial \psi_{1n}^\epsilon}{\partial t}||_0^2 + \frac{\Gamma}{\nu^2}||\frac{\partial \psi_{2n}^\epsilon}{\partial t}||_0^2 + 2\frac{\sigma x_0^2}{\lambda_1^2}||\frac{\partial \mathbf{A}_n^\epsilon}{\partial t}||_0^2 \\
&\quad \le \big(||\psi_{1n}^\epsilon||_{0,4}^2 + \frac{\Gamma}{\nu^2}||\psi_{2n}^\epsilon||_{0,4}^2\big)||\phi_c||_{0,4}^2 + ||(\mathbf{H}_e)_t||_0^2 + \mathcal{E}_n^\epsilon.
\end{aligned} \tag{3.36}$$

Integrating the above inequality w.r.t. time over $[0,T]$ and by using Young's inequality, we get

$$\begin{aligned}
&||\frac{\partial \psi_{1n}^\epsilon}{\partial t}||^2_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))} + \frac{\Gamma}{\nu^2}||\frac{\partial \psi_{2n}^\epsilon}{\partial t}||^2_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))} + 2\frac{\sigma x_0^2}{\lambda_1^2}||\frac{\partial \mathbf{A}_n^\epsilon}{\partial t}||^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))} \\
&\quad \le \mathcal{E}_n^\epsilon(0) - \mathcal{E}_n^\epsilon(T) + \varepsilon_1\big(||\psi_{1n}^\epsilon||^4_{\mathcal{L}^4(0,T;\mathcal{L}^4(\Omega))} + ||\psi_{2n}^\epsilon||^4_{\mathcal{L}^4(0,T;\mathcal{L}^4(\Omega))}\big) \\
&\qquad + C_{\varepsilon_1}||\phi_c||^4_{L^4(0,T;L^4(\Omega))} + ||(\mathbf{H}_e)_t||^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))} + \int_0^T \mathcal{E}_n^\epsilon(t)dt. \\
&\quad \le \mathcal{E}_n^\epsilon(0) + \varepsilon_1\big(||\psi_{1n}^\epsilon||^4_{\mathcal{L}^4(0,T;\mathcal{L}^4(\Omega))} + ||\psi_{2n}^\epsilon||^4_{\mathcal{L}^4(0,T;\mathcal{L}^4(\Omega))}\big) \\
&\qquad + C_{\varepsilon_1}||\phi_c||^4_{L^4(0,T;L^4(\Omega))} + ||(\mathbf{H}_e)_t||^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))} + e^T TE \quad \text{(by corollary 3.2.5)} \\
&\quad \le E + e^T TE,
\end{aligned}$$

here the constant ε_1 appeared in the above inequalities is the same constant picked in the proof of the lemma 3.2.4. Note that the terms, excluding the last term in the second inequality above are exactly the same as those in the right hand side of the inequality (3.29) in lemma 3.2.3 from which and lemma 3.2.4 with the choice of ε_1, we equal these terms to the constant E defined in corollary 3.2.5. ∎

Lemma 3.2.7 *Suppose the assumptions in lemma 3.2.4 hold and* $\mathbf{H}_e \in \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))$, *then for any* $\epsilon > 0$, $n > 0$ *and* $T > 0$,

$$\epsilon||\text{div}\mathbf{A}_n^\epsilon||^2_{L^\infty(0,T;L^2(\Omega))} + \frac{1}{2}||\text{curl}\mathbf{A}_n^\epsilon||^2_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))} \leq \left(e^T E + ||\mathbf{H}_e||^2_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))}\right), \tag{3.37}$$

where the constant E *is defined in corollary 3.2.5. Therefore, the sequence* $\{\mathbf{A}_n^\epsilon\}$ *is uniformly bounded in* $\mathbf{L}^\infty(0,T;\mathbf{H}_n^1(\Omega))$, *independent of* n *but dependent on* ϵ.

Proof From equation (3.27), we have for any $t \in (0,T]$,

$$\begin{aligned}\mathcal{E}_n^\epsilon &\geq \epsilon||\text{div}\mathbf{A}_n^\epsilon||_0^2 + ||\text{curl}\mathbf{A}_n^\epsilon - \mathbf{H}_e||_0^2 \\ &\geq \epsilon||\text{div}\mathbf{A}_n^\epsilon||_0^2 + \frac{1}{2}||\text{curl}\mathbf{A}_n^\epsilon||_0^2 - ||\mathbf{H}_e||_0^2.\end{aligned}$$

Then corollary 3.2.5 gives the inequality (3.37). Now since the norm for the space $\mathbf{H}_n^1(\Omega)$ is equal to $||\text{div}\mathbf{A}_n^\epsilon||_0 + ||\text{curl}\mathbf{A}_n^\epsilon||_0$, see (2.23), we have

$$||\mathbf{A}_n^\epsilon||^2_{\mathbf{L}^\infty(0,T;\mathbf{H}_n^1(\Omega))} \leq \frac{1}{\min\{\epsilon, \frac{1}{2}\}}\left(e^T E + ||\mathbf{H}_e||^2_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))}\right).$$

Thus if $\mathbf{H}_e \in \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))$, the sequence $\{\mathbf{A}_n^\epsilon\}$ is uniformly bounded in $\mathbf{L}^\infty(0,T;\mathbf{H}_n^1(\Omega))$, independent of n but dependent on ϵ. ∎

Lemma 3.2.8 *Suppose the assumptions in lemma 3.2.7 hold and* $\mathbf{A}_c \in \mathbf{L}^\infty(0,T;\mathbf{L}^4(\Omega))$, *then for any* $\epsilon > 0$, $n > 0$ *and* $T > 0$,

$$\begin{aligned}&||\nabla\psi_{1n}^\epsilon||^2_{\mathcal{L}^\infty(0,T;\mathcal{L}^2(\Omega))} + ||\nabla\psi_{2n}^\epsilon||^2_{\mathcal{L}^\infty(0,T;\mathcal{L}^2(\Omega))} \\ &\quad \leq C_1\left[e^T E + C_2\left(||\psi_{1n}^\epsilon||^2_{\mathcal{L}^\infty(0,T;\mathcal{L}^4(\Omega))} + ||\psi_{2n}^\epsilon||^2_{\mathcal{L}^\infty(0,T;\mathcal{L}^4(\Omega))}\right)||(\mathbf{A}_n^\epsilon + \mathbf{A}_c)||^2_{\mathbf{L}^\infty(0,T;\mathbf{L}^4(\Omega))}\right],\end{aligned} \tag{3.38}$$

where the constant E *is defined in corollary 3.2.5, and* C_1 *and* C_2 *are independent of* ϵ *and* n. *Therefore, the sequences* $\{\psi_{1n}^\epsilon\}$ *and* $\{\psi_{2n}^\epsilon\}$ *are uniformly bounded in* $\mathcal{L}^\infty(0,T;\mathcal{H}^1(\Omega))$, *independent of* n *but dependent on* ϵ.

Proof

$$\begin{aligned}
&\int_\Omega \left| \left(i\frac{\xi_1}{x_0}\nabla + \frac{x_0}{\lambda_1}(\mathbf{A}_n^\epsilon + \mathbf{A}_c) \right) \psi_{1n}^\epsilon \right|^2 d\Omega \\
&\quad = \int_\Omega \Big\{ \big(\frac{\xi_1}{x_0}\big)^2 |\nabla \psi_{1n}^\epsilon|^2 + \big(\frac{x_0}{\lambda_1}\big)^2 |(\mathbf{A}_n^\epsilon + \mathbf{A}_c)|^2 |\psi_{1n}^\epsilon|^2 \\
&\qquad + i\frac{\xi_1}{\lambda_1}(\mathbf{A}_n^\epsilon + \mathbf{A}_c) \cdot (\psi_{1n}^{\epsilon *}\nabla\psi_{1n}^\epsilon - \psi_{1n}^\epsilon \nabla \psi_{1n}^{\epsilon *}) \Big\} d\Omega \\
&\quad \geq \int_\Omega \Big\{ \big(\frac{\xi_1}{x_0}\big)^2 |\nabla \psi_{1n}^\epsilon|^2 - 2\frac{\xi_1}{\lambda_1} |\psi_{1n}^{\epsilon *}\nabla\psi_{1n}^\epsilon| \cdot (\mathbf{A}_n^\epsilon + \mathbf{A}_c) \Big\} d\Omega \\
&\quad \geq \big(\frac{\xi_1}{x_0}\big)^2 ||\nabla\psi_{1n}^\epsilon||_0^2 - \frac{1}{\varepsilon}\big(\frac{\xi_1}{\lambda_1}\big)^2 ||\psi_{1n}^\epsilon||_{0,4}^2 ||\mathbf{A}_n^\epsilon + \mathbf{A}_c||_{0,4}^2 - \varepsilon ||\nabla\psi_{1n}^\epsilon||_0^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_\Omega \left| \left(i\frac{\xi_2}{x_0}\nabla + \nu\frac{x_0}{\lambda_2}(\mathbf{A}_n^\epsilon + \mathbf{A}_c) \right) \psi_{2n}^\epsilon \right|^2 d\Omega \\
&\quad \geq \big(\frac{\xi_2}{x_0}\big)^2 ||\nabla\psi_{2n}^\epsilon||_0^2 - \frac{1}{\varepsilon}\big(\nu\frac{\xi_1}{\lambda_1}\big)^2 ||\psi_{2n}^\epsilon||_{0,4)}^2 ||\mathbf{A}_n^\epsilon + \mathbf{A}_c||_{0,4}^2 - \varepsilon ||\nabla\psi_{2n}^\epsilon||_0^2.
\end{aligned}$$

From the energy functional (3.27), we have for $t \in (0, T]$,

$$\int_\Omega \left[\left| \left(i\frac{\xi_1}{x_0}\nabla + \frac{x_0}{\lambda_1}(\mathbf{A}_n^\epsilon + \mathbf{A}_c) \right) \psi_{1n}^\epsilon \right|^2 + \left| \left(i\frac{\xi_2}{x_0}\nabla + \nu\frac{x_0}{\lambda_2}(\mathbf{A}_n^\epsilon + \mathbf{A}_c) \right) \psi_{2n}^\epsilon \right|^2 \right] d\Omega \leq \mathcal{E}_n^\epsilon.$$

Combining the above inequalities, we get for $t \in (0, T]$,

$$\begin{aligned}
&\big[\big(\frac{\xi_1}{x_0}\big)^2 - \varepsilon\big] ||\nabla\psi_{1n}^\epsilon||_0^2 + \big[\big(\frac{\xi_2}{x_0}\big)^2 - \varepsilon\big] ||\nabla\psi_{2n}^\epsilon||_0^2 \\
&\quad \leq e^T E + \frac{1}{\varepsilon}\big(\frac{\xi_1}{\lambda_1}\big)^2 ||\psi_{1n}^\epsilon||_{0,4}^2 ||\mathbf{A}_n^\epsilon + \mathbf{A}_c||_{0,4}^2 + \frac{1}{\varepsilon}\big(\nu\frac{\xi_1}{\lambda_1}\big)^2 ||\psi_{2n}^\epsilon||_{0,4}^2 ||\mathbf{A}_n^\epsilon + \mathbf{A}_c||_{0,4}^2.
\end{aligned}$$

Now by lemma 3.2.4, $\{\psi_{1n}^\epsilon\}$ and $\{\psi_{2n}^\epsilon\}$ are uniformly bounded in $\mathcal{L}^\infty(0, T; \mathcal{L}^4(\Omega))$. Also by lemma 3.2.7, $\{\mathbf{A}_n^\epsilon\}$ is uniformly bounded in $\mathbf{L}^\infty(0, T; \mathbf{H}_n^1(\Omega))$, dependent on ϵ. So if in addition $\mathbf{A}_c \in \mathbf{L}^\infty(0, T; \mathbf{L}^4(\Omega))$, then by choosing ε small enough, we arrive at the inequality (3.38) and thus the lemma is proved. ∎

Up to this point, we summarize the requirement of the spaces for ϕ_c, $\mathbf{A}_c$ and $\mathbf{H}_e$. We need $\phi_c \in L^4(0, T; L^4)$ and $\mathbf{A}_c \in \mathbf{L}^\infty(0, T; \mathbf{L}^4(\Omega))$. For $\mathbf{H}_e$, we require $\mathbf{H}_e \in \mathbf{H}^1(0, T; \mathbf{L}^2(\Omega)) \cap \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))$. For the initial conditions, we need $\psi_{i0} \in \mathcal{H}^1(\Omega)$ and $\mathbf{A}_0$, $\mathbf{A}_c(\mathbf{x}, 0) \in \mathbf{H}_n^1(\Omega)$, and also $\mathbf{H}_e(\mathbf{x}, 0) \in \mathbf{L}^2(\Omega)$.

Hereafter unless otherwise stated, we will assume that the following regularity assumptions are satisfied throughout the rest of this work:

RA1: Assume that $\psi_{i0} \in \mathcal{H}^1(\Omega)$, $\mathbf{A}_0$, $\mathbf{A}_c(\mathbf{x}, 0) \in \mathbf{H}^1_n(\Omega)$, $\mathbf{H}_e(\mathbf{x}, 0) \in \mathbf{L}^2(\Omega)$, $\phi_c \in L^4(0, T; H^1(\Omega))$, $\mathbf{A}_c \in \mathbf{L}^\infty(0, T; \mathbf{H}^1_n(\Omega))$ ($\mathbf{A}_c \cdot \mathbf{n} = 0$, see equation (2.60)) and $\mathbf{H}_e \in \mathbf{H}^1(0, T; \mathbf{L}^2(\Omega)) \cap \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))$.

Now with all the *a priori* estimates established, the remark following lemma 3.2.1 says that $T_n = T$.

Corollary 3.2.9 *Given $\epsilon > 0$, $n > 0$ and $T > 0$, there exists a unique global solution $(\psi^\epsilon_{1n}, \psi^\epsilon_{2n}, \mathbf{A}^\epsilon_n) \in \mathcal{V} \times \mathcal{V} \times \mathbf{V}$ satisfying the problem* (WP^ϵ_n) *in* $[0, T]$.

Corollary 3.2.10 *Given $\epsilon > 0$ and $T > 0$, the sequence $\{(\psi^\epsilon_{1n}, \psi^\epsilon_{2n}, \mathbf{A}^\epsilon_n)\}$ is uniformly bounded in $\mathcal{V} \times \mathcal{V} \times \mathbf{V}$, independent of n only.*

The following standard lemma and corollary can be found, for example, in [2] and [46].

Lemma 3.2.11 (Aubin-Lions) *Let B be a Banach space and B_i, $i = 0, 1$, be Hilbert spaces. Suppose that $B_0 \hookrightarrow\hookrightarrow B$, i.e., the embedding is compact, and suppose that $B \hookrightarrow B_1$, i.e., the embedding is continuous, then*

$$L^p(0, T; B_0) \cap W^{1,q}(0, T; B_1) \hookrightarrow\hookrightarrow L^p(0, T; B) \qquad \forall 1 < p, q < \infty.$$

Corollary 3.2.12 *Given $\epsilon > 0$ and $T > 0$, there exists an element $(\psi^\epsilon_1, \psi^\epsilon_2, \mathbf{A}^\epsilon) \in \mathcal{V} \times \mathcal{V} \times \mathbf{V}$, and subsequences $\{\psi^\epsilon_{1n_k}\}$, $\{\psi^\epsilon_{2n_k}\}$ and $\{\mathbf{A}^\epsilon_{n_k}\}$ such that as $n \to \infty$,*

$$\begin{aligned}
\psi^\epsilon_{1n_k} &\rightharpoonup \psi^\epsilon_1 \text{ weakly (and } \overset{*}{\rightharpoonup} \text{ weakly}^*\text{) in } \mathcal{V}, \\
\psi^\epsilon_{2n_k} &\rightharpoonup \psi^\epsilon_2 \text{ weakly (and } \overset{*}{\rightharpoonup} \text{ weakly}^*\text{) in } \mathcal{V}, \\
\mathbf{A}^\epsilon_{n_k} &\rightharpoonup \mathbf{A}^\epsilon \text{ weakly (and } \overset{*}{\rightharpoonup} \text{ weakly}^*\text{) in } \mathbf{V},
\end{aligned}$$

and

$$\begin{aligned}
\psi^\epsilon_{1n_k} &\to \psi^\epsilon_1 \text{ strongly in } \mathcal{L}^{\mathrm{p}}(0, \mathrm{T}; \mathcal{L}^{\mathrm{q}}(\Omega)), \\
\psi^\epsilon_{2n_k} &\to \psi^\epsilon_2 \text{ strongly in } \mathcal{L}^{\mathrm{p}}(0, \mathrm{T}; \mathcal{L}^{\mathrm{q}}(\Omega)), \\
\mathbf{A}^\epsilon_{n_k} &\to \mathbf{A}^\epsilon \text{ strongly in } \mathbf{L}^{\mathrm{p}}(0, \mathrm{T}; \mathbf{L}^{\mathrm{q}}(\Omega)),
\end{aligned}$$

where $p \in (1, \infty)$, and $q \in (1, \infty)$ for $d = 2$ and $q \in (1, 6)$ for $d = 3$.

With the weak (weak*) and strong convergence of the limits, we can pass to the limit $n_k \to \infty$ to prove that the limits together form a solution satisfying the modified problem (WP$^\epsilon$).

Theorem 3.2.13 *Given $\epsilon > 0$ and $T > 0$, the weak (weak*) limit $(\psi_1^\epsilon, \psi_2^\epsilon, \mathbf{A}^\epsilon)$ in corollary 3.2.12 is a solution of the problem* (WP$^\epsilon$).

Proof We want to show that when passing to the limit $n_k \to \infty$, the system equations (3.16)-(3.21) in the problem (WP$_n^\epsilon$) converge to the system equations (3.9)-(3.14) in the problem (WP$^\epsilon$), respectively. Once we have shown the convergence of the following nonlinear terms, it remains to be a standard procedure to show the convergence of the system equations and the initial conditions, see for example [45], [46]. For convenience, we will write an element of a subsequence, for example, $\psi_{1n_k}^\epsilon$ simple as ψ_{1n}^ϵ.

For $\varphi \in \mathcal{C}_0^\infty[0,T]$ and $\tilde{\psi} \in \mathcal{Z}_n$,

$$
\begin{aligned}
(I)\ & \left| \int_0^T \left(|\psi_{1n}^\epsilon|^2 \psi_{1n}^\epsilon - |\psi_1^\epsilon|^2 \psi_1^\epsilon,\ \tilde{\psi} \right) \varphi \, dt \right| \\
&\le \int_0^T \int_\Omega \Big[\left(\big| |\psi_{1n}^\epsilon|^2 - |\psi_1^\epsilon|^2 \big| \right) \ |\psi_{1n}^\epsilon| + |\psi_1^\epsilon|^2 \, |\psi_{1n}^\epsilon - \psi_1^\epsilon| \Big] |\varphi \tilde{\psi}| \, d\Omega dt \\
&= \int_0^T \int_\Omega \Big[\left(\big| |\psi_{1n}^\epsilon| - |\psi_1^\epsilon| \big| \right) \ (|\psi_{1n}^\epsilon| + |\psi_1^\epsilon|) \, |\psi_{1n}^\epsilon| + |\psi_1^\epsilon|^2 \, |\psi_{1n}^\epsilon - \psi_1^\epsilon| \Big] |\varphi \tilde{\psi}| \, d\Omega dt \\
&\le \int_0^T ||\psi_{1n}^\epsilon - \psi_1^\epsilon||_0 \Big[(||\psi_{1n}^\epsilon||_1 + ||\psi_1^\epsilon||_1) \, ||\psi_{1n}^\epsilon||_1 + ||\psi_1^\epsilon||_1^2 \Big] ||\varphi \tilde{\psi}||_1 \, dt \\
&\le ||\psi_{1n}^\epsilon - \psi_1^\epsilon||_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))} \Big[\left(||\psi_{1n}^\epsilon||_{\mathcal{L}^\infty(0,T;\mathcal{H}^1(\Omega))} + ||\psi_1^\epsilon||_{\mathcal{L}^\infty(0,T;\mathcal{H}^1(\Omega))} \right) ||\psi_{1n}^\epsilon||_{\mathcal{L}^\infty(0,T;\mathcal{H}^1(\Omega))} \\
&\qquad + ||\psi_1^\epsilon||^2_{\mathcal{L}^\infty(0,T;\mathcal{H}^1(\Omega))} \Big] ||\varphi \tilde{\psi}||_{\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega))} \\
&\to 0 \qquad \text{as } n \to \infty.
\end{aligned}
$$

Here in the above second inequality, we have use the Sobolev embedding $\mathcal{H}^1(\Omega) \hookrightarrow \mathcal{L}^q(\Omega)$, for $1 \le q \le 6$. The convergence in the last inequality is justified by the fact that $\psi_{1n}^\epsilon \to \psi_1^\epsilon$ strongly in $\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))$ by corollary 3.2.12, and that $\psi_{1n}^\epsilon, \psi_1^\epsilon \in \mathcal{L}^\infty(0,T;\mathcal{H}^1(\Omega))$ by lemma 3.2.8 and corollary 3.2.12, respectively.

For the convergence of the nonlinear term in

$$
\left(-i\frac{\xi_1}{x_0} \nabla \psi_{1n}^\epsilon - \frac{x_0}{\lambda_1} (\mathbf{A}_{\mathbf{n}}^\epsilon + \mathbf{A}_c) \psi_{1n}^\epsilon,\ -i\frac{\xi_1}{x_0} \nabla \tilde{\psi} - \frac{x_0}{\lambda_1} (\mathbf{A}_n^\epsilon + \mathbf{A}_c) \tilde{\psi} \right),
$$

we have, after dropping the constants,

$$
\begin{aligned}
(II)\ &\left|\int_0^T \int_\Omega \big[\,(\mathbf{A}_n^\epsilon + \mathbf{A}_c)\cdot\nabla\psi_{1n}^\epsilon - (\mathbf{A}^\epsilon + \mathbf{A}_c)\cdot\nabla\psi_1^\epsilon\,\big]\varphi\tilde{\psi}\; d\Omega dt\right| \\
&\le \int_0^T \int_\Omega \Big[\big|(\nabla\psi_{1n}^\epsilon - \nabla\psi_1^\epsilon)\cdot(\mathbf{A}^\epsilon + \mathbf{A}_c)\varphi\tilde{\psi}\big| + \big|\nabla\psi_{1n}^\epsilon\cdot(\mathbf{A}_n^\epsilon - \mathbf{A}^\epsilon)\varphi\tilde{\psi}\big|\Big]\; d\Omega dt \\
&\le \int_0^T \Big((\nabla\psi_{1n}^\epsilon - \nabla\psi_1^\epsilon),\ (\mathbf{A}^\epsilon + \mathbf{A}_c)\varphi\tilde{\psi}\Big)\; dt \\
&\quad + ||\nabla\psi_{1n}^\epsilon||_{\mathcal{L}^\infty(0,T;\mathcal{L}^2(\Omega))}\, ||\mathbf{A}_n^\epsilon - \mathbf{A}^\epsilon||_{\mathbf{L}^2(0,T;\mathbf{L}^4)}\, ||\varphi\tilde{\psi}||_{\mathcal{L}^2(0,T;\mathcal{H}^1)}\, dt \\
&\to 0 \qquad \text{as } n\to\infty.
\end{aligned}
$$

In the above last inequality, the first right hand side (R.H.S.) term converges to zero by the fact that $\psi_{1n_k}^\epsilon \overset{*}{\rightharpoonup} \psi_1^\epsilon$ in $\mathcal{L}^\infty(0,T;\mathcal{H}^1(\Omega))$ as shown in corollary 3.2.12, and note that now by the regularity of $\mathbf{A}^\epsilon$ and $\mathbf{A}_c$, we have $(\mathbf{A}^\epsilon + \mathbf{A}_c)\varphi\tilde{\psi} \in \mathcal{L}^\infty(0,T;\mathcal{L}^2(\Omega))^d \subset \mathcal{L}^1(0,T;\mathcal{L}^2(\Omega))^d$, here $d = 2,3$. The second R.H.S. term converges to zero because $\mathbf{A}_n^\epsilon \to \mathbf{A}^\epsilon$ strongly in $\mathbf{L}^2(0,\mathrm{T};\mathbf{L}^4(\Omega))$, again from corollary 3.2.12.

$$
\begin{aligned}
(III)\ &\left|\int_0^T \int_\Omega \big[\,(\mathbf{A}_n^\epsilon + \mathbf{A}_c)\psi_{1n}^\epsilon - (\mathbf{A}^\epsilon + \mathbf{A}_c)\psi_1^\epsilon\,\big]\cdot\varphi\nabla\tilde{\psi}\; d\Omega dt\right| \\
&\le \int_0^T \int_\Omega \Big[\big|(\mathbf{A}_n^\epsilon - \mathbf{A}^\epsilon)\cdot\psi_{1n}^\epsilon\varphi\nabla\tilde{\psi}\big| + \big|(\psi_{1n}^\epsilon - \psi_1^\epsilon)(\mathbf{A}^\epsilon - \mathbf{A}_c)\cdot\varphi\nabla\tilde{\psi}\big|\Big]\; d\Omega dt \\
&\le ||\mathbf{A}_n^\epsilon - \mathbf{A}^\epsilon||_{\mathbf{L}^2(0,T;\mathbf{L}^4)}\, ||\psi_{1n}^\epsilon||_{\mathcal{L}^\infty(0,T;\mathcal{L}^4(\Omega))}\, ||\varphi\nabla\tilde{\psi}||_{\mathcal{L}^2(0,T;\mathcal{L}^2)}\, dt \\
&\quad + ||\psi_{1n}^\epsilon - \psi_1^\epsilon||_{\mathcal{L}^2(0,T;\mathcal{L}^4(\Omega))}\, ||\mathbf{A}^\epsilon - \mathbf{A}_c||_{\mathbf{L}^\infty(0,T;\mathbf{L}^4)}\, ||\varphi\nabla\tilde{\psi}||_{\mathcal{L}^2(0,T;\mathcal{L}^2)} \\
&\to 0 \qquad \text{as } n\to\infty.
\end{aligned}
$$

Again, the above convergence is justified by lemma 3.2.7, lemma 3.2.8 and corollary 3.2.12.

Now for the boundary term, first note that the trace operator

$$\gamma : \mathcal{L}^q(0,T;\mathcal{H}^1(\Omega)) \to \mathcal{L}^q(0,T;\mathcal{H}^{\frac{1}{2}}(\partial\Omega))$$

defined upon the usual trace operator γ_0 as

$$\gamma(\psi(t)) = \gamma_0(\psi(t)) := \psi(t)|_{\partial\Omega} \quad \text{in the trace sense, } \forall \mathrm{t} \in [0,\mathrm{T}],$$

is linear and continuous. Indeed, if $\psi_n \to \psi$ in $\mathcal{L}^q(0,T;\mathcal{H}^1(\Omega))$ as $n\to\infty$, then by the trace theorem, we have

$$\int_0^T ||\gamma_0\psi_n - \gamma_0\psi||^q_{\mathcal{H}^{\frac{1}{2}}(\partial\Omega))} \le C\int_0^T ||\psi_n - \psi||^q_{\mathcal{H}^1(\Omega))} \to 0 \quad \text{as } n\to\infty.$$

Therefore the operator γ is also weakly continuous [53]. As a result, we have

$$\psi_{1n}^{\epsilon} \rightharpoonup \psi_1^{\epsilon} \text{ in } \mathcal{L}^2(0,\mathrm{T};\mathcal{H}^1(\Omega)) \implies \gamma(\psi_{1\mathrm{n}}^{\epsilon}) \rightharpoonup \gamma(\psi_1^{\epsilon}) \text{ in } \mathcal{L}^2(0,\mathrm{T};\mathcal{H}^{\frac{1}{2}}(\partial\Omega)). \tag{3.39}$$

So by the weak convergence, and note that $\gamma(\varphi\tilde{\psi}) \in \mathcal{L}^2(0,T;\mathcal{L}^2(\partial\Omega)) \subset \mathcal{L}^2(0,T;\mathcal{H}^{-\frac{1}{2}}(\partial\Omega))$, we have

$$(IV)\ \left|\int_0^T \left\langle\, \left[\,\gamma_0(\psi_{1n}^{\epsilon}) - \gamma_0(\psi_1^{\epsilon})\,\right],\ \gamma_0(\varphi\tilde{\psi})\,\right\rangle_{\partial\Omega} dt\right| \to 0 \qquad \text{as } n \to \infty,$$

here $\langle\cdot,\ \cdot\rangle_{\partial\Omega}$ denotes the duality pairing on the boundary $\partial\Omega$.

The convergence of rest of the other terms can be derived either analogous to the proof in (I) or in (II). For instance, following the steps in (I), we get

$$\left|\int_0^T \int_\Omega \left[|\mathbf{A}_n^{\epsilon} + \mathbf{A}_c|^2\psi_{1n}^{\epsilon} - |\mathbf{A}^{\epsilon} + \mathbf{A}_c|^2\psi_1^{\epsilon}\right]\varphi\tilde{\psi}\ d\Omega dt\right| \to 0 \qquad \text{as } n \to \infty. \qquad \blacksquare$$

Lemma 3.2.14 *Given $\epsilon > 0$ and $T > 0$, let $(\psi_1^{\epsilon 1}, \psi_2^{\epsilon 1}, \mathbf{A}^{\epsilon 1})$ and $(\psi_1^{\epsilon 2}, \psi_2^{\epsilon 2}, \mathbf{A}^{\epsilon 2})$ be any two solutions of the problem* (WP$^{\epsilon}$)*; let $(\psi_1^{\epsilon 1}(\mathbf{x},0), \psi_2^{\epsilon 1}(\mathbf{x},0), \mathbf{A}^{\epsilon 1}(\mathbf{x},0)) = (\psi_{10}^1, \psi_{20}^1, \mathbf{A}_0^1)$ and $(\psi_1^{\epsilon 2}(\mathbf{x},0), \psi_2^{\epsilon 2}(\mathbf{x},0), \mathbf{A}^{\epsilon 2}(\mathbf{x},0)) = (\psi_{10}^2, \psi_{20}^2, \mathbf{A}_0^2)$ be any two corresponding initial conditions. Also let $\tilde{\psi}_1^{\epsilon} = \psi_1^{\epsilon 1} - \psi_1^{\epsilon 2}$, $\tilde{\psi}_2^{\epsilon} = \psi_2^{\epsilon 1} - \psi_2^{\epsilon 2}$, and $\tilde{\mathbf{A}}^{\epsilon} = \mathbf{A}^{\epsilon 1} - \mathbf{A}^{\epsilon 2}$, then for $t \in [0,T]$,*

$$\begin{aligned} &||\tilde{\psi}_1^{\epsilon}(t)||_0^2 + ||\tilde{\psi}_2^{\epsilon}(t)||_0^2 + ||\tilde{\mathbf{A}}^{\epsilon}(t)||_0^2 \\ &\qquad \le C(T)\left(||\psi_{10}^1 - \psi_{10}^2||_0^2 + ||\psi_{20}^1 - \psi_{20}^2||_0^2 + ||\mathbf{A}_0^1 - \mathbf{A}_0^2||_0^2\right), \end{aligned} \tag{3.40}$$

where the constant $C(T)$ is dependent of T but independent of ϵ.

Proof Let the weak form (3.9) of the problem (WP$^{\epsilon}$) be denoted as $G1((\psi_1^{\epsilon}, \psi_2^{\epsilon}, \mathbf{A}^{\epsilon});\ \tilde{\psi})$, then $G1((\psi_1^{\epsilon 1}, \psi_2^{\epsilon 1}, \mathbf{A}^{\epsilon 1});\ \tilde{\psi}_1^{\epsilon}) - G1((\psi_1^{\epsilon 2}, \psi_2^{\epsilon 2}, \mathbf{A}^{\epsilon 2});\ \tilde{\psi}_1^{\epsilon})$ gives

$$\begin{aligned} &\int_0^t \int_\Omega \Big\{ \frac{\partial\tilde{\psi}_1^{\epsilon}}{\partial t}\tilde{\psi}_1^{\epsilon *} + i\phi_c|\tilde{\psi}_1^{\epsilon}|^2 + \left((|\psi_1^{\epsilon 1}|^2 - \mathcal{T}_1)\psi_1^{\epsilon 1} - (|\psi_1^{\epsilon 2}|^2 - \mathcal{T}_1)\psi_1^{\epsilon 2}\right)\tilde{\psi}_1^{\epsilon *} \\ &\quad + (\frac{\xi_1}{x_0})^2|\nabla\tilde{\psi}_1^{\epsilon}|^2 + i\frac{\xi_1}{\lambda_1}(\mathbf{A}^{\epsilon 1} + \mathbf{A}_c)\cdot(\tilde{\psi}_1^{\epsilon *}\nabla\psi_1^{\epsilon 1} - \psi_1^{\epsilon 1}\nabla\tilde{\psi}_1^{\epsilon *}) \\ &\quad - i\frac{\xi_1}{\lambda_1}(\mathbf{A}^{\epsilon 2} + \mathbf{A}_c)\cdot(\tilde{\psi}_1^{\epsilon *}\nabla\psi_1^{\epsilon 2} - \psi_1^{\epsilon 2}\nabla\tilde{\psi}_1^{\epsilon *}) \\ &\quad + (\frac{x_0}{\lambda_1})^2|\mathbf{A}^{\epsilon 1} + \mathbf{A}_c|^2\psi_1^{\epsilon 1}\tilde{\psi}_1^{\epsilon *} - (\frac{x_0}{\lambda_1})^2|\mathbf{A}^{\epsilon 2} + \mathbf{A}_c|^2\psi_1^{\epsilon 2}\tilde{\psi}_1^{\epsilon *} + \eta\,\tilde{\psi}_2^{\epsilon}\tilde{\psi}_1^{\epsilon *}\Big\} d\Omega ds \\ &\quad + \int_0^t \int_{\partial\Omega} \gamma_1\frac{\xi_1^2}{x_0^2}|\tilde{\psi}_1^{\epsilon}|^2 d\partial\Omega ds = 0. \end{aligned}$$

Taking the real part of the above equation, we get for $t \in [0,T]$ that

$$
\begin{aligned}
&\frac{1}{2}\int_0^t \frac{d}{dt}||\tilde{\psi}_1^\epsilon||_0^2\,ds + (\frac{\xi_1}{x_0})^2\int_0^t ||\nabla\tilde{\psi}_1^\epsilon||_0^2\,ds \\
&\quad\le \int_0^t\int_\Omega \Re\Big\{ \big((|\psi_1^{\epsilon 2}|^2 - \mathcal{T}_1)\psi_1^{\epsilon 2} - (|\psi_1^{\epsilon 1}|^2 - \mathcal{T}_1)\psi_1^{\epsilon 1}\big)\,\tilde{\psi}_1^{\epsilon *} \\
&\quad - i\frac{\xi_1}{\lambda_1}(\mathbf{A}^{\epsilon 1} + \mathbf{A}_c)\cdot(\tilde{\psi}_1^{\epsilon *}\nabla\psi_1^{\epsilon 1} - \psi_1^{\epsilon 1}\nabla\tilde{\psi}_1^{\epsilon *}) \\
&\quad + i\frac{\xi_1}{\lambda_1}(\mathbf{A}^{\epsilon 2} + \mathbf{A}_c)\cdot(\tilde{\psi}_1^{\epsilon *}\nabla\psi_1^{\epsilon 2} - \psi_1^{\epsilon 2}\nabla\tilde{\psi}_1^{\epsilon *}) \\
&\quad - (\frac{x_0}{\lambda_1})^2|\mathbf{A}^{\epsilon 1} + \mathbf{A}_c|^2\psi_1^{\epsilon 1}\tilde{\psi}_1^{\epsilon *} + (\frac{x_0}{\lambda_1})^2|\mathbf{A}^{\epsilon 2} + \mathbf{A}_c|^2\psi_1^{\epsilon 2}\tilde{\psi}_1^{\epsilon *} - \,\eta\,\tilde{\psi}_2^\epsilon\tilde{\psi}_1^{\epsilon *}\Big\}d\Omega ds \\
&\quad\le \int_0^t\int_\Omega \Re\Big\{ -\big[(|\psi_1^{\epsilon 1}|^2 - \mathcal{T}_1)\tilde{\psi}_1^\epsilon + (\psi_1^{\epsilon 1 *}\tilde{\psi}_1^\epsilon + \psi_1^{\epsilon 2}\tilde{\psi}_1^{\epsilon *})\psi_1^{\epsilon 2}\big]\tilde{\psi}_1^{\epsilon *} \\
&\quad - i\frac{\xi_1}{\lambda_1}\tilde{\mathbf{A}}^\epsilon\cdot(\tilde{\psi}_1^{\epsilon *}\nabla\psi_1^{\epsilon 1} - \psi_1^{\epsilon 1}\nabla\tilde{\psi}_1^{\epsilon *}) - i\frac{\xi_1}{\lambda_1}(\mathbf{A}^{\epsilon 2} + \mathbf{A}_c)\cdot(\tilde{\psi}_1^{\epsilon *}\nabla\tilde{\psi}_1^\epsilon - \tilde{\psi}_1^\epsilon\nabla\tilde{\psi}_1^{\epsilon *}) \\
&\quad - (\frac{x_0}{\lambda_1})^2\big[\tilde{\mathbf{A}}^\epsilon\cdot(\mathbf{A}^{\epsilon 1} + \mathbf{A}^{\epsilon 2} + 2\mathbf{A}_c)\psi_1^{\epsilon 1}\tilde{\psi}_1^{\epsilon *} + |\mathbf{A}^{\epsilon 2} + \mathbf{A}_c|^2|\tilde{\psi}_1^\epsilon|^2\big] - \,\eta\,\tilde{\psi}_2^\epsilon\tilde{\psi}_1^{\epsilon *}\Big\}d\Omega ds \\
&\quad\le \int_0^t\Big\{|\mathcal{T}_1|\,||\tilde{\psi}_1^\epsilon||_0^2 + ||\psi_1^{\epsilon 1}||_{0,4}||\psi_1^{\epsilon 2}||_{0,4}||\tilde{\psi}_1^\epsilon||_{0,4}^2 + \,|\eta|\,||\tilde{\psi}_2^\epsilon||_0||\tilde{\psi}_1^\epsilon||_0 \\
&\quad + \frac{\xi_1}{\lambda_1}\Big[||\tilde{\mathbf{A}}^\epsilon||_{0,4}\,||\tilde{\psi}_1^\epsilon||_{0,4}\,||\nabla\psi_1^{\epsilon 1}||_0 + ||\tilde{\mathbf{A}}^\epsilon||_{0,4}\,||\psi_1^{\epsilon 1}||_{0,4}\,||\nabla\tilde{\psi}_1^\epsilon||_0 \\
&\quad + 2||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_{0,4}\,||\tilde{\psi}_1^\epsilon||_{0,4}\,||\nabla\tilde{\psi}_1^\epsilon||_0\Big] \\
&\quad + (\frac{x_0}{\lambda_1})^2||\tilde{\mathbf{A}}^\epsilon||_{0,4}\,||\mathbf{A}^{\epsilon 1} + \mathbf{A}^{\epsilon 2} + 2\mathbf{A}_c||_{0,4}\,||\psi_1^{\epsilon 1}||_{0,4}\,||\tilde{\psi}_1^\epsilon||_{0,4}\Big\}ds \\
&\quad\le \int_0^t\Big\{|\mathcal{T}_1|\,||\tilde{\psi}_1^\epsilon||_0^2 + ||\psi_1^{\epsilon 1}||_{0,4}||\psi_1^{\epsilon 2}||_{0,4}||\tilde{\psi}_1^\epsilon||_{0,4}^2 + \,\frac{|\eta|}{2}(||\tilde{\psi}_2^\epsilon||_0^2 + ||\tilde{\psi}_1^\epsilon||_0^2) \\
&\quad + \frac{\xi_1}{2\lambda_1}\Big[||\tilde{\mathbf{A}}^\epsilon||_{0,4}^2||\nabla\psi_1^{\epsilon 1}||_0^2 + \,||\tilde{\psi}_1^\epsilon||_{0,4}^2\, + \frac{1}{\varepsilon}||\tilde{\mathbf{A}}^\epsilon||_{0,4}^2\,||\psi_1^{\epsilon 1}||_{0,4}^2\, + \varepsilon||\nabla\tilde{\psi}_1^\epsilon||_0^2 \\
&\quad + \frac{1}{\varepsilon}||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_{0,4}^2\,||\tilde{\psi}_1^\epsilon||_{0,4}^2 + \,2\varepsilon||\nabla\tilde{\psi}_1^\epsilon||_0^2\Big] \\
&\quad + (\frac{x_0}{\lambda_1})^2\Big[||\tilde{\mathbf{A}}^\epsilon||_{0,4}^2\,(||\mathbf{A}^{\epsilon 1} + \mathbf{A}_c||_{0,4}^2 + ||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_{0,4}^2)\,||\psi_1^{\epsilon 1}||_{0,4}^2 + \,||\tilde{\psi}_1^\epsilon||_{0,4}^2\Big]\Big\}ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_0^t \Big\{ |\mathcal{T}_1|\,||\tilde{\psi}_1^\epsilon||_0^2 + ||\psi_1^{\epsilon 1}||_{0,4}||\psi_1^{\epsilon 2}||_{0,4}\big(C_{\varepsilon'}||\tilde{\psi}_1^\epsilon||_0^2 + \varepsilon'||\nabla\tilde{\psi}_1^\epsilon||_0^2\big) + \frac{|\eta|}{2}(||\tilde{\psi}_2^\epsilon||_0^2 + ||\tilde{\psi}_1^\epsilon||_0^2) \\
&\quad + \frac{\xi_1}{2\lambda_1}\Big[\big(||\nabla\psi_1^{\epsilon 1}||_0^2 + \frac{1}{\varepsilon}\,||\psi_1^{\epsilon 1}||_{0,4}^2\big)\,\big(D_{\varepsilon''}||\tilde{\mathbf{A}}^\epsilon||_0^2 + \varepsilon''||\nabla\tilde{\mathbf{A}}^\epsilon||_0^2\big) \\
&\quad + \big(1 + \frac{1}{\varepsilon}||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_{0,4}^2\big)\,\big(C_{\varepsilon'}||\tilde{\psi}_1^\epsilon||_0^2 + \varepsilon'||\nabla\tilde{\psi}_1^\epsilon||_0^2\big) + 3\varepsilon||\nabla\tilde{\psi}_1^\epsilon||_0^2\Big] \\
&\quad + \big(\frac{x_0}{\lambda_1}\big)^2\Big[(||\mathbf{A}^{\epsilon 1} + \mathbf{A}_c||_{0,4}^2 + ||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_{0,4}^2)\,||\psi_1^{\epsilon 1}||_{0,4}^2\big(D_{\varepsilon''}||\tilde{\mathbf{A}}^\epsilon||_0^2 + \varepsilon''||\nabla\tilde{\mathbf{A}}^\epsilon||_0^2\big) \\
&\quad + \big(C_{\varepsilon'}||\tilde{\psi}_1^\epsilon||_0^2 + \varepsilon'||\nabla\tilde{\psi}_1^\epsilon||_0^2\big)\Big]\Big\} ds.
\end{aligned}
$$

In the last inequality, we have used the following Sobolev inequality

$$
\begin{aligned}
||u||_{0,4}^2 &\le C||u||_0\,||u||_1 \\
&= C||u||_0\,(||u||_0^2 + ||\nabla u||_0^2)^{\frac{1}{2}} \\
&\le C^2\frac{1}{2\varepsilon}||u||_0^2 + \frac{\varepsilon}{2}\,(||u||_0^2 + ||\nabla u||_0^2) \\
&\le C_\varepsilon||u||_0^2 + \frac{\varepsilon}{2}||\nabla u||_0^2.
\end{aligned}
$$

Rearranging the terms gives

$$
\begin{aligned}
&\frac{1}{2}\int_0^t \frac{d}{dt}||\tilde{\psi}_1^\epsilon||_0^2\,ds \\
&\quad + \int_0^t \Big\{\Big[\big(\frac{\xi_1}{x_0}\big)^2 - \varepsilon'||\psi_1^{\epsilon 1}||_1||\psi_1^{\epsilon 2}||_1 \\
&\quad - \frac{\xi_1}{2\lambda_1}\Big(3\varepsilon + \varepsilon'\big(1 + \frac{1}{\varepsilon}||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_1^2\big)\Big) - \varepsilon'\big(\frac{x_0}{\lambda_1}\big)^2\Big]\,||\nabla\tilde{\psi}_1^\epsilon||_0^2 \\
&\quad - \varepsilon''\Big[\frac{\xi_1}{2\lambda_1}\big(||\nabla\psi_1^{\epsilon 1}||_0^2 + \frac{1}{\varepsilon}\,||\psi_1^{\epsilon 1}||_1^2\big) \\
&\quad + \big(\frac{x_0}{\lambda_1}\big)^2(||\mathbf{A}^{\epsilon 1} + \mathbf{A}_c||_1^2 + ||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_1^2)\,||\psi_1^{\epsilon 1}||_1^2\Big]||\nabla\tilde{\mathbf{A}}^\epsilon||_0^2\Big\} ds \\
&\le \int_0^t \Big\{\Big[|\mathcal{T}_1| + \frac{|\eta|}{2} + C_{\varepsilon'}\Big(||\psi_1^{\epsilon 1}||_1||\psi_1^{\epsilon 2}||_1 + \frac{\xi_1}{2\lambda_1}\big(1 + \frac{1}{\varepsilon}||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_1^2\big) + \big(\frac{x_0}{\lambda_1}\big)^2\Big)\Big]||\tilde{\psi}_1^\epsilon||_0^2 \\
&\quad + D_{\varepsilon''}\Big[\frac{\xi_1}{2\lambda_1}\big(||\nabla\psi_1^{\epsilon 1}||_0^2 + \frac{1}{\varepsilon}\,||\psi_1^{\epsilon 1}||_1^2\big) + \big(\frac{x_0}{\lambda_1}\big)^2(||\mathbf{A}^{\epsilon 1} + \mathbf{A}_c||_1^2 \\
&\quad + ||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_1^2)\,||\psi_1^{\epsilon 1}||_1^2\Big]||\tilde{\mathbf{A}}^\epsilon||_0^2 + \frac{|\eta|}{2}||\tilde{\psi}_2^\epsilon||_0^2\Big\} ds.
\end{aligned}
\tag{3.41}
$$

Similarly, for $\tilde{\psi}_2^\epsilon$, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^t \frac{d}{dt}||\tilde{\psi}_2^\epsilon||_0^2\,ds \\
&\quad + \int_0^t \Bigg\{ \Bigg[\big(\frac{\xi_2}{x_0}\big)^2 - \varepsilon'||\psi_2^{\epsilon 1}||_1||\psi_2^{\epsilon 2}||_1 - \nu\frac{\xi_2}{2\lambda_2}\Bigg(3\varepsilon + \varepsilon'\big(1+\frac{1}{\varepsilon}||\mathbf{A}^{\epsilon 2}+\mathbf{A}_c||_1^2\big)\Bigg) \\
&\qquad - \varepsilon'\big(\nu\frac{x_0}{\lambda_2}\big)^2\Bigg]\,||\nabla\tilde{\psi}_2^\epsilon||_0^2 \\
&\qquad - \varepsilon''\Bigg[\nu\frac{\xi_2}{2\lambda_2}\big(||\nabla\psi_2^{\epsilon 1}||_0^2 + \frac{1}{\varepsilon}\,||\psi_2^{\epsilon 1}||_1^2\big) \\
&\qquad + \big(\nu\frac{x_0}{\lambda_2}\big)^2(||\mathbf{A}^{\epsilon 1}+\mathbf{A}_c||_1^2 + ||\mathbf{A}^{\epsilon 2}+\mathbf{A}_c||_1^2)\,||\psi_2^{\epsilon 1}||_1^2\Bigg]||\nabla\tilde{\mathbf{A}}^\epsilon||_0^2\Bigg\}ds \\
&\le \int_0^t \Bigg\{ \Bigg[|\mathcal{T}_2| + \nu^2\frac{|\eta|}{2} + C_{\varepsilon'}\Bigg(||\psi_2^{\epsilon 1}||_1||\psi_2^{\epsilon 2}||_1 + \nu\frac{\xi_2}{2\lambda_2}\big(1+\frac{1}{\varepsilon}||\mathbf{A}^{\epsilon 2}+\mathbf{A}_c||_1^2\big) \ + \big(\nu\frac{x_0}{\lambda_2}\big)^2\Bigg)\Bigg]||\tilde{\psi}_2^\epsilon||_0^2 \\
&\quad + D_{\varepsilon''}\Bigg[\nu\frac{\xi_2}{2\lambda_2}\big(||\nabla\psi_2^{\epsilon 1}||_0^2 + \frac{1}{\varepsilon}\,||\psi_2^{\epsilon 1}||_1^2\big) \ + \big(\nu\frac{x_0}{\lambda_1}\big)^2(||\mathbf{A}^{\epsilon 1}+\mathbf{A}_c||_1^2 \\
&\quad + ||\mathbf{A}^{\epsilon 2}+\mathbf{A}_c||_1^2)\,||\psi_2^{\epsilon 1}||_1^2\Bigg]||\tilde{\mathbf{A}}^\epsilon||_0^2 + \nu^2\frac{|\eta|}{2}||\tilde{\psi}_1^\epsilon||_0^2\Bigg\}ds. \qquad (3.42)
\end{aligned}$$

As for $\tilde{\mathbf{A}}^\epsilon$, first let the weak form (3.11) of the problem (WP^ϵ) be denoted as $G2((\psi_1^\epsilon, \psi_2^\epsilon, \mathbf{A}^\epsilon);\, \tilde{\mathbf{A}})$, then $G2((\psi_1^{\epsilon 1}, \psi_2^{\epsilon 1}, \mathbf{A}^{\epsilon 1});\, \tilde{\mathbf{A}}^\epsilon) - G2((\psi_1^{\epsilon 2}, \psi_2^{\epsilon 2}, \mathbf{A}^{\epsilon 2});\, \tilde{\mathbf{A}}^\epsilon)$ gives

$$\begin{aligned}
&\sigma\frac{x_0^2}{2\lambda_1^2}\int_0^t \frac{d}{dt}||\tilde{\mathbf{A}}^\epsilon||_0^2\,ds + \int_0^t \big[\,\epsilon||\mathrm{div}\tilde{\mathbf{A}}^\epsilon||_0^2 + ||\mathrm{curl}\tilde{\mathbf{A}}^\epsilon||_0^2\big]\,ds \\
&\quad + \int_0^t \Bigg\{\Re\Bigg(i\frac{1}{\kappa_1}(\nabla\psi_1^{\epsilon 1}\psi_1^{\epsilon 1*} - \nabla\psi_1^{\epsilon 2}\psi_1^{\epsilon 2*}),\ \tilde{\mathbf{A}}^\epsilon\Bigg) \\
&\quad + \frac{x_0^2}{\lambda_1^2}\Big(|\psi_1^{\epsilon 1}|^2(\mathbf{A}^{\epsilon 1}+\mathbf{A}_c) - |\psi_1^{\epsilon 2}|^2(\mathbf{A}^{\epsilon 2}+\mathbf{A}_c),\ \tilde{\mathbf{A}}^\epsilon\Big) \\
&\quad + \Re\Bigg(i\frac{1}{\nu}\frac{1}{\kappa_2}(\nabla\psi_2^{\epsilon 1}\psi_2^{\epsilon 1*} - \nabla\psi_2^{\epsilon 2}\psi_2^{\epsilon 2*}),\ \tilde{\mathbf{A}}^\epsilon\Bigg) \\
&\quad + \frac{x_0^2}{\lambda_2^2}\Big(|\psi_2^{\epsilon 1}|^2(\mathbf{A}^{\epsilon 1}+\mathbf{A}_c) - |\psi_2^{\epsilon 2}|^2(\mathbf{A}^{\epsilon 2}+\mathbf{A}_c),\ \tilde{\mathbf{A}}^\epsilon\Big)\Bigg\}ds = 0.
\end{aligned}$$

This gives

$$
\begin{aligned}
&\sigma\frac{x_0^2}{2\lambda_1^2}\int_0^t \frac{d}{dt}||\tilde{\mathbf{A}}^\epsilon||_0^2\,ds + \int_0^t \big[\,\epsilon||\mathrm{div}\tilde{\mathbf{A}}^\epsilon||_0^2 + ||\mathrm{curl}\tilde{\mathbf{A}}^\epsilon||_0^2\big]ds \\
&\leq \int_0^t \Bigg\{ -\Re\left(i\frac{1}{\kappa_1}(\nabla\psi_1^{\epsilon 1}\tilde{\psi}_1^{\epsilon *} + \nabla\tilde{\psi}_1^{\epsilon}\psi_1^{\epsilon 2*}),\ \tilde{\mathbf{A}}^\epsilon\right) \\
&\quad - \frac{x_0^2}{\lambda_1^2}\Big(|\psi_1^{\epsilon 1}|^2\tilde{\mathbf{A}}^\epsilon + (|\psi_1^{\epsilon 1}|^2 - |\psi_1^{\epsilon 2}|^2)(\mathbf{A}^{\epsilon 2} + \mathbf{A}_c),\ \tilde{\mathbf{A}}^\epsilon\Big) \\
&\quad - \Re\left(i\frac{1}{\nu}\frac{1}{\kappa_2}(\nabla\psi_2^{\epsilon 1}\tilde{\psi}_2^{\epsilon *} - \nabla\tilde{\psi}_2^{\epsilon}\psi_2^{\epsilon 2*}),\ \tilde{\mathbf{A}}^\epsilon\right) \\
&\quad - \frac{x_0^2}{\lambda_2^2}\Big(|\psi_2^{\epsilon 1}|^2\tilde{\mathbf{A}}^\epsilon + (|\psi_2^{\epsilon 1}|^2 - |\psi_2^{\epsilon 2}|^2)(\mathbf{A}^{\epsilon 2} + \mathbf{A}_c),\ \tilde{\mathbf{A}}^\epsilon\Big)\Bigg\}ds \\
&\leq \int_0^t \Bigg\{ \frac{1}{\kappa_1}\Big[||\nabla\psi_1^{\epsilon 1}||_0||\tilde{\psi}_1^{\epsilon}||_{0,4} + ||\nabla\tilde{\psi}_1^{\epsilon}||_0\,||\psi_1^{\epsilon 2}||_{0,4}\Big]\,||\tilde{\mathbf{A}}^\epsilon||_{0,4} \\
&\quad + \frac{x_0^2}{\lambda_1^2}||\tilde{\psi}_1^{\epsilon}||_{0,4}\big(||\psi_1^{\epsilon 1}||_{0,4} + ||\psi_1^{\epsilon 2}||_{0,4}\big)||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_{0,4}\,||\tilde{\mathbf{A}}^\epsilon||_{0,4} \\
&\quad + \frac{1}{\nu}\frac{1}{\kappa_2}\Big[||\nabla\psi_2^{\epsilon 1}||_0||\tilde{\psi}_2^{\epsilon}||_{0,4} + ||\nabla\tilde{\psi}_2^{\epsilon}||_0\,||\psi_2^{\epsilon 2}||_{0,4}\Big]\,||\tilde{\mathbf{A}}^\epsilon||_{0,4} \\
&\quad + \frac{x_0^2}{\lambda_2^2}||\tilde{\psi}_2^{\epsilon}||_{0,4}\big(||\psi_2^{\epsilon 1}||_{0,4} + ||\psi_2^{\epsilon 2}||_{0,4}\big)||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_{0,4}\,||\tilde{\mathbf{A}}^\epsilon||_{0,4}\Bigg\}ds \\
&\leq \int_0^t \Bigg\{ \frac{1}{2\kappa_1}\Big[||\nabla\psi_1^{\epsilon 1}||_0^2\left(D_{\varepsilon''}||\tilde{\mathbf{A}}^\epsilon||_0^2 + \varepsilon''||\nabla\tilde{\mathbf{A}}^\epsilon||_0^2\right) + \left(C_{\varepsilon'}||\tilde{\psi}_1^{\epsilon}||_0^2 + \varepsilon'||\nabla\tilde{\psi}_1^{\epsilon}||_0^2\right) \\
&\quad + \frac{1}{\varepsilon}||\psi_1^{\epsilon 2}||_1^2\left(D_{\varepsilon''}||\tilde{\mathbf{A}}^\epsilon||_0^2 + \varepsilon''||\nabla\tilde{\mathbf{A}}^\epsilon||_0^2\right) + \varepsilon||\nabla\tilde{\psi}_1^{\epsilon}||_0^2)\Big] \\
&\quad + \frac{x_0^2}{\lambda_1^2}\Big[\big(||\psi_1^{\epsilon 1}||_1^2 + ||\psi_1^{\epsilon 2}||_1^2\big)||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_1^2\left(D_{\varepsilon''}||\tilde{\mathbf{A}}^\epsilon||_0^2 + \varepsilon''||\nabla\tilde{\mathbf{A}}^\epsilon||_0^2\right) \\
&\quad + \left(C_{\varepsilon'}||\tilde{\psi}_1^{\epsilon}||_0^2 + \varepsilon'||\nabla\tilde{\psi}_1^{\epsilon}||_0^2\right)\Big] \\
&\quad + \frac{1}{\nu}\frac{1}{2\kappa_2}\Big[||\nabla\psi_2^{\epsilon 1}||_0^2\left(D_{\varepsilon''}||\tilde{\mathbf{A}}^\epsilon||_0^2 + \varepsilon''||\nabla\tilde{\mathbf{A}}^\epsilon||_0^2\right) + \left(C_{\varepsilon'}||\tilde{\psi}_2^{\epsilon}||_0^2 + \varepsilon'||\nabla\tilde{\psi}_2^{\epsilon}||_0^2\right) \\
&\quad + \frac{1}{\varepsilon}||\psi_2^{\epsilon 2}||_1^2\left(D_{\varepsilon''}||\tilde{\mathbf{A}}^\epsilon||_0^2 + \varepsilon''||\nabla\tilde{\mathbf{A}}^\epsilon||_0^2\right) + \varepsilon||\nabla\tilde{\psi}_2^{\epsilon}||_0^2)\Big] \\
&\quad + \frac{x_0^2}{\lambda_2^2}\Big[\big(||\psi_2^{\epsilon 1}||_1^2 + ||\psi_2^{\epsilon 2}||_1^2\big)||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_1^2\left(D_{\varepsilon''}||\tilde{\mathbf{A}}^\epsilon||_0^2 + \varepsilon''||\nabla\tilde{\mathbf{A}}^\epsilon||_0^2\right) \\
&\quad + \left(C_{\varepsilon'}||\tilde{\psi}_2^{\epsilon}||_0^2 + \varepsilon'||\nabla\tilde{\psi}_2^{\epsilon}||_0^2\right)\Big]\Bigg\}ds.
\end{aligned}
$$

Rearranging the terms gives

$$
\begin{aligned}
&\sigma\frac{x_0^2}{2\lambda_1^2}\int_0^t \frac{d}{dt}||\tilde{\mathbf{A}}^\epsilon||_0^2\,ds + \int_0^t \Bigg\{\Bigg[\epsilon C - \varepsilon''\Bigg(\frac{1}{2\kappa_1}\big(||\nabla\psi_1^{\epsilon 1}||_0^2 + \frac{1}{\varepsilon}||\psi_1^{\epsilon 2}||_1^2\big) \\
&\quad + \frac{x_0^2}{\lambda_1^2}\big(||\psi_1^{\epsilon 1}||_1^2 + ||\psi_1^{\epsilon 2}||_1^2\big)||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_1^2 + \frac{1}{\nu}\frac{1}{2\kappa_2}\big(||\nabla\psi_2^{\epsilon 1}||_0^2 + \frac{1}{\varepsilon}||\psi_2^{\epsilon 2}||_1^2\big) \\
&\quad + \frac{x_0^2}{\lambda_2^2}\big(||\psi_2^{\epsilon 1}||_1^2 + ||\psi_2^{\epsilon 2}||_1^2\big)||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_1^2\Bigg)\Bigg]||\nabla\tilde{\mathbf{A}}^\epsilon||_0^2 \\
&\quad - \big(\frac{1}{2\kappa_1}(\varepsilon' + \varepsilon) + \frac{x_0^2}{\lambda_1^2}\varepsilon'\big)||\nabla\tilde{\psi}_1^\epsilon||_0^2 - \big(\frac{1}{\nu}\frac{1}{2\kappa_2}(\varepsilon' + \varepsilon) + \frac{x_0^2}{\lambda_2^2}\varepsilon'\big)||\nabla\tilde{\psi}_2^\epsilon||_0^2\Bigg\}ds \\
&\le \int_0^t \Bigg\{D_{\varepsilon''}\Bigg[\frac{1}{2\kappa_1}\big(||\nabla\psi_1^{\epsilon 1}||_0^2 + \frac{1}{\varepsilon}||\psi_1^{\epsilon 2}||_1^2\big) + \frac{x_0^2}{\lambda_1^2}\big(||\psi_1^{\epsilon 1}||_1^2 + ||\psi_1^{\epsilon 2}||_1^2\big)||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_1^2 \\
&\quad + \frac{1}{\nu}\frac{1}{2\kappa_2}\big(||\nabla\psi_2^{\epsilon 1}||_0^2 + \frac{1}{\varepsilon}||\psi_2^{\epsilon 2}||_1^2\big) + \frac{x_0^2}{\lambda_2^2}\big(||\psi_2^{\epsilon 1}||_1^2 + ||\psi_2^{\epsilon 2}||_1^2\big)||\mathbf{A}^{\epsilon 2} + \mathbf{A}_c||_1^2\Bigg]||\tilde{\mathbf{A}}^\epsilon||_0^2 \\
&\quad + C_{\varepsilon'}\Bigg[\big(\frac{1}{2\kappa_1} + \frac{x_0^2}{\lambda_1^2}\big)||\tilde{\psi}_1^\epsilon||_0^2 + \big(\frac{1}{\nu}\frac{1}{2\kappa_2} + \frac{x_0^2}{\lambda_2^2}\big)||\tilde{\psi}_2^\epsilon||_0^2\Bigg]\Bigg\}ds. \qquad (3.43)
\end{aligned}
$$

In the left hand side (L.H.S.) of the last inequality, we have used the following inequality which is from (2.23)

$$\epsilon C||\nabla\tilde{\mathbf{A}}^\epsilon||_0^2 \;\le\; \epsilon||\mathrm{div}\tilde{\mathbf{A}}^\epsilon||_0^2 + ||\mathrm{curl}\tilde{\mathbf{A}}^\epsilon||_0^2 \qquad \forall \mathbf{A} \in \mathbf{H}_n^1(\Omega).$$

Now choose ε, ε' and ε'' small enough such that the integrands in the L.H.S. of the inequalities (3.41), (3.42) and (3.43) are all positive, then combining these three inequalities, we get for $t \in [0, T]$ that

$$
\begin{aligned}
&\frac{1}{2}\Bigg[||\tilde{\psi}_1^\epsilon(t)||_0^2 + ||\tilde{\psi}_2^\epsilon(t)||_0^2 + \sigma\frac{x_0^2}{2\lambda_1^2}||\tilde{\mathbf{A}}^\epsilon(t)||_0^2\Bigg] \\
&\quad + \int_0^t \Big\{C_1'||\nabla\tilde{\psi}_1^\epsilon(t)||_0^2 + C_2'||\nabla\tilde{\psi}_2^\epsilon(t)||_0^2 + C_\epsilon'||\nabla\tilde{\mathbf{A}}^\epsilon(t)||_0^2\Big\}ds \\
&\le \frac{1}{2}\Bigg[||\tilde{\psi}_1^\epsilon(0)||_0^2 + ||\tilde{\psi}_2^\epsilon(0)||_0^2 + \sigma\frac{x_0^2}{2\lambda_1^2}||\tilde{\mathbf{A}}^\epsilon(0)||_0^2\Bigg] \\
&\quad + \int_0^t \Big\{C_3'||\tilde{\psi}_1^\epsilon(t)||_0^2 + C_4'||\tilde{\psi}_2^\epsilon(t)||_0^2 + C_5'||\tilde{\mathbf{A}}^\epsilon(t)||_0^2\Big\}ds, \qquad (3.44)
\end{aligned}
$$

where all the constants are greater than zero and bounded independent of time. They are bounded independent of time because all the norms appearing in these constants involve only $||\psi_i^{\epsilon j}||_m$ and/or $||\mathbf{A}^{\epsilon j}||_m$, where $i, j, m = 0, 1$; and recall that $\psi_1^\epsilon,\ \psi_2^\epsilon \in \mathcal{L}^\infty(0, T; \mathcal{H}^1(\Omega))$ and $\mathbf{A}^\epsilon \in \mathbf{L}^\infty(0, T; \mathbf{H}^1(\Omega))$. Also note that all these constants, except C_ε', are implicitly dependent on ϵ, this is because the bound of the norm $||\mathbf{A}^\epsilon||_{\mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega))}$ is dependent of ϵ. In theorem 3.2.22, we will show that this bound is actually independent of ϵ, thus the constants on the R.H.S. of the inequality (3.44) are actually independent of ϵ. This fact is reflected in the statement of this lemma. Applying Gronwall's inequality in integral form to the inequality (3.44) gives the continuous data dependency of the solution $(\psi_1^\epsilon, \psi_2^\epsilon, \mathbf{A}^\epsilon)$ on the initial data, as expressed in (3.40). ∎

By setting $(\psi_{10}^1, \psi_{20}^1, \mathbf{A}_0^1) = (\psi_{10}^2, \psi_{20}^2, \mathbf{A}_0^2)$, we conclude that all weakly (and weakly*) convergence subsequences have the same limit. This gives us the following theorem.

Theorem 3.2.15 *Given $\epsilon > 0$ and $T > 0$, the sequence $\{(\psi_{1n}^\epsilon, \psi_{2n}^\epsilon, \mathbf{A}_n^\epsilon)\}$ converges weakly (and weakly*) in $\mathcal{V} \times \mathcal{V} \times \mathbf{V}$ to the unique solution $(\psi_1^\epsilon, \psi_2^\epsilon, \mathbf{A}^\epsilon)$ of the problem* (WP$^\epsilon$).

We want to stress that the uniqueness of $(\psi_1^\epsilon, \psi_2^\epsilon, \mathbf{A}^\epsilon)$ depends on the choice of a particular $\mathbf{A}_c$ which in turn depends on the choice of $\mathbf{j}_c$. Recall that $\mathbf{j}_c$ as a divergence free lifting function of $\mathbf{j}_a \cdot \mathbf{n}|_{\partial\Omega}$ is not unique.

Next we state that the solutions ψ_1^ϵ and ψ_2^ϵ satisfy the "maximum" principle.

Theorem 3.2.16 *For any $\epsilon > 0$ and $T > 0$, let $(\psi_1^\epsilon, \psi_2^\epsilon, \mathbf{A}^\epsilon)$ be a solution of the problem* (WP$^\epsilon$), *and let $\Upsilon = \max\{|\mathcal{T}_1|, |\mathcal{T}_2|\}$, $\eta^* = \max\{|\eta|, |\eta|\nu^2\}$ and $a = \sqrt{4\eta^* + \Upsilon}$. Suppose $|\psi_1^\epsilon(\mathbf{x}, 0)| = |\psi_{10}| \le a$ and $|\psi_2^\epsilon(\mathbf{x}, 0)| = |\psi_{20}| \le a$ a.e. in Ω, then $|\psi_1^\epsilon(\mathbf{x}, t)| \le a$ and $|\psi_2^\epsilon(\mathbf{x}, t)| \le a$ a.e. in $\Omega \times [0, T]$.*

Proof Let $\psi_1 = f_1 \exp(i\theta_1)$ and $\psi_2 = f_2 \exp(i\theta_2)$, where $f_1 = |\psi_1|$ and $f_2 = |\psi_2|$, and θ_1 and θ_2 are the phases of ψ_1 and ψ_2, respectively. Then we have

$$\begin{aligned}
&\Re\left\{\left(\frac{\partial\psi_1^\epsilon}{\partial t} + i\,\phi_a\psi_1^\epsilon\right) + \left(|\psi_1^\epsilon|^2 - \mathcal{T}_1\right)\psi_1^\epsilon + \left(-i\frac{\xi_1}{x_0}\nabla - \frac{x_0}{\lambda_1}\mathbf{A}\right)^2\psi_1^\epsilon + \eta\psi_2^\epsilon\right\} \\
&\quad = (f_1)_t - \frac{\xi_1}{x_0}\triangle f_1 + |\frac{x_0}{\lambda_1}\mathbf{A} - \frac{\xi_1}{x_0}\nabla\theta_1|^2 f_1 + (f_1^2 - 1)f_1 + \eta f_2\cos(\theta_2 - \theta_1) \\
&\quad = 0 \qquad \text{in } \Omega \times (0, \mathrm{T}),
\end{aligned} \tag{3.45}$$

$$\begin{aligned}
&\Re\left\{\Gamma\left(\frac{\partial\psi_2^\epsilon}{\partial t} + i\,\phi_a\psi_2^\epsilon\right) + \left(|\psi_2^\epsilon|^2 - \mathcal{T}_2\right)\psi_2^\epsilon + \left(-i\frac{\xi_2}{x_0}\nabla - \nu\frac{x_0}{\lambda_2}\mathbf{A}\right)^2\psi_2^\epsilon + \eta\nu^2\psi_1\right\} \\
&\quad = \Gamma(f_2)_t - \frac{\xi_2}{x_0}\triangle f_2 + |\nu\frac{x_0}{\lambda_2}\mathbf{A} - \frac{\xi_2}{x_0}\nabla\theta_2|^2 f_2 + (f_2^2 - 1)f_2 + \eta\nu^2 f_1\cos(\theta_2 - \theta_1) \\
&\quad = 0 \qquad \text{in } \Omega \times (0, \mathrm{T}).
\end{aligned} \tag{3.46}$$

And f_1 and f_2 also satisfy the real parts of the boundary and initial conditions, namely, the following boundary and initial conditions:

$$\nabla f_1 \cdot \mathbf{n} = -\gamma_1 f_1 \qquad \text{on } \partial\Omega \times (0, \mathrm{T}), \tag{3.47}$$

$$\nabla f_2 \cdot \mathbf{n} = -\gamma_2 f_2 \qquad \text{on } \partial\Omega \times (0, \mathrm{T}), \tag{3.48}$$

$$f_1(x, 0) = |\psi_1^\epsilon(x, 0)| \qquad \text{in } \Omega, \tag{3.49}$$

$$f_2(x, 0) = |\psi_2^\epsilon(x, 0)| \qquad \text{in } \Omega. \tag{3.50}$$

Recall that $\gamma_i \in L^\infty(\partial\Omega)$ and $\gamma_i(\mathbf{x}) \geq 0$ on $\partial\Omega$.

Assume momentarily that a, with $a \geq 1$, is an unknown constant to be determined. Define time-independent subdomain $\Omega_+ = \{f_1 > a \text{ or } f_2 > a\}$ for almost all $t \in [0, T]$, and define time-dependent subdomain $\Omega_+^t = \{f_1(t) > a \text{ or } f_2(t) > a\}$ for a specific $t \in [0, T]$. Then for a specific $t \in [0, T]$, $\Omega_+^t = \Omega_1^t \cup \Omega_2^t \cup \Omega_3^t$, where $\Omega_1^t = \{f_1(t) > a \text{ and } f_2(t) > a\}$, $\Omega_2^t = \{f_1(t) > a \text{ and } f_2(t) \leq a\}$ and $\Omega_3^t = \{f_1(t) \leq a \text{ and } f_2(t) > a\}$.

We want to find an appropriate value of a such that the measure $|\Omega_+| = 0$. Multiplying both sides of equations (3.45) and (3.46) by $(f_1 - a)_+$ and $(f_2 - a)_+$ respectively, then integrating over Ω, we get

$$\begin{aligned}
0 &= \frac{1}{2}\frac{d}{dt}\int_\Omega ((f_1 - a)_+)^2 d\Omega + \frac{\xi_1}{x_0}\int_\Omega |\nabla (f_1 - a)_+|^2 d\Omega \\
&\quad + \int_\Omega |\frac{x_0}{\lambda_1}\mathbf{A} - \frac{\xi_1}{x_0}\nabla\theta_1|^2 f_1 (f_1 - a)_+ d\Omega \\
&\quad + \int_\Omega \left[(f_1^2 - \mathcal{T}_1) f_1 + \eta f_2 \cos(\theta_2 - \theta_1)\right] (f_1 - a)_+ d\Omega \\
&\quad - \int_{\partial\Omega} \nabla f_1 (f_1 - a)_+ \cdot \mathbf{n}\, dS \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega ((f_1 - a)_+)^2 d\Omega + \frac{\xi_1}{x_0}\int_\Omega |\nabla (f_1 - a)_+|^2 d\Omega \\
&\quad + \int_\Omega |\frac{x_0}{\lambda_1}\mathbf{A} - \frac{\xi_1}{x_0}\nabla\theta_1|^2 f_1 (f_1 - a)_+ d\Omega \\
&\quad + \int_\Omega \left[(f_1^2 - \mathcal{T}_1) f_1 + \eta f_2 \cos(\theta_2 - \theta_1)\right] (f_1 - a)_+ d\Omega \\
&\quad + \int_{\partial\Omega_N} \gamma_1 f_1 (f_1 - a)_+\, dS, \text{ by boundary condition (3.47).}
\end{aligned} \tag{3.51}$$

And similarly,

$$
\begin{aligned}
0 \;=\;& \frac{1}{2}\frac{d}{dt}\int_\Omega \Gamma((f_2-a)_+)^2 d\Omega + \frac{\xi_2}{x_0}\int_\Omega |\nabla (f_2-a)_+|^2 d\Omega \\
&+\int_\Omega |\nu\frac{x_0}{\lambda_2}\mathbf{A} - \frac{\xi_2}{x_0}\nabla\theta_2|^2 f_2(f_2-a)_+ d\Omega \\
&+\int_\Omega \left[(f_2^2-\mathcal{T}_2)f_2 + \eta\nu^2 f_1\cos(\theta_2-\theta_1)\right](f_2-a)_+ d\Omega \\
&+\int_{\partial\Omega_N} \gamma_2 f_2(f_2-a)_+\, dS,\ \text{by boundary condition (3.48).} \qquad (3.52)
\end{aligned}
$$

Then by adding equations (3.51) and (3.52), dropping some positive terms including the boundary integrals, and replacing $\cos(\theta_2-\theta_1)$ with -1, we get

$$
\begin{aligned}
0 \;\geq\;& \frac{1}{2}\frac{d}{dt}\int_\Omega \left[((f_1-a)_+)^2 + \Gamma((f_2-a)_+)^2\right] d\Omega \\
&+\int_\Omega \left[(f_1^2-\mathcal{T}_1)f_1(f_1-a)_+ + (f_2^2-\mathcal{T}_2)f_2(f_2-a)_+\right] d\Omega \\
&-\int_\Omega \eta^*\left[f_2(f_1-a)_+ + f_1(f_2-a)_+\right] d\Omega \qquad (3.53)
\end{aligned}
$$

For the integrand in the second term of the above inequality (3.53), we have

$$
\sum_{\mathrm{i}=1}^{2}(f_i^2-\mathcal{T}_i)f_i(f_i-a)_+ \;\geq\; \sum_{\mathrm{i}=1}^{2}(f_i^2-\Upsilon)f_i(f_i-a)_+. \qquad (3.54)
$$

Applying the Chebyshev's sum inequality $\frac{1}{n}\sum_{\mathrm{i}=1}^n a_i b_i \geq (\frac{1}{n}\sum_{\mathrm{i}=1}^n a_i)(\frac{1}{n}\sum_{\mathrm{i}=1}^n b_i)$ twice to the above inequality (3.54), we get

$$
\sum_{\mathrm{i}=1}^{2}(f_i^2-\mathcal{T}_i)f_i(f_i-a)_+ \;\geq\; \frac{1}{4}\left[\sum_{\mathrm{i}=1}^{2}(f_i^2-\Upsilon)\right]\left[\sum_{\mathrm{i}=1}^{2} f_i\right]\left[\sum_{\mathrm{i}=1}^{2}(f_i-a)_+\right]. \qquad (3.55)
$$

For the integrand in the third term of the inequality (3.53), we get

$$
\begin{aligned}
-\eta^*\left[f_2(f_1-a)_+ + f_1(f_2-a)_+\right] \;&\geq\; -\eta^*\left[f_1^2+f_2^2\right]^{\frac{1}{2}}\left[((f_1-a)_+)^2 + ((f_2-a)_+)^2\right]^{\frac{1}{2}} \\
&\geq\; -2\eta^*\left[f_1+f_2\right]\left[(f_1-a)_+ + (f_2-a)_+\right]. \qquad (3.56)
\end{aligned}
$$

In the above inequality (3.56), the first inequality is obtained by applying the Cauchy-Schwarz inequality $\sum a_i b_i \leq (\sum a_i^2)^{\frac{1}{2}}(\sum b_i^2)^{\frac{1}{2}}$, and the second inequality is by applying the inequality $|a+b|^{\frac{1}{2}} \leq 2^{\frac{1}{2}}(|a|^{\frac{1}{2}} + |b|^{\frac{1}{2}})$.

Combining these two inequalities, (3.53) becomes

$$
\begin{aligned}
0 \;\geq\;& \frac{1}{2}\frac{d}{dt}\int_\Omega \left[((f_1-a)_+)^2+\Gamma((f_2-a)_+)^2\right] d\Omega \\
&+\int_\Omega \left[\frac{1}{4}\left((f_1^2-\Upsilon)+(f_2^2-\Upsilon)\right)-2\eta^*\right][f_1+f_2] \\
&\quad \times \left[(f_1-a)_+ + (f_2-a)_+\right]\, d\Omega. \qquad (3.57)
\end{aligned}
$$

Now we investigate the conditions (values of a) that make the last integral in the above inequality (3.57) greater or equal to zero, in particular, in the term $\frac{1}{4}\left((f_1^2-\Upsilon)+(f_2^2-\Upsilon)\right)-2\eta^*$.

If for some $t \in [0,T]$, the integration domain is Ω_1^t, i.e., both $f_1(t)$ and $f_2(t) > a \geq 1$, then we need

$$
\inf_{\Omega_1^t}\{(f_1^2+f_2^2-2\Upsilon)\} \geq 8\eta^*.
$$

We set $\inf f_1 = \inf f_2 = a$ in Ω_1^t and this gives

$$
a^2-\Upsilon \geq 4\eta^*.
$$

Thus we require

$$
a \geq \sqrt{4\eta^*+\Upsilon}. \qquad (3.58)
$$

Therefore, if we set $a=\sqrt{4\eta^*+\Upsilon}$, then the last integrand in (3.57) is always greater or equal to zero in Ω_1^t.

Now we want to consider the case in Ω_2^t and Ω_3^t. First suppose for some $t \in [0,T]$, the integration domain is in Ω_2^t, i.e., $f_1(t) > a \geq 1$ and $f_2(t) \leq a$, then from equation (3.53) all terms involving f_2 vanish so we have

$$
\begin{aligned}
0 \;\geq\;& \frac{1}{2}\frac{d}{dt}\int_\Omega ((f_1-a)_+)^2 d\Omega + \int_\Omega \left[(f_1^2-\mathcal{T}_1)f_1-\eta^* f_2\right](f_1-a)_+ d\Omega \\
\geq\;& \frac{1}{2}\frac{d}{dt}\int_\Omega ((f_1-a)_+)^2 d\Omega + \int_\Omega \left[(f_1^2-\Upsilon)f_1-\eta^* a\right](f_1-a)_+ d\Omega. \qquad (3.59)
\end{aligned}
$$

Here we have replaced $\max f_2$ in Ω_2^t with a in the last integral. We set $\inf f_1 = a \geq 1$ in Ω_2^t, and use the value of a we obtained previously, i.e., $a=\sqrt{4\eta^*+\Upsilon}$, then we have

$$
\begin{aligned}
\inf_{\Omega_2^t}\{(f_1^2-\Upsilon)f_1-\eta^* a\} &= (a^2-\Upsilon)a-\eta^* a \\
&= (4\eta^*-\eta^*)a \\
&\geq 0. \qquad (3.60)
\end{aligned}
$$

Therefore in the case of Ω_2^t, we have

$$0 \;\geq\; \frac{1}{2}\frac{d}{dt}\int_\Omega ((f_1-a)_+)^2 d\Omega. \tag{3.61}$$

This same argument for Ω_2^t but now with $f_1 \leq a$ and $f_2 > a \geq 1$ gives us the same result for the case in Ω_3^t, i.e.,

$$0 \;\geq\; \frac{1}{2}\frac{d}{dt}\int_\Omega \Gamma((f_2-a)_+)^2 d\Omega. \tag{3.62}$$

Combining all the above three cases, we conclude that for almost all $t \in [0,T]$,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left[((f_1-a)_+(\mathbf{x},t))^2 + \Gamma((f_2-a)_+(\mathbf{x},t))^2\right] d\Omega \\
&\quad = \frac{d}{dt}\Big[\int_\Omega \left[((f_1-a)_+(\mathbf{x},t))^2 + \Gamma((f_2-a)_+(\mathbf{x},t))^2\right] \chi_{\Omega_1^t}(\mathbf{x},t)\, d\Omega \\
&\qquad\quad + \int_\Omega ((f_1-a)_+(\mathbf{x},t))^2 \chi_{\Omega_2^t}(\mathbf{x},t)\, d\Omega + \int_\Omega \Gamma((f_2-a)_+(\mathbf{x},t))^2 \chi_{\Omega_3^t}(\mathbf{x},t)\, d\Omega\Big] \\
&\quad \leq 0,
\end{aligned} \tag{3.63}$$

where $\chi_{\Omega_i^t}(\mathbf{x},t)$ is the characteristics function dependent on time, since Ω_i^t is time-dependent.

Integrating (3.63) w.r.t. time from 0 to t, we get

$$\begin{aligned}
&\int_\Omega \left[((f_1-a)_+(\mathbf{x},t))^2 + \Gamma((f_2-a)_+(\mathbf{x},t))^2\right] d\Omega \\
&\qquad \leq \int_\Omega \left[((f_1-a)_+(\mathbf{x},0))^2 + \Gamma((f_2-a)_+(\mathbf{x},0))^2\right] d\Omega.
\end{aligned} \tag{3.64}$$

But the R.H.S. of equation (3.64) is equal to zero, since by assumption $f_1(\mathbf{x},0) \leq a$ and $f_2(\mathbf{x},0) \leq a$. This implies that $|\Omega_+| = 0$. Therefore we must have $f_i(\mathbf{x},t) = |\psi_i(\mathbf{x},t)| \leq a$ a.e. in $\Omega \times [0,T]$, for both $i = 1,2$. This completes the proof. ∎

From lemma 3.2.7 we know that $||\text{div}\mathbf{A}^\epsilon||_{L^\infty(0,T;L^2)(\Omega)}$ is not uniformly bounded w.r.t. ϵ and hence $\{(\psi_1^\epsilon, \psi_2^\epsilon, \mathbf{A}^\epsilon)\}$ is not uniformly bounded in $\mathcal{V}\times\mathcal{V}\times\mathbf{V}$. In order to pass to the limit $\epsilon \to 0$ to obtain a solution for the original problem (WP), we need to establish a uniform bound on $||\text{div}\mathbf{A}^\epsilon||_{L^\infty(0,T;L^2)(\Omega)}$. This implies that $||\mathbf{A}^\epsilon||_{\mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega))}$ is uniformly bounded, independent of ϵ which in turn, from lemma 3.2.8 implies that $\{\psi_1^\epsilon\}$ and $\{\psi_2^\epsilon\}$ are uniformly bounded in $\mathcal{L}^\infty(0,T;\mathcal{H}^1(\Omega))$, independent of ϵ. Finally, this says that $\{(\psi_1^\epsilon, \psi_2^\epsilon, \mathbf{A}^\epsilon)\}$ is uniformly bounded in $\mathcal{V}\times\mathcal{V}\times\mathbf{V}$, independent of ϵ.

We will prove that $||\text{div}\mathbf{A}^\epsilon||_{L^\infty(0,T;L^2)}$ is uniformly bounded in theorem 3.2.22. But first we need to establish some results of higher regularities for later proofs. By expanding and

rearranging the terms in the PDEs (2.71)-(2.73), the differential equations corresponding to the problem (WP$^\epsilon$) can be rewritten as the following two boundary-value problems:
Problem (P1): For $i = 1, 2$,

$$
\begin{aligned}
-\frac{\xi_i^2}{x_0^2}\triangle\psi_i^\epsilon &= f_i \qquad \text{in } \Omega, \times(0,T), &(3.65)\\
\frac{\partial\psi_i^\epsilon}{\partial n} &= -\gamma_i\psi_i^\epsilon \quad \text{on } \partial\Omega\times(0,T), &(3.66)
\end{aligned}
$$

where $\gamma_i(\mathbf{x}) \geq 0$, for $\mathbf{x}\in\partial\Omega$ and $\gamma_i \in L^\infty(\partial\Omega)$. And

$$
\begin{aligned}
f_1 &= -\left(\frac{\partial\psi_1^\epsilon}{\partial t} + i\,\phi_c\psi_1^\epsilon\right) - (|\psi_1^\epsilon|^2 - \mathcal{T}_1)\psi_1^\epsilon - i\frac{\xi_1}{\lambda_1}\big((\mathbf{A}^\epsilon+\mathbf{A}_c)\cdot\nabla\psi_1^\epsilon - \nabla\cdot(\psi_1^\epsilon(\mathbf{A}^\epsilon+\mathbf{A}_c))\big)\\
&\quad -\frac{x_0^2}{\lambda_1^2}|\mathbf{A}^\epsilon+\mathbf{A}_c|^2\psi_1^\epsilon - \eta\psi_2^\epsilon, &(3.67)\\
f_2 &= -\Gamma\left(\frac{\partial\psi_2^\epsilon}{\partial t} + i\,\phi_c\psi_2^\epsilon\right) - (|\psi_2^\epsilon|^2 - \mathcal{T}_2)\psi_2^\epsilon - i\frac{\xi_2}{\lambda_2}\big((\mathbf{A}^\epsilon+\mathbf{A}_c)\cdot\nabla\psi_2^\epsilon - \nabla\cdot(\psi_2^\epsilon(\mathbf{A}^\epsilon+\mathbf{A}_c))\big)\\
&\quad -\frac{x_0^2}{\lambda_2^2}|\mathbf{A}^\epsilon+\mathbf{A}_c|^2\psi_2^\epsilon - \eta\psi_1^\epsilon. &(3.68)
\end{aligned}
$$

And for $\mathbf{A}^\epsilon$,
Problem (P2):

$$
\begin{aligned}
-\epsilon\nabla\mathrm{div}\mathbf{A}^\epsilon + \mathrm{curl}(\mathrm{curl}\mathbf{A}^\epsilon - \mathbf{H}_e) &= \mathbf{g} \quad \Omega\times(0,T), &(3.69)\\
\mathbf{A}^\epsilon\cdot\mathbf{n} &= 0 \quad \text{on } \partial\Omega, \times(0,T), &(3.70)\\
(\mathrm{curl}\mathbf{A}^\epsilon - \mathbf{H}_e)\times\mathbf{n} &= \mathbf{0} \quad \text{on } \partial\Omega\times(0,T), &(3.71)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{g} &= -\sigma\frac{x_0^2}{\lambda_1^2}\frac{\partial\mathbf{A}^\epsilon}{\partial t}\\
&+ i\frac{1}{2\kappa_1}(\psi_1^\epsilon\nabla\psi_1^{\epsilon*} - \psi_1^{\epsilon*}\nabla\psi_1^\epsilon) - \frac{x_0^2}{\lambda_1^2}|\psi_1^\epsilon|^2(\mathbf{A}^\epsilon+\mathbf{A}_c)\\
&+ i\frac{1}{\nu}\frac{1}{2\kappa_2}(\psi_2^\epsilon\nabla\psi_2^{\epsilon*} - \psi_2^{\epsilon*}\nabla\psi_2^\epsilon) - \frac{x_0^2}{\lambda_2^2}|\psi_2^\epsilon|^2(\mathbf{A}^\epsilon+\mathbf{A}_c). &(3.72)
\end{aligned}
$$

Clearly the unique solution $(\psi_1^\epsilon, \psi_2^\epsilon, \mathbf{A}^\epsilon)$ of the weak problem (WP$^\epsilon$) in $[0,T]$ is weak solutions of the above boundary-value problems (P1) and (P2). Since $(\psi_1^\epsilon, \psi_2^\epsilon, \mathbf{A}^\epsilon) \in \mathcal{V}\times\mathcal{V}\times\mathbf{V}$, together with the regularity assumptions made in **RA1** and note that the compatibility condition $\int_\Omega f_i\,d\Omega = \langle\gamma_i\psi_i^\epsilon,\,1\rangle_{\partial\Omega}$ is satisfied by the weak form with $\tilde{\psi} = 1$, then after two successive applications of the theory of H^2-regularity on elliptic BVPs on a convex polygon Ω (see e.g., [44]), we obtain for $i = 1, 2$, $f_i \in \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega))$, $\partial\psi_i^\epsilon/\partial n|_{\partial\Omega_j} \in \mathcal{L}^2(0, T; \mathcal{H}^{\frac{1}{2}}(\partial\Omega_j))$, where $\partial\Omega_j \in \{\text{edges of the polygon}\}$, and thus $\psi_i^\epsilon \in \mathcal{L}^2(0, T; \mathcal{H}^2(\Omega))$.

With this new regularity result of ψ_i^ϵ, plus the regularity assumptions **RA1**, we can see from equation (3.72) that $\mathbf{g} \in \mathbf{L}^2(0, T; \mathbf{L}^2(\Omega))$. Then by using the result of Helmholtz decomposition of $\mathbf{L}^2(\Omega)$ (see e.g.,[44]), we obtain the following lemma.

Lemma 3.2.17 *For a.e. $t \in [0, T]$, assume $\mathbf{g} \in \mathbf{L}^2(\Omega)$ and $\mathbf{H}_e \in \mathbf{H}^1(\Omega)$. Then the problem (P2) has a unique weak solution $\mathbf{A}^\epsilon \in \mathbf{H}_n^1(\Omega)$ such that $\operatorname{div} \mathbf{A}^\epsilon \in H^1(\Omega)$ and $\operatorname{curl}\mathbf{A}^\epsilon \in \mathbf{H}^1(\Omega)$.*

Proof The proof is similar to that of Theorem 2.1 in [17] with minor modification to take into account the regularization coefficient ϵ. ∎

Again using Helmholtz decomposition, for a.e. $t \in [0, T]$, $\mathbf{g}$ can be decomposed as

$$\mathbf{g} = \nabla q + \operatorname{curl} \theta, \tag{3.73}$$

where $q \in H^1(\Omega)/\mathbb{R}$ is the only solution of

$$(\nabla q, \nabla v) = (\mathbf{g}, \nabla v) \qquad \forall v \in H^1(\Omega), \tag{3.74}$$

and $\theta \in H_0^1(\Omega)$ (in $\mathbb{R}^2$ case) is the only solution of

$$(\operatorname{curl} \theta, \operatorname{curl} w) = (\mathbf{g} - \nabla q, \operatorname{curl} w) \qquad \forall w \in H_0^1(\Omega). \tag{3.75}$$

Comparing to equations (3.69), we set

$$q = -\epsilon \operatorname{div}\mathbf{A}^\epsilon, \tag{3.76}$$

$$\theta = \operatorname{curl}\mathbf{A}^\epsilon - H_e, \tag{3.77}$$

and provided $H_e \in L^2(0, T; H^1(\Omega))$ (here in 2D, we assumed that $\mathbf{H}_e = H_e\mathbf{z}$), then we can see that by the results in theorem (3.2.17) and by virtue of the lemma 3.2.20 below, equation (3.69)-(3.71) satisfies equation (3.74). From the Neumann problem (3.74), we get

$$\begin{aligned} ||\epsilon \operatorname{div}\mathbf{A}^\epsilon||_1 &\leq ||\nabla q||_0 \\ &\leq ||\mathbf{g}||_0, \end{aligned} \tag{3.78}$$

where the first inequality are obtained by the fact stated after the inequality (2.52), and the second inequality (3.78) follows directly from equation (3.74). We can also see that equation (3.69) with (3.76)-(3.77) satisfies (3.75). From the fact that in $\mathbb{R}^2$, $(\mathbf{curl}\, u,\, \mathbf{curl}\, v) = (\nabla u,\, \nabla v)$ for all $u,\ v \in H^1(\Omega)$, we obtain from the Dirichlet problem (3.75) and Poincaré inequality that

$$\begin{aligned} ||\mathrm{curl}\mathbf{A}^\epsilon||_1 &\leq ||\nabla\theta||_0 \\ &\leq C||\mathbf{g}||_0. \end{aligned} \tag{3.79}$$

Therefore, from the H^1 norm estimates (3.78) and (3.79), we get $\mathrm{div}\mathbf{A}^\epsilon \in L^2(0,T;H^1(\Omega))$ and $\mathrm{curl}\mathbf{A}^\epsilon \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega))$.

Next we want to show that $\partial\mathbf{A}^\epsilon/\partial t \in L^2(0,T;\mathbf{H}^1(\Omega))$. We start with a lemma.

Lemma 3.2.18 *Given a fixed $\epsilon > 0$, assume $\mathbf{f} \in \mathbf{L}^2(0,T;(\mathbf{H}^1_n(\Omega))')$, $\mathbf{H} \in \mathbf{L}^2(0,T;\mathbf{H}(\mathrm{curl};\Omega))$ and $\mathbf{A}(0) \in \mathbf{L}^2(\Omega)$. Then the following IBVP has a unique weak solution $\mathbf{A} \in \mathbf{L}^2(0,T;\mathbf{H}^1_n(\Omega)) \cap \mathbf{H}^1(0,T;(\mathbf{H}^1_n(\Omega))')$.*

$$\frac{\partial\mathbf{A}}{\partial t} - \epsilon\nabla(\mathrm{div}\,\mathbf{A}) + \mathrm{curl}^2\mathbf{A} = \mathbf{f} + \mathrm{curl}\,\mathbf{H} \qquad \text{in } \Omega \tag{3.80}$$

$$\mathbf{A}\cdot\mathbf{n} = 0 \qquad \text{on } \partial\Omega, \tag{3.81}$$

$$(\mathrm{curl}\,\mathbf{A} - \mathbf{H})\times\mathbf{n} = 0 \qquad \text{on } \partial\Omega, \tag{3.82}$$

$$\mathbf{A}(0) = \mathbf{A}_0 \qquad \text{in } \Omega. \tag{3.83}$$

Proof We demonstrate the proof by using Rothe's method (see, e.g. [47] and [45]) which is useful to find higher regularity for time-dependent problem. First consider the following problem

$$\begin{aligned} -\epsilon\nabla(\mathrm{div}\,\mathbf{A}) + \mathrm{curl}^2\mathbf{A} + \tau\mathbf{A} &= \mathbf{f} + \mathrm{curl}\,\mathbf{H} \qquad \text{in } \Omega \\ \mathbf{A}\cdot\mathbf{n} &= 0 \qquad \text{on } \partial\Omega, \\ (\mathrm{curl}\,\mathbf{A} - \mathbf{H})\times\mathbf{n} &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where τ is a real positive constant. Denote $\mathbf{V} = \mathbf{H}^1_n(\Omega)$. Define $\tilde{\mathbf{L}} : \mathbf{V} \mapsto \mathbf{V}'$ as

$$\langle\tilde{\mathbf{L}}(\mathbf{A}),\, \mathbf{B}\rangle = \epsilon(\mathrm{div}\,\mathbf{A},\, \mathrm{div}\,\mathbf{B}) + (\mathrm{curl}\mathbf{A},\, \mathrm{curl}\,\mathbf{B}) + \tau(\mathbf{A},\, \mathbf{B}) \quad \forall\mathbf{A},\ \mathbf{B} \in \mathbf{V}.$$

Then the weak form of the above problem can be written as

$$\langle\tilde{\mathbf{L}}(\mathbf{A}),\, \mathbf{B}\rangle = \langle\mathbf{f},\, \mathbf{B}\rangle + \langle\mathbf{H},\, \mathrm{curl}\,\mathbf{B}\rangle \qquad \forall\mathbf{B} \in \mathbf{V},$$

By Lax-Milgram's theorem, the above problem has a unique weak solution $\mathbf{A} \in \mathbf{V}$.

Let $k = T/n$, for $n \geq 1$, and $t_m = mk$. Define $\mathbf{L} : \mathbf{L}(0, T; \mathbf{V}) \mapsto \mathbf{L}(0, T; \mathbf{V}')$ as

$$\langle \mathbf{L}(\mathbf{A}),\, \mathbf{B} \rangle = \epsilon(\operatorname{div} \mathbf{A},\, \operatorname{div} \mathbf{B}) + (\operatorname{curl}\mathbf{A},\, \operatorname{curl} \mathbf{B}).$$

Also define $\mathbf{f}^m, \mathbf{H}^m \in \mathbf{V}'$, $1 \leq m \leq n$ as

$$\begin{aligned} \mathbf{f}^m &= \frac{1}{k} \int_{t_{m-1}}^{t_m} \mathbf{f}(t) dt, \\ \mathbf{H}^m &= \frac{1}{k} \int_{t_{m-1}}^{t_m} \mathbf{H}(t) dt. \end{aligned}$$

Define the following problem from which we want to use Rothe's method to show that it's solution when pass to limit is a solution of the problem (3.80)-(3.83).

Problem (FP): Find $\mathbf{A}^m \in \mathbf{V}$, for $m = 1, \cdots, n$, such that

$$\begin{aligned} \frac{\mathbf{A}^m - \mathbf{A}^{m-1}}{k} + \mathbf{L}(\mathbf{A}^m) &= \mathbf{F}^m \qquad \text{in } \mathbf{V}', \\ \mathbf{A}^0 &= \mathbf{A}_0, \end{aligned} \tag{3.84}$$

where $\langle \mathbf{F}^m,\, \mathbf{B} \rangle = \langle \mathbf{f}^m,\, \mathbf{B} \rangle + \langle \mathbf{H}^m,\, \operatorname{curl} \mathbf{B} \rangle$, for all $\mathbf{B} \in \mathbf{V}$.

Then by the result above, there is a unique solution $\mathbf{A}^m \in \mathbf{V}$. Thus this problem is well-defined.

Next we get *a priori* estimate independent of k. By applying the identity $2(a - b, a) = |a|^2 - |b|^2 + |a - b|^2$ to the following equation

$$(\mathbf{A}^m - \mathbf{A}^{m-1},\, \mathbf{A}^m) + k\epsilon(\operatorname{div} \mathbf{A}^m,\, \operatorname{div} \mathbf{A}^m) + k(\operatorname{curl} \mathbf{A}^m,\, \operatorname{curl} \mathbf{A}^m) = k\langle \mathbf{f}^m,\, \mathbf{A}^m \rangle + (\mathbf{H}^m,\, \operatorname{curl}\mathbf{A}^m)$$

we get

$$\begin{aligned} &||\mathbf{A}^m||_0^2 - ||\mathbf{A}^{m-1}||_0^2 + ||\mathbf{A}^m - \mathbf{A}^{m-1}||_0^2 + 2k\epsilon||\operatorname{div}\mathbf{A}^m||_0^2 + 2k||\operatorname{curl} \mathbf{A}^m||_0^2 \\ &\quad \leq \frac{k}{\delta}||\mathbf{f}^m||_{\mathbf{V}'} + \delta k||\mathbf{A}^m||_{\mathbf{V}} + k||\mathbf{H}^m||_0^2 + k||\operatorname{curl} \mathbf{A}^m||_0^2 \\ &\quad \leq \frac{k}{\delta}||\mathbf{f}^m||_{\mathbf{V}'} + \delta k C(||\operatorname{div} \mathbf{A}^m||_0^2 + ||\operatorname{curl} \mathbf{A}^m||_0^2) + k||\mathbf{H}^m||_0^2 + k||\operatorname{curl} \mathbf{A}^m||_0^2. \end{aligned}$$

By choosing $\delta = \epsilon/C$ and using the discrete Gronwall's inequality, we get

$$\begin{aligned} &\max_{1 \leq m \leq n} ||\mathbf{A}^m||_0^2 + \sum_1^n ||\mathbf{A}^m - \mathbf{A}^{m-1}||_0^2 + \epsilon k \sum_1^n ||\operatorname{div} \mathbf{A}^m||_0^2 + (1 - \epsilon)k \sum_1^n ||\operatorname{curl} \mathbf{A}^m||_0^2 \\ &\quad \leq ||\mathbf{A}^0||_0^2 + \frac{kC}{\epsilon} \sum_1^n ||\mathbf{f}^m||_{\mathbf{V}'}^2 + k \sum_1^n ||\mathbf{H}^m||_0^2 \\ &\quad \leq ||\mathbf{A}^0||_0^2 + \frac{C}{\epsilon}||\mathbf{f}||_{\mathbf{L}^2(0,T;\mathbf{V}')}^2 + ||\mathbf{H}||_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))}^2, \end{aligned} \tag{3.85}$$

where we have used the inequality (see [45])

$$k\sum_1^n ||\mathbf{f}^m||^2_{\mathbf{V}'} \leq \int_0^T ||\mathbf{f}(t)||^2_{\mathbf{V}'}dt, \qquad k\sum_1^n ||\mathbf{H}^m||^2_0 \leq \int_0^T ||\mathbf{H}(t)||^2_0 dt$$

From (3.84), we get

$$\begin{aligned} \left|\left|\frac{\mathbf{A}^m - \mathbf{A}^{m-1}}{k}\right|\right|_{\mathbf{V}'} &\leq ||\mathbf{L}(\mathbf{A}^m)||_{\mathbf{V}'} + ||\mathbf{F}^m||_{\mathbf{V}'} \\ &\leq \epsilon||\mathrm{div}\,\mathbf{A}^m||_0 + ||\mathrm{curl}\,\mathbf{A}^m||_0 + ||\mathbf{f}^m||_{\mathbf{V}'} + ||\mathbf{H}^m||_0. \end{aligned} \tag{3.86}$$

Using the result in (3.85) and summing the inequality above from $m = 1$ to n, we get

$$k\sum_1^n \left|\left|\frac{\mathbf{A}^m - \mathbf{A}^{m-1}}{k}\right|\right|^2_{\mathbf{V}'} \leq C_\epsilon(||\mathbf{A}^0||^2_0 + ||\mathbf{f}||^2_{\mathbf{L}^2(0,T;\mathbf{V}')} + ||\mathbf{H}||^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))}). \tag{3.87}$$

Define $\mathbf{A}_n : [0,T] \mapsto \mathbf{V}$ as $\mathbf{A}_n(t) = \mathbf{A}^m$, for $t \in [(m-1)k, mk]$, $m = 1, \cdots, n$. Also define $\tilde{\mathbf{A}}_n : [0,T] \mapsto \mathbf{V}$ as $\tilde{\mathbf{A}}_n(t) = \mathbf{A}^m - (t_m - t)(\mathbf{A}^m - \mathbf{A}^{m-1})/k$, for $t \in [(m-1)k\, mk]$, $m = 1, \cdots, n$. Note that

$$\frac{\partial \tilde{\mathbf{A}}_n}{\partial t}(t) = \frac{\mathbf{A}^m - \mathbf{A}^{m-1}}{k} \qquad \text{for } (m-1)k < t < mk.$$

Then from the estimates (3.85) and (3.87), we have

$$\begin{aligned} &\{\mathbf{A}_n\} \text{ and } \{\tilde{\mathbf{A}}_n\} \quad \text{uniformly bounded in } \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega)) \cap \mathbf{L}^2(0,T;\mathbf{V}), \\ &\{\frac{\partial(\tilde{\mathbf{A}}_n)}{\partial t}\} \quad \text{uniformly bounded in } \mathbf{L}^2(0,T;\mathbf{V}'). \end{aligned}$$

As a result, there are subsequences $\{\mathbf{A}_{n_k}\}$ of $\{\mathbf{A}_n\}$ and $\{\tilde{\mathbf{A}}_{n_k}\}$ of $\{\tilde{\mathbf{A}}_n\}$ such that

$$\mathbf{A}_{n_k} \rightharpoonup \mathbf{A} \quad \text{in } \mathbf{L}^2(0,T;\mathbf{V}), \tag{3.88}$$

$$\overset{*}{\rightharpoonup} \mathbf{A} \quad \text{in } \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega)), \tag{3.89}$$

$$\tilde{\mathbf{A}}_{n_k} \rightharpoonup \tilde{\mathbf{A}} \quad \text{in } \mathbf{L}^2(0,T;\mathbf{V}), \tag{3.90}$$

$$\overset{*}{\rightharpoonup} \tilde{\mathbf{A}} \quad \text{in } \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega)), \tag{3.91}$$

$$\frac{\partial \tilde{\mathbf{A}}_{n_k}}{\partial t} \rightharpoonup (\tilde{\mathbf{A}})_t \quad \text{in } \mathbf{L}^2(0,T;\mathbf{V}'). \tag{3.92}$$

Now we show that $\mathbf{A}_n \to \tilde{\mathbf{A}}_n$ in $\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))$. Indeed, since

$$\tilde{\mathbf{A}}_n - \mathbf{A}_n = \frac{t - mk}{k}(\mathbf{A}^m - \mathbf{A}^{m-1}) \qquad \text{for } (m-1)k \leq t \leq mk,$$

this gives

$$\sum_{1}^{n}\int_{(m-1)k}^{mk}||\tilde{\mathbf{A}}_n-\mathbf{A}_n||_0^2dt=\sum_{1}^{n}\frac{k}{3}||\mathbf{A}^m-\mathbf{A}^{m-1}||_0^2.$$

By the estimate in (3.85), the right hand side turns to zero as $k\to 0$, i.e., as $n\to\infty$.

From this result, we know that the weak limits are the same, i.e., $\mathbf{A}=\tilde{\mathbf{A}}$.

Now we are ready to pass to the limit $n\to\infty$ in equation (3.84) of problem (**FP**). Equation (3.84) can be rewritten as

$$\frac{\partial\tilde{\mathbf{A}}_n}{\partial t}+\mathbf{L}(\mathbf{A}_n)=\mathbf{F}_n, \tag{3.93}$$

where $\mathbf{F}_n$ is defined as $\mathbf{F}_n(t)=\mathbf{F}^m$ for $(m-1)k\le t\le mk$, $m=1,\cdots,n$. It can be shown that $\langle\mathbf{F}_n,\,\mathbf{B}\rangle\to\langle\mathbf{f},\,\mathbf{B}\rangle+(\mathbf{H},\,\mathrm{curl}\mathbf{B})$ as $n\to\infty$ for any $\mathbf{B}\in\mathbf{L}^2(0,T;\mathbf{V})$ (see [45]).

Now by the weak convergence of $\mathbf{A}_n$ and $(\tilde{\mathbf{A}})_t$ in $\mathbf{L}(0,T;\mathbf{V}')$, a passage of the limit $n\to\infty$ in equation (3.93) gives

$$\frac{\partial\mathbf{A}}{\partial t}+\mathbf{L}(\mathbf{A})=\mathbf{F}, \tag{3.94}$$

where $\mathbf{F}$ is defined as $\langle\mathbf{F},\,\mathbf{B}\rangle=\langle\mathbf{f},\,\mathbf{B}\rangle+(\mathbf{H},\,\mathrm{curl}\mathbf{B})$ for $\mathbf{B}\in\mathbf{L}^2(0,T;\mathbf{V})$.

Finally, because of (3.90) and (3.92), $\tilde{\mathbf{A}}_n\in\mathbf{C}([0,T];\mathbf{L}^2(\Omega))$, thus

$$\langle\tilde{\mathbf{A}}_n(t),\Phi\rangle\to\langle\mathbf{A}(t),\Phi\rangle\qquad\forall\Phi\in\mathbf{V}',\quad\forall t\in[0,T],$$

this gives

$$\mathbf{A}(0)=\tilde{\mathbf{A}}_n(0)=\mathbf{A}_0. \tag{3.95}$$

Together the (3.94) and (3.95) show that $\mathbf{A}\in\mathbf{L}^2(0,T;\mathbf{H}_n^1(\Omega))\cap\mathbf{H}^1(0,T;(\mathbf{H}_n^1(\Omega))')$ is a weak solution of the problem (3.80)-(3.83). We omit the standard uniqueness proof. ∎

Now with lemma 3.2.18, we are ready to show $\partial\mathbf{A}^\epsilon/\partial t\in L^2(0,T;\mathbf{H}^1(\Omega))$. Consider the following IBVP

$$\begin{aligned}
\sigma\frac{x_0^2}{\lambda_1^2}\frac{\partial\,\Phi}{\partial t}-\epsilon\nabla(\mathrm{div}\,\Phi)+\mathrm{curl}^2\Phi &= \mathbf{f}_t+\mathrm{curl}\,(\mathbf{H}_e)_t \qquad \text{in }\Omega && (3.96)\\
\Phi\cdot\mathbf{n} &= 0 \qquad \text{on }\partial\Omega, && (3.97)\\
(\mathrm{curl}\,\Phi-(\mathbf{H}_e)_t)\times\mathbf{n} &= 0 \qquad \text{on }\partial\Omega, && (3.98)\\
\Phi(0) &= \Phi_0 \qquad \text{in }\Omega, && (3.99)
\end{aligned}$$

where

$$
\begin{aligned}
\mathbf{f} &= i\frac{1}{2\kappa_1}\left(\psi_1^\epsilon\nabla\psi_1^{\epsilon*} - \psi_1^{\epsilon*}\nabla\psi_1^\epsilon\right) - \frac{x_0^2}{\lambda_1^2}|\psi_1^\epsilon|^2(\mathbf{A}^\epsilon + \mathbf{A}_c) \\
&\quad + i\frac{1}{\nu}\frac{1}{2\kappa_2}\left(\psi_2^\epsilon\nabla\psi_2^{\epsilon*} - \psi_2^{\epsilon*}\nabla\psi_2^\epsilon\right) - \frac{x_0^2}{\lambda_2^2}|\psi_2^\epsilon|^2(\mathbf{A}^\epsilon + \mathbf{A}_c). \\
\mathbf{f}_t &= -\frac{1}{\kappa_1}\Im\left((\psi_1^\epsilon)_t\nabla\psi_1^{\epsilon*} + \psi_1^\epsilon\nabla(\psi_1^{\epsilon*})_t\right) - \frac{1}{\nu}\frac{1}{\kappa_2}\Im\left((\psi_2^\epsilon)_t\nabla\psi_2^{\epsilon*} - \psi_2^\epsilon\nabla(\psi_2^{\epsilon*})_t\right) \\
&\quad - 2\frac{x_0^2}{\lambda_1^2}\Re((\psi_1^\epsilon)_t\psi^{\epsilon*})(\mathbf{A}^\epsilon + \mathbf{A}_c) - \frac{x_0^2}{\lambda_1^2}|\psi_1^\epsilon|^2((\mathbf{A}^\epsilon)_t + (\mathbf{A}_c)_t) \\
&\quad - 2\frac{x_0^2}{\lambda_2^2}\Re((\psi_2^\epsilon)_t\psi^{\epsilon*})(\mathbf{A}^\epsilon + \mathbf{A}_c) - \frac{x_0^2}{\lambda_2^2}|\psi_2^\epsilon|^2((\mathbf{A}^\epsilon)_t + (\mathbf{A}_c)_t), \\
\Phi_0 &= \mathbf{f}(0) + \epsilon\nabla(\operatorname{div}\mathbf{A}^\epsilon(0)) - \operatorname{curl}^2\mathbf{A}^\epsilon(0) + \operatorname{curl}\mathbf{H}_e(0).
\end{aligned}
$$

From the regularity results we obtained so far, we know that $\psi_i^\epsilon \in \mathcal{L}(0,T;\mathcal{H}^2(\Omega)) \cap \mathcal{L}^\infty(\Omega \times (0,T)) \cap \mathcal{H}^1(0,T;\mathcal{L}^2(\Omega))$ and $\mathbf{A}^\epsilon \in \mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{H}^1(0,T;\mathbf{L}^2(\Omega))$. Assume $\mathbf{A}_c \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{H}^1(0,T;\mathbf{L}^2(\Omega))$ also, we can show that $\mathbf{f}_t \in \mathbf{L}^2(0,T;(\mathbf{H}_n^1(\Omega))')$. Further assume $(\mathbf{H}_e)_t \in \mathbf{L}^2(0,T;H(\text{curl};\Omega))$, then the right hand side of equation (3.96) belongs $\mathbf{L}^2(0,T;(\mathbf{H}_n^1(\Omega))')$. For the initial conditions Φ_0, first assume that $\mathbf{A}_c(0) \in \mathbf{H}(\text{div}_0;\Omega)$, then $\text{div}\mathbf{A}_c(0) = 0$. Also if $\mathbf{j}_c(0) \in \mathbf{H}(\text{div}_0;\Omega)$, then $\text{curl}^2\mathbf{A}_c(0) = \text{curl}\,\mathbf{H}_c(0) \in \mathbf{L}^2(\Omega)$, where $\mathbf{H}_c(0) \in \mathbf{H}^1(\Omega)$ is a solution of $\text{curl}\,\mathbf{H}_c(0) = \mathbf{j}_c(0)$. The existence of $\mathbf{H}_c(0)$ is guaranteed since $\text{div}\mathbf{j}_c(0) = 0$ and $\langle \mathbf{j}_c(0)\cdot\mathbf{n},\, 1\rangle_{\partial\Omega} = 0$, see [44]. Now if we assume $\mathbf{A}_0 \in \mathbf{H}^2(\Omega)$, then since $\mathbf{A}^\epsilon(0) = \mathbf{A}_0 - \mathbf{A}_c(0)$, we can see that $\nabla(\text{div}\,\mathbf{A}^\epsilon(0)) - \text{curl}^2\mathbf{A}^\epsilon(0) \in \mathbf{L}^2(\Omega)$. Further assume that $\psi_i^\epsilon(0) \in \mathcal{H}^2(\Omega)$, $\mathbf{H}_e(0) \in \mathbf{H}(\text{curl};\Omega)$ and $\mathbf{A}_c(0) \in \mathbf{H}^1(\Omega)$, then we can show that $\Phi(0) = \Phi_0 \in \mathbf{L}^2(\Omega)$. Therefore, an application of lemma 3.2.18 gives us that the IBVP (3.96)-(3.99) has a unique solution $\Phi \in \mathbf{L}^2(0,T;\mathbf{H}_n^1(\Omega)) \cap (\mathbf{H}^1(0,T;\mathbf{H}_n^1(\Omega))')$. It can be shown that $(\mathbf{A}^\epsilon)_t = \Phi$, see, for example, [50].

By this new results together with an additional assumption that $\phi_c \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and $\phi_c(0) \in H^1(\Omega)$, the same method presented above can also be applied to equations (3.65)-(3.68) to get $\partial\psi_i^\epsilon/\partial t \in \mathcal{L}^2(0,T;\mathcal{H}^1(\Omega))$. Consider the following IBVP

$$
\begin{aligned}
\frac{\partial\theta}{\partial t} - \frac{\xi_1^2}{x_0^2}\triangle\theta &= (f_1)_t \quad \text{in } \Omega, && (3.100)\\
\frac{\partial\theta}{\partial n} + \gamma_1\varphi &= 0 \quad \text{on } \partial\Omega, && (3.101)\\
\theta(0) &= \theta_0 \quad \text{in } \Omega, && (3.102)
\end{aligned}
$$

where

$$
\begin{aligned}
f_1 &= -i\,\phi_c\psi_1^\epsilon - (|\psi_1^\epsilon|^2 - \mathcal{T}_1)\psi_1^\epsilon - i\frac{\xi_1}{\lambda_1}\big((\mathbf{A}^\epsilon + \mathbf{A}_c)\cdot\nabla\psi_1^\epsilon - \nabla\cdot(\psi_1^\epsilon(\mathbf{A}^\epsilon + \mathbf{A}_c))\big) \\
&\quad - \frac{x_0^2}{\lambda_1^2}|\mathbf{A}^\epsilon + \mathbf{A}_c|^2\psi_1^\epsilon - \eta\psi_2^\epsilon, \\
(f_1)_t &= -i((\phi_c)_t\psi_1^\epsilon + \phi_c(\psi_1^\epsilon)_t) - (|\psi_1^\epsilon|^2 - \mathcal{T}_1)(\psi_1^\epsilon)_t - 2\Re((\psi_1^\epsilon)_t\psi_1^{\epsilon*}) \\
&\quad - i\frac{2\xi_1}{\lambda_1}\big[((\mathbf{A}^\epsilon)_t + (\mathbf{A}_c)_t)\cdot\nabla\psi_1^\epsilon + (\mathbf{A}^\epsilon + \mathbf{A}_c)\cdot\nabla(\psi_1^\epsilon)_t\big] \\
&\quad - i\frac{\xi_1}{\lambda_1}\big[(\psi_1^\epsilon)_t\nabla\cdot(\mathbf{A}^\epsilon + \mathbf{A}_c) + \psi_1^\epsilon\nabla\cdot((\mathbf{A}^\epsilon)_t + (\mathbf{A}_c)_t)\big] \\
&\quad - \frac{x_0^2}{\lambda_1^2}\big[\,|\mathbf{A}^\epsilon + \mathbf{A}_c|^2(\psi_1^\epsilon)_t + 2((\mathbf{A}^\epsilon)_t + (\mathbf{A}_c)_t)(\mathbf{A}^\epsilon + \mathbf{A}_c)\psi_1^\epsilon\big] - \eta(\psi_2^\epsilon)_t, \\
\theta_0 &= f_1(0) + \frac{\xi_1^2}{x_0^2}\triangle\psi_1^\epsilon(0).
\end{aligned}
$$

With the above additional assumptions on ϕ_c and $\phi_c(0)$, we can show that $(f_1)_t \in \mathcal{L}^2(0,T;(\mathcal{H}^1(\Omega))')$ and $\theta_0 \in \mathcal{L}^2(\Omega)$. Then by the standard theory of parabolic IBVP with homogeneous Robin boundary condition, the IBVP (3.100)-(3.102) has a unique solution $\theta \in \mathcal{L}^2(0,T;\mathcal{H}^1(\Omega)) \cap \mathcal{H}^1(0,T;(\mathcal{H}^1(\Omega))')$. We can show that $(\psi_1^\epsilon)_t = \theta$. Similarly, we get $(\psi_2^\epsilon)_t \in \mathcal{L}^2(0,T;\mathcal{H}^1(\Omega)) \cap \mathcal{H}^1(0,T;(\mathcal{H}^1(\Omega))')$.

Summarizing all the regularity results we have obtained, we arrive at the following theorem.

Theorem 3.2.19 *In addition to the regularity assumptions* **RA1**, *assume that* $\mathbf{A}_c \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{H}^1(0,T;\mathbf{L}^2(\Omega))$, $\mathbf{H}_e \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{H}^1(0,T;\mathbf{H}(\mathrm{curl};\Omega))$, $\mathbf{A}_c(0) \in \mathbf{H}(\mathrm{div}_0;\Omega)$, $\mathbf{j}_c(0) \in \mathbf{H}(\mathrm{div}_0;\Omega)$, $\mathbf{A}_0 \in \mathbf{H}^2(\Omega)$, $\psi_i^\epsilon(0) = \psi_{i0} \in \mathcal{H}^2(\Omega)$ *and* $|\psi_{i0}| \le a$ *for* $i = 1,2$, *where* a *is defined in Theorem 3.2.16. Then given* $\epsilon > 0$ *and* $T > 0$,

$$
\begin{aligned}
\psi_1^\epsilon,\, \psi_2^\epsilon &\in \mathcal{L}^2(0,T;\mathcal{H}^2(\Omega)), &(3.103)\\
\mathrm{curl}\mathbf{A}^\epsilon &\in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)), &(3.104)\\
\mathrm{div}\mathbf{A}^\epsilon &\in L^2(0,T;H^1(\Omega)), &(3.105)\\
\frac{\partial\mathbf{A}^\epsilon}{\partial t} &\in \mathbf{L}^2(0,T;\mathbf{H}_n^1(\Omega)) \cap (\mathbf{H}^1(0,T;(\mathbf{H}_n^1(\Omega))'). &(3.106)
\end{aligned}
$$

If in addition, $\phi_c \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ *and* $\phi_c(0) \in H^1(\Omega)$, *then we also have*

$$
\frac{\partial\psi_1^\epsilon}{\partial t},\ \frac{\partial\psi_2^\epsilon}{\partial t} \in \mathcal{L}^2(0,T;\mathcal{H}^1(\Omega)) \cap \mathcal{H}^1(0,T;(\mathcal{H}^1(\Omega))'). \tag{3.107}
$$

Moreover, the bound of the respective norm of each function depends on ϵ.

These regularity results will be used in lemma 3.2.21 to find the uniform bound of $\mathrm{div}\mathbf{A}^\epsilon$ in $L^\infty(0,T;L^2(\Omega))$, independent of ϵ. The proof of this lemma follows that of lemma 3.11 in Du's paper [2]. We first show the following lemma.

Lemma 3.2.20 *Recall that* $\mathbf{H}(\mathrm{curl};\Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) | \, \mathrm{curl}\,\mathbf{u} \in \mathbf{L}^2(\Omega) \text{ in } \Omega\}$ *and* $\mathbf{H}_0(\mathrm{curl};\Omega) = \{\mathbf{u} \in \mathbf{H}(\mathrm{curl};\Omega) | \, \mathbf{u} \times \mathbf{n}|_{\partial\Omega} = 0\}$. *The norm for* $\mathbf{H}(\mathrm{curl};\Omega)$ *is* $||\mathbf{u}||_{\mathbf{H}(\mathrm{curl};\Omega)} = (||\mathbf{u}||^2_{\mathbf{L}^2(\Omega)} + ||\mathrm{curl}\,\mathbf{u}||^2_{\mathbf{L}^2(\Omega)})^{1/2}$. *Then for* $\mathbf{u} \in \mathbf{H}_0(\mathrm{curl};\Omega)$ *and* $\mathbf{v} \in \mathbf{H}(\mathrm{curl};\Omega)$, *we have*

$$\int_\Omega \mathrm{curl}\,\mathbf{u} \cdot \mathbf{v}\, d\Omega = \int_\Omega \mathbf{u} \cdot \mathrm{curl}\,\mathbf{v}\, d\Omega. \tag{3.108}$$

Proof We quote a fact that the closure of $\mathbf{C}^\infty(\overline{\Omega})$ in the $\mathbf{H}(\mathrm{curl};\Omega)$ norm is the space $\mathbf{H}(\mathrm{curl};\Omega)$, see, e.g., [57]. Let the sequence $\{\Phi_n\} \subset \mathbf{C}^\infty(\overline{\Omega})$ be such that $\Phi_n \to \mathbf{v}$, then by integration by parts we have

$$\int_\Omega \mathrm{curl}\,\mathbf{u} \cdot \Phi_n\, d\Omega = \int_\Omega \mathbf{u} \cdot \mathrm{curl}\,\Phi_n\, d\Omega + \int_{\partial\Omega} \mathbf{n} \times \mathbf{u} \cdot \Phi_n\, d\partial\Omega.$$

But since $\mathbf{u} \in \mathbf{H}_0(\mathrm{curl};\Omega)$, the above boundary integral vanishes. So by the continuity of each integral on the function Φ_n in Ω, we have

$$\int_\Omega \mathrm{curl}\,\mathbf{u} \cdot \mathbf{v}\, d\Omega - \int_\Omega \mathbf{u} \cdot \mathrm{curl}\,\mathbf{v}\, d\Omega = \lim_{n\to\infty} \left\{ \int_\Omega \mathrm{curl}\,\mathbf{u} \cdot \Phi_n\, d\Omega - \int_\Omega \mathbf{u} \cdot \mathrm{curl}\,\Phi_n\, d\Omega \right\} = 0. \quad \blacksquare$$

Lemma 3.2.21 *Suppose the assumptions in theorem 3.2.19 hold, then given* $\epsilon > 0$ *and* $T > 0$, *we have for* $t \in [0,T]$,

$$\begin{aligned} &\sigma \frac{x_0^2}{\lambda_1^2} ||\mathrm{div}\mathbf{A}^\epsilon(t)||_{L^2(\Omega)} + \epsilon \int_0^t ||\nabla \mathrm{div}\mathbf{A}^\epsilon(s)||_{\mathbf{L}^2(\Omega)} ds \\ &\quad = \int_0^t \Re \left\{ \left(\frac{\partial \psi_1^\epsilon(s)}{\partial t} + i\,\phi_c(s)\psi_1^\epsilon(s),\; i\frac{x_0^2}{\xi_1\lambda_1} \mathrm{div}\mathbf{A}^\epsilon(s)\psi_1^\epsilon(s) \right) \right\} ds \\ &\quad\quad + \Gamma \int_0^t \Re \left\{ \left(\frac{\partial \psi_2^\epsilon(s)}{\partial t} + i\,\phi_c(s)\psi_2^\epsilon(s),\; i\frac{x_0^2}{\xi_2\lambda_2} \mathrm{div}\mathbf{A}^\epsilon(s)\psi_2^\epsilon(s) \right) \right\} ds. \end{aligned} \tag{3.109}$$

Proof Let the test function in the weak form (3.9) in the problem (WP^ϵ) be $\tilde{\psi} = i\frac{x_0^2}{\xi_1\lambda_1}\psi_1^\epsilon \nabla \cdot \mathbf{A}^\epsilon$. Note that by theorem 3.2.16, ψ_i^ϵ is uniformly bounded in $\Omega \times [0,T]$ and so by theorem 3.2.19, we know that $\psi_1^\epsilon \nabla \cdot \mathbf{A}^\epsilon \in \mathcal{L}^2(0,T;\mathcal{H}^1(\Omega))$. Then taking the real parts, we have

$$
\begin{aligned}
&\Re\left\{-i\frac{\xi_1}{x_0}\nabla\psi_1^\epsilon - \frac{x_0}{\lambda_1}(\mathbf{A}^\epsilon+\mathbf{A}_c)\psi_1^\epsilon,\ -i\frac{\xi_1}{x_0}\nabla(i\frac{x_0^2}{\xi_1\lambda_1}\psi_1^\epsilon\nabla\cdot\mathbf{A}^\epsilon) - \frac{x_0}{\lambda_1}(\mathbf{A}^\epsilon+\mathbf{A}_c)\cdot(i\frac{x_0^2}{\xi_1\lambda_1}\psi_1^\epsilon\nabla\cdot\mathbf{A}^\epsilon)\right\}\\
&\quad= \Re\int_\Omega\left\{\big[-i\frac{\xi_1}{x_0}\nabla\psi_1^\epsilon - \frac{x_0}{\lambda_1}(\mathbf{A}^\epsilon+\mathbf{A}_c)\psi_1^\epsilon\big]\cdot\big[\frac{x_0}{\lambda_1}(\psi_1^{\epsilon*}\nabla(\nabla\cdot\mathbf{A}^\epsilon)+\nabla\psi_1^{\epsilon*}\nabla\cdot\mathbf{A}^\epsilon)\right.\\
&\qquad\left.+ i\frac{x_0^3}{\xi_1\lambda_1^2}(\mathbf{A}^\epsilon+\mathbf{A}_c)\psi_1^{\epsilon*}\nabla\cdot\mathbf{A}^\epsilon)\big]\right\}d\Omega\\
&\quad= \Re\int_\Omega\left\{\frac{x_0}{\lambda_1}\psi_1^{\epsilon*}\big(-i\frac{\xi_1}{x_0}\nabla\psi_1^\epsilon - \frac{x_0}{\lambda_1}(\mathbf{A}^\epsilon+\mathbf{A}_c)\psi_1^\epsilon\big)\cdot\nabla(\nabla\cdot\mathbf{A}^\epsilon)\right.\\
&\qquad - i\frac{\xi_1}{\lambda_1}|\nabla\psi_1^\epsilon|^2\nabla\cdot\mathbf{A}^\epsilon - \frac{x_0^2}{\lambda_1^2}(\mathbf{A}^\epsilon+\mathbf{A}_c)\cdot\psi_1^\epsilon\nabla\psi_1^{\epsilon*}\nabla\cdot\mathbf{A}^\epsilon\\
&\qquad\left.+\frac{x_0^2}{\lambda_1^2}(\mathbf{A}^\epsilon+\mathbf{A}_c)\cdot\psi_1^{\epsilon*}\nabla\psi_1^\epsilon\nabla\cdot\mathbf{A}^\epsilon - i\frac{x_0^4}{\xi_1\lambda_1^3}|\mathbf{A}^\epsilon+\mathbf{A}_c|^2|\psi_1^\epsilon|^2\nabla\cdot\mathbf{A}^\epsilon\right\}d\Omega\\
&\quad= \Re\int_\Omega\left\{\frac{x_0}{\lambda_1}\psi_1^{\epsilon*}\big(-i\frac{\xi_1}{x_0}\nabla\psi_1^\epsilon - \frac{x_0}{\lambda_1}(\mathbf{A}^\epsilon+\mathbf{A}_c)\psi_1^\epsilon\big)\cdot\nabla(\nabla\cdot\mathbf{A}^\epsilon)\right.\\
&\qquad\left.+\frac{x_0^2}{\lambda_1^2}(\mathbf{A}^\epsilon+\mathbf{A}_c)\cdot\big(\psi_1^{\epsilon*}\nabla\psi_1^\epsilon - \psi_1^\epsilon\nabla\psi_1^{\epsilon*}\big)\nabla\cdot\mathbf{A}^\epsilon\right\}d\Omega\\
&\quad= \Re\int_\Omega\frac{x_0}{\lambda_1}\psi_1^{\epsilon*}\big(-i\frac{\xi_1}{x_0}\nabla\psi_1^\epsilon - \frac{x_0}{\lambda_1}(\mathbf{A}^\epsilon+\mathbf{A}_c)\psi_1^\epsilon\big)\cdot\nabla(\nabla\cdot\mathbf{A}^\epsilon)\,d\Omega.
\end{aligned}
$$

Similarly, substituting $\tilde{\psi} = i\frac{1}{\nu}\frac{x_0^2}{\xi_2\lambda_2}\psi_2^\epsilon\nabla\cdot\mathbf{A}^\epsilon$ in the weak form (3.10), we get

$$
\begin{aligned}
&\Re\left\{-i\frac{\xi_2}{x_0}\nabla\psi_2^\epsilon - \nu\frac{x_0}{\lambda_2}(\mathbf{A}^\epsilon+\mathbf{A}_c)\psi_2^\epsilon,\ -i\frac{\xi_2}{x_0}\nabla(i\frac{x_0^2}{\xi_2\lambda_2}\psi_2^\epsilon\nabla\cdot\mathbf{A}^\epsilon) - \nu\frac{x_0}{\lambda_2}(\mathbf{A}^\epsilon+\mathbf{A}_c)\cdot(i\frac{x_0^2}{\xi_2\lambda_2}\psi_2^\epsilon\nabla\cdot\mathbf{A}^\epsilon)\right\}\\
&\quad= \Re\int_\Omega\frac{1}{\nu}\frac{x_0}{\lambda_2}\psi_2^{\epsilon*}\big(-i\frac{\xi_2}{x_0}\nabla\psi_2^\epsilon - \nu\frac{x_0}{\lambda_2}(\mathbf{A}^\epsilon+\mathbf{A}_c)\psi_2^\epsilon\big)\cdot\nabla(\nabla\cdot\mathbf{A}^\epsilon)\,d\Omega.
\end{aligned}
$$

For the two η coupling terms, when adding them together, we get

$$
\begin{aligned}
&\eta\,\Re\big(\psi_2^\epsilon,\ i\frac{x_0^2}{\xi_1\lambda_1}\psi_1^\epsilon\nabla\cdot\mathbf{A}^\epsilon\big) + \nu^2\eta\,\Re\big(\psi_1^\epsilon,\ i\frac{1}{\nu}\frac{x_0^2}{\xi_2\lambda_2}\psi_2^\epsilon\nabla\cdot\mathbf{A}^\epsilon\big)\\
&\quad= \eta\,\Re\left\{\int_\Omega i\left[\frac{x_0^2}{\xi_1\lambda_1}\psi_1^{\epsilon*}\psi_2^\epsilon + \frac{\xi_2^2\lambda_2^2}{\xi_1^2\lambda_1^2}\,\frac{\xi_1\lambda_1}{\xi_2\lambda_2}\,\frac{x_0^2}{\xi_2\lambda_2}\psi_1^\epsilon\psi_2^{\epsilon*}\right]\nabla\cdot\mathbf{A}^\epsilon\,d\Omega\right\}\\
&\quad= \eta\,\Re\left\{\int_\Omega i\frac{x_0^2}{\xi_1\lambda_1}\left[\psi_1^{\epsilon*}\psi_2^\epsilon + \psi_1^\epsilon\psi_2^{\epsilon*}\right]\nabla\cdot\mathbf{A}^\epsilon\,d\Omega\right\}\\
&\quad= 0.
\end{aligned}
$$

We also have for $j = 1, 2$,

$$
\begin{aligned}
&\Re\left\{(|\psi_j^\epsilon|^2 - T_j)\psi_j^\epsilon,\ i\frac{x_0^2}{\xi_j\lambda_j}\psi_j^\epsilon\nabla\cdot\mathbf{A}^\epsilon\right\} = 0,\\
&\Re\left\{\psi_j^\epsilon,\ i\frac{x_0^2}{\xi_j\lambda_j}\psi_j^\epsilon\nabla\cdot\mathbf{A}^\epsilon\right\}_{\partial\Omega} = 0.
\end{aligned}
$$

Therefore, we get for $t \in [0,T]$,

$$\int_0^t \Re\big(\frac{\partial \psi_1^\epsilon}{\partial t} + i\phi_c\psi_1^\epsilon,\ i\frac{x_0^2}{\xi_1\lambda_1}\psi_1^\epsilon \nabla\cdot\mathbf{A}^\epsilon\big) + \Gamma\Re\big(\frac{\partial \psi_2^\epsilon}{\partial t} + i\phi_c\psi_2^\epsilon,\ i\frac{1}{\nu}\frac{x_0^2}{\xi_2\lambda_2}\psi_2^\epsilon \nabla\cdot\mathbf{A}^\epsilon\big)ds$$
$$= -\int_0^t \Re\int_\Omega \Big[\frac{x_0}{\lambda_1}\psi_1^{\epsilon*}\big(-i\frac{\xi_1}{x_0}\nabla\psi_1^\epsilon - \frac{x_0}{\lambda_1}(\mathbf{A}^\epsilon+\mathbf{A}_c)\psi_1^\epsilon\big)\cdot\nabla(\nabla\cdot\mathbf{A}^\epsilon)$$
$$+ \frac{1}{\nu}\frac{x_0}{\lambda_2}\psi_2^{\epsilon*}\big(-i\frac{\xi_2}{x_0}\nabla\psi_2^\epsilon - \nu\frac{x_0}{\lambda_2}(\mathbf{A}^\epsilon+\mathbf{A}_c)\psi_2^\epsilon\big)\cdot\nabla(\nabla\cdot\mathbf{A}^\epsilon)\Big]d\Omega ds. \tag{3.110}$$

Next we multiply the PDE for $\mathbf{A}^\epsilon$, repeated below, by $-\nabla(\nabla\cdot\mathbf{A}^\epsilon)$ and integrate on Ω,

$$\begin{aligned}
\sigma\frac{x_0^2}{\lambda_1^2}\frac{\partial\mathbf{A}^\epsilon}{\partial t} - \epsilon\nabla(\nabla\cdot\mathbf{A}^\epsilon) &= -\mathrm{curl}^2\mathbf{A}^\epsilon + \mathrm{curl}\,\mathbf{H}_e \\
&+ \Re\big[\frac{x_0}{\lambda_1}\psi_1^{\epsilon*}\big(-i\frac{\xi_1}{x_0}\nabla - \frac{x_0}{\lambda_1}(\mathbf{A}^\epsilon+\mathbf{A}_c)\big)\psi_1^\epsilon\big] \\
&+ \Re\big[\frac{1}{\nu}\frac{x_0}{\lambda_2}\psi_2^{\epsilon*}\big(-i\frac{\xi_2}{x_0}\nabla - \nu\frac{x_0}{\lambda_2}(\mathbf{A}^\epsilon+\mathbf{A}_c)\big)\psi_2^\epsilon\big]
\end{aligned}$$

then through integrations by parts, we obtain for $t \in [0,T]$ that

$$\int_0^t \sigma\frac{x_0^2}{\lambda_1^2}\langle\frac{\partial\nabla\cdot\mathbf{A}^\epsilon}{\partial t},\ \nabla\cdot\mathbf{A}^\epsilon\rangle ds + \epsilon\int_0^t ||\nabla(\nabla\cdot\mathbf{A}^\epsilon)||_0^2 ds$$
$$= \int_0^t\int_\Omega (\mathrm{curl}\mathbf{A}^\epsilon - \mathbf{H}_e)\cdot\ \mathrm{curl}(\nabla(\nabla\cdot\mathbf{A}^\epsilon))\,d\Omega ds$$
$$-\int_0^t \Re\int_\Omega \Big[\frac{x_0}{\lambda_1}\psi_1^{\epsilon*}\big(-i\frac{\xi_1}{x_0}\nabla\psi_1^\epsilon - \frac{x_0}{\lambda_1}(\mathbf{A}^\epsilon+\mathbf{A}_c)\psi_1^\epsilon\big)\cdot\nabla(\nabla\cdot\mathbf{A}^\epsilon)$$
$$+ \frac{1}{\nu}\frac{x_0}{\lambda_2}\psi_2^{\epsilon*}\big(-i\frac{\xi_2}{x_0}\nabla\psi_2^\epsilon - \nu\frac{x_0}{\lambda_2}(\mathbf{A}^\epsilon+\mathbf{A}_c)\psi_2^\epsilon\big)\cdot\nabla(\nabla\cdot\mathbf{A}^\epsilon)\Big]\,d\Omega ds.$$

Since we have proved in theorem 3.2.19 that $\partial\mathbf{A}/\partial t \in \mathbf{L}^2(0,T;\mathbf{H}_n^1(\Omega))$, the integration by parts involving the time derivative term can be justified. The integral involving the curl operator on the R.H.S. of the inequality above is obtained by the integration by parts showed in lemma 3.2.20. This integral vanishes by the identity $\mathrm{curl}\nabla\varphi = 0$. By the regularity results (3.105)-(3.106), we have for a.e. $t \in [0,T]$ that $\langle\partial\nabla\cdot\mathbf{A}^\epsilon(t)/\partial t,\ \nabla\cdot\mathbf{A}^\epsilon(t)\rangle = (d/2dt)||\nabla\cdot\mathbf{A}^\epsilon(t)||_0^2$, see, for example, [53]. Therefore by using equation (3.110), for $t \in [0,T]$, we get

$$
\begin{aligned}
&\sigma\frac{x_0^2}{2\lambda_1^2}\big[\,||\nabla\cdot\mathbf{A}^\epsilon(t)||_0^2+||\nabla\cdot\mathbf{A}^\epsilon(0)||_0^2\,\big]+\epsilon\int_0^t||\nabla(\nabla\cdot\mathbf{A}^\epsilon)||_0^2\,ds\\
&=\int_0^t\Re\big(\frac{\partial\psi_1^\epsilon}{\partial t}+i\phi_c\psi_1^\epsilon,\ i\frac{x_0^2}{\xi_1\lambda_1}\psi_1^\epsilon\nabla\cdot\mathbf{A}^\epsilon\big)ds+\Gamma\int_0^t\Re\big(\frac{\partial\psi_2^\epsilon}{\partial t}+i\phi_c\psi_2^\epsilon,\ i\frac{1}{\nu}\frac{x_0^2}{\xi_2\lambda_2}\psi_2^\epsilon\nabla\cdot\mathbf{A}^\epsilon\big)ds\\
&\le\frac{x_0^2}{\xi_1\lambda_1}\Bigg[||\psi_1^\epsilon||_{\mathcal{L}^\infty(\Omega\times[0,T])}\int_0^t||\frac{\partial\psi_1^\epsilon}{\partial t}||_0\,ds+||\psi_1^\epsilon||^2_{\mathcal{L}^\infty(\Omega\times[0,T])}\int_0^t||\phi_c||_0\,ds\Bigg]||\nabla\cdot\mathbf{A}^\epsilon||_0ds\\
&\quad+\frac{\Gamma}{\nu}\frac{x_0^2}{\xi_2\lambda_2}\Bigg[||\psi_2^\epsilon||_{\mathcal{L}^\infty(\Omega\times[0,T])}\int_0^t||\frac{\partial\psi_2^\epsilon}{\partial t}||_0\,ds+||\psi_2^\epsilon||^2_{\mathcal{L}^\infty(\Omega\times[0,T])}\int_0^t||\phi_c||_0ds\Bigg]||\nabla\cdot\mathbf{A}^\epsilon||_0\,ds\\
&\le C\int_0^t\Bigg[||\frac{\partial\psi_1^\epsilon}{\partial t}||_0+||\phi_c||_0\Bigg]^2ds+\int_0^t||\nabla\cdot\mathbf{A}^\epsilon||_0^2\,ds\\
&\quad+C\int_0^t\Bigg[||\frac{\partial\psi_2^\epsilon}{\partial t}||_0+||\phi_c||_0\Bigg]^2ds+\int_0^t||\nabla\cdot\mathbf{A}^\epsilon||_0^2\,ds,
\end{aligned}
\tag{3.111}
$$

here we have used the fact from theorem 3.2.16 that $\psi_i^\epsilon\in\mathcal{L}^\infty(\Omega\times[0,T])$ to obtain the constant C which is independent of ϵ. This proves the lemma. ∎

Theorem 3.2.22 *Suppose the assumptions in theorem 3.2.19 hold and* $\mathbf{A}_0\in\mathbf{H}_n^1(\mathrm{div};\Omega)$, *then for any* $\epsilon>0$ *and* $T>0$,

$$
\begin{aligned}
&||\mathrm{div}\mathbf{A}^\epsilon||_{L^\infty(0,T;L^2(\Omega))}\\
&\quad\le C(1+DTe^{DT})\Bigg[||\frac{\partial\psi_1^\epsilon}{\partial t}||^2_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))}+||\frac{\partial\psi_2^\epsilon}{\partial t}||^2_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))}+2||\phi_c||^2_{L^2(0,T;L^2(\Omega))}\Bigg],
\end{aligned}
\tag{3.112}
$$

where the constant C and D are independent of ϵ. Since from lemma 3.2.6, both $\{\partial\psi_1^\epsilon/\partial t\}$ and $\{\partial\psi_2^\epsilon/\partial t\}$ are uniformly bounded in $\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))$ independent of ϵ, so $\{\mathrm{div}\mathbf{A}^\epsilon\}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$, independent of ϵ. As a result, $\{(\psi_1^\epsilon,\psi_2^\epsilon,\mathbf{A}^\epsilon)\}$ is uniformly bounded in $\mathcal{V}\times\mathcal{V}\times\mathbf{V}$, independent of ϵ.

Proof Applying Gronwall's inequality in integral form to the inequality (3.111), we get for a.e. $t\in[0,T]$,

$$
\begin{aligned}
||\nabla\cdot\mathbf{A}^\epsilon||_{L^\infty(0,T;L^2(\Omega))}\ \le\ &C(1+DTe^{DT})\Bigg(||\nabla\cdot\mathbf{A}^\epsilon(0)||_0^2+||\frac{\partial\psi_1^\epsilon}{\partial t}||^2_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))}\\
&+||\frac{\partial\psi_2^\epsilon}{\partial t}||^2_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))}+2||\phi_c||^2_{L^2(0,T;L^2(\Omega))}\Bigg)
\end{aligned}
$$

By the assumption of $\mathbf{A}_0$ and since $\mathbf{A}^\epsilon(0)=\mathbf{A}_0$, we have $\mathrm{div}\mathbf{A}^\epsilon(0)=0$. This gives the inequality (3.112). By using the same a priori estimates for the problem $(\mathrm{WP}_\mathrm{n}^\epsilon)$ as before, but now for $\{(\psi_1^\epsilon, \psi_2^\epsilon, \mathbf{A}^\epsilon)\}$, we get the uniform boundedness of the solution in $\mathcal{V} \times \mathcal{V} \times \mathbf{V}$ $\blacksquare$

In theorem 3.2.19 and theorem 3.2.22 we see that in addition to the regularity assumptions **RA1**, we also need $\mathbf{A}_c \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{H}^1(0,T;\mathbf{L}^2(\Omega))$, $\phi_c \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$, $\mathbf{H}_e \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{H}^1(0,T;\mathbf{H}(\text{curl};\Omega))$, $\mathbf{A}_0 \in \mathbf{H}_n^1(\text{div};\Omega) \cap \mathbf{H}^2(\Omega)$, $\mathbf{j}_c(0) \in \mathbf{H}(\text{div}_0;\Omega)$, $\mathbf{A}_c(0) \in \mathbf{H}(\text{div}_0;\Omega)$, $\phi_c(0) \in H^1(\Omega)$, $\psi_{i0}(0) \in \mathcal{H}^2(\Omega)$ with $|\psi_{i0}| \leq a$ for $i = 1, 2$. Therefore, hereafter unless otherwise stated, we will assume that the following updated regularity assumptions are satisfied throughout the rest of this work:

RA2: Assume that $\psi_{i0} \in \mathcal{H}^2(\Omega)$ with $|\psi_{i0}| \leq a$ for $i = 1, 2$, where a is defined in Theorem 3.2.16, $\mathbf{A}_0 \in \mathbf{H}_n^1(\text{div};\Omega) \cap \mathbf{H}^2(\Omega)$, $\mathbf{j}_c(\mathbf{x}, 0) \in \mathbf{H}(\text{div}_0;\Omega)$, $\mathbf{A}_c(\mathbf{x}, 0) \in \mathbf{H}(\text{div}_0;\Omega)$, $\phi_c(\mathbf{x}, 0) \in H^1(\Omega)$. We also assume $\mathbf{A}_c \in \mathbf{L}^\infty(0,T;\mathbf{H}_n^1(\Omega)) \cap \mathbf{H}^1(0,T;\mathbf{L}^2(\Omega))$, $\phi_c \in L^4(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and $\mathbf{H}_e \in \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega)) \cap \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{H}^1(0,T;\mathbf{H}(\text{curl};\Omega))$. The last relation gives $\mathbf{H}_e(\mathbf{x}, 0) \in \mathbf{H}(\text{curl};\Omega)$.

Remark Now up to this point the regularity requirements are all clear and we want to derive the regularity requirement of the Type-A applied current $\mathbf{j}_a \cdot \mathbf{n}|_{\partial\Omega}$ such that that regularity requirements for both $\mathbf{A}_c$ and ϕ_c as listed in **RA2** can all be satisfied. From the norm estimates in inequalities (2.52) and (2.55), (2.62), (2.64), and the comments following them, we can see that if $\mathbf{j}_c \in \mathbf{H}^1(0,T;\mathbf{H}(\text{div}_0;\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{H}(\text{div}_0;\Omega))$, then we have $\phi_c \in H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap C([0,T];H^1(\Omega))$, and also $\mathbf{A}_c \in \mathbf{H}^1(0,T;\mathbf{H}_n^1(\text{div};\Omega)) \cap \mathbf{L}^\infty(0,\mathrm{T};\mathbf{H}_\mathrm{n}^1(\text{div};\Omega)) \cap \mathbf{C}([0,\mathrm{T}];\mathbf{H}_\mathrm{n}^1(\text{div};\Omega))$. These regularities imply $\phi_c(\mathbf{x}, 0) \in H^1(\Omega)$ and $\mathbf{A}_c(\mathbf{x}, 0) \in \mathbf{H}_n^1(\text{div};\Omega)$. Lastly, from the norm estimates (2.46) and (2.47), we see that it is enough to have the Type-A applied current satisfying the regularity $\mathbf{j}_a \cdot \mathbf{n}|_{\partial\Omega} \in H^1(0,T;H^{-\frac{1}{2}}(\partial\Omega)) \cap L^\infty(0,T;H^{-\frac{1}{2}}(\partial\Omega))$ in order to obtain $\mathbf{j}_c \in \mathbf{H}^1(0,T;\mathbf{H}(\text{div}_0;\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{H}(\text{div}_0;\Omega))$. The last relation implies $\mathbf{j}_c(\mathbf{x}, 0) \in \mathbf{H}(\text{div}_0;\Omega)$. However, to guarantee that the internal current $\mathbf{j}_c$ is divergence free as a solution to the BVP (2.39)-(2.40), we need to define $\mathbf{j}_a|_{\partial\Omega} \in H^1(0,T;H^{\frac{1}{2}}(\partial\Omega)) \cap L^\infty(0,T;H^{\frac{1}{2}}(\partial\Omega))$.

The regularity assumptions **RA** found at the beginning of our analysis section summarized all the regularity requirements we needed in our analysis.

Now we are ready to pass to the limit $\epsilon \to 0$. Analogous to corollary 3.2.12 with the role of n replaced by ϵ, we obtain a weak (weak*) and strong limit $(\psi_1, \psi_2, \mathbf{A})$ as $\epsilon_k \to 0$.

Lemma 3.2.23 *Given $T > 0$, there exists an element $(\psi_1, \psi_2, \mathbf{A}) \in \mathcal{V} \times \mathcal{V} \times \mathbf{V}$, and sebsequences $\{\psi_1^{\epsilon_k}\}$, $\{\psi_2^{\epsilon_k}\}$ and $\{\mathbf{A}^{\epsilon_k}\}$ such that as $\epsilon_k \to 0$,*

$$
\begin{aligned}
\psi_1^{\epsilon_k} &\rightharpoonup \psi_1 \text{ weakly (and } \stackrel{*}{\rightharpoonup} \text{ weakly}^*\text{) in } \mathcal{V}, \\
\psi_2^{\epsilon_k} &\rightharpoonup \psi_2 \text{ weakly (and } \stackrel{*}{\rightharpoonup} \text{ weakly}^*\text{) in } \mathcal{V}, \\
\mathbf{A}^{\epsilon_k} &\rightharpoonup \mathbf{A} \text{ weakly (and } \stackrel{*}{\rightharpoonup} \text{ weakly}^*\text{) in } \mathbf{V}, \\
\psi_1^{\epsilon_k} &\to \psi_1 \text{ strongly in } \mathcal{L}^{\mathrm{P}}(0, \mathrm{T}; \mathcal{L}^{\mathrm{q}}(\Omega)), \\
\psi_2^{\epsilon_k} &\to \psi_2 \text{ strongly in } \mathcal{L}^{\mathrm{P}}(0, \mathrm{T}; \mathcal{L}^{\mathrm{q}}(\Omega)), \\
\mathbf{A}^{\epsilon_k} &\to \mathbf{A} \text{ strongly in } \mathbf{L}^{\mathrm{P}}(0, \mathrm{T}; \mathbf{L}^{\mathrm{q}}(\Omega)),
\end{aligned}
$$

where $p \in (1, \infty)$, and $q \in (1, \infty)$ for $d = 2$ and $q \in (1, 6)$ for $d = 3$.

Observe that with the uniform boundedness of the $\text{div}\mathbf{A}^\epsilon$ term asserted in theorem 3.2.22, the term involving $\epsilon\text{div}\mathbf{A}^\epsilon$ in the modified problem (WP$^\epsilon$) now tends to zero as $\epsilon \to 0$. Following the proof of theorem 3.2.13 in passing to the limit, now with $\epsilon_k \to 0$, and we can show that $(\psi_1, \psi_2, \mathbf{A})$ is a solution for the original problem (WP).

Theorem 3.2.24 *Given $T > 0$, the weak (weak*) limit $(\psi_1, \psi_2, \mathbf{A})$ in lemma 3.2.23 is a solution of the original problem* (WP)*.*

Following the proof of theorem 3.2.15, we can show that the solution $(\psi_1, \psi_2, \mathbf{A})$ also satisfies the "maximum" principle as stated below.

Theorem 3.2.25 *For any $\epsilon > 0$ and $T > 0$, let $(\psi_1, \psi_2, \mathbf{A})$ be a solution of the problem* (WP)*, and let $\Upsilon = \max\{|\mathcal{T}_1|, |\mathcal{T}_2|\}$, $\eta^* = \max\{|\eta|, |\eta|\nu^2\}$ and $a = \sqrt{4\eta^* + \Upsilon}$. Suppose $|\psi_1(\mathbf{x}, 0)| = |\psi_{10}| \le a$ and $|\psi_2(\mathbf{x}, 0)| = |\psi_{20}| \le a$ a.e. in Ω, then $|\psi_1(\mathbf{x}, t)| \le a$ and $|\psi_2(\mathbf{x}, t)| \le a$ a.e. in $\Omega \times [0, T]$.*

Theorem 3.2.26 *Given $T > 0$,*

$$
\begin{aligned}
\psi_1,\, \psi_2 &\in \mathcal{L}^2(0, T; \mathcal{H}^2(\Omega)), \\
\frac{\partial \psi_1}{\partial t}, \frac{\partial \psi_2}{\partial t} &\in \mathcal{L}^2(0, T; \mathcal{H}^1(\Omega)) \cap \mathcal{H}^1(0, T; (\mathcal{H}^1(\Omega))'), \\
\text{curl}\mathbf{A} &\in \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega)), \\
\text{div}\mathbf{A} &\in L^\infty(0, T; L^2(\Omega)), \\
\frac{\partial \text{div}\mathbf{A}}{\partial t} &\in L^2(0, T; L^2(\Omega)).
\end{aligned}
$$

Moreover, the bounds of the norms are independent of ϵ.

Proof Each of the first four relations follows from theorem 3.2.22 and the weak or weak* convergence of its ϵ counterpart sequence. We now show that $\partial(\text{div}\mathbf{A})/\partial t \in L^2(0,T;L^2(\Omega))$, with bound of its norm independent of ϵ. By taking the divergence of equation (2.73) with $\eta_1 = 0$, we get

$$
\begin{aligned}
\sigma \frac{x_0^2}{\lambda_1^2}\frac{\partial \text{div}\mathbf{A}}{\partial t} = & -\frac{1}{\kappa_1}\Im\left(\psi_1 \nabla^2 \psi_1^*\right) - \frac{x_0^2}{\lambda_1^2}\left[2\Re\left(\psi_1 \nabla \psi_1^*\right)\cdot(\mathbf{A}+\mathbf{A}_c) + |\psi_1|^2 \operatorname{div}(\mathbf{A}+\mathbf{A}_c)\right] \\
- & \frac{1}{\kappa_2}\Im\left(\psi_2 \nabla^2 \psi_2^*\right) - \frac{x_0^2}{\lambda_2^2}\left[2\Re\left(\psi_2 \nabla \psi_2^*\right)\cdot(\mathbf{A}+\mathbf{A}_c) + |\psi_2|^2 \operatorname{div}(\mathbf{A}+\mathbf{A}_c)\right].
\end{aligned} \tag{3.113}
$$

From theorem 3.2.25, we have $\psi_1,\ \psi_2 \in \mathcal{L}^\infty(\Omega\times(0,T))$ and from previous regularity results, we have $\psi_1,\ \psi_2 \in \mathcal{L}^2(0,T;\mathcal{H}^2(\Omega))$, $\mathbf{A}\in\mathbf{L}^2(0,T;\mathbf{H}_n^1(\Omega))$ and $\mathbf{A}_c\in\mathbf{L}^2(0,T;\mathbf{H}_n^1(\Omega))$, with bounds of their norms independent of ϵ. From these regularities, the terms on the right hand side of equation (3.113) belong to $L^2(0,T;L^2(\Omega))$ with bound of its norm independent of ϵ. ∎

By using the result of theorem 3.2.25 and following the proof of theorem 3.2.14 with some minor modifications, we arrive at the following theorem.

Theorem 3.2.27 *Given $T>0$, let $(\psi_1^1,\psi_2^1,\mathbf{A}^1)$ and $(\psi_1^2,\psi_2^2,\mathbf{A}^2)$ be any two solutions of the problem* (WP) *with initial conditions $(\psi_{10}^1,\psi_{20}^1,\mathbf{A}_0^1)$ and $(\psi_{10}^2,\psi_{20}^2,\mathbf{A}_0^2)$ respectively, then for $t\in[0,T]$,*

$$
\begin{aligned}
&||\psi_1^1(t)-\psi_1^2(t)||_0^2 + ||\psi_2^1(t)-\psi_2^2(t)||_0^2 + ||\mathbf{A}^1(t)-\mathbf{A}^2(t)||_{\mathbf{L}^2(\Omega)}^2 \\
&\qquad \le C\left(||\psi_{10}^1-\psi_{10}^2||_0^2 + ||\psi_{20}^1-\psi_{20}^2||_0^2 + ||\mathbf{A}_0^1-\mathbf{A}_0^2||_0^2\right).
\end{aligned} \tag{3.114}
$$

Corollary 3.2.28 *Given $T>0$, the sequence $\{(\psi_1^\epsilon,\psi_2^\epsilon,\mathbf{A}^\epsilon)\}$ converges weakly (and weakly*) in $\mathcal{V}\times\mathcal{V}\times\mathbf{V}$ to the unique solution $(\psi_1,\psi_2,\mathbf{A})$ of the problem* (WP).

As in lemma 3.2.14, we want to stress that the uniqueness of $(\psi_1,\psi_2,\mathbf{A})$ depends on the choice of $\mathbf{A}_c$ which in turn depends on the choice of $\mathbf{j}_c$. As stated at the beginning of section 3.1, we are eventually interested in the total solution $(\psi_1,\psi_2,\overline{\mathbf{A}}) = (\psi_1,\psi_2,\mathbf{A}+\mathbf{A}_c)$. From the regularity of $\mathbf{A}_c$ as stated in the regularity assumption **RA2**, we find the total solution $(\psi_1,\psi_2,\overline{\mathbf{A}})\in\mathcal{V}\times\mathcal{V}\times\mathbf{V}$ by adding up $\overline{\mathbf{A}}$ as $\mathbf{A}+\mathbf{A}_c$.

Theorem 3.2.29 *Given $T > 0$, there exist a weak total solution $(\psi_1, \psi_2, \overline{\mathbf{A}}) = (\psi_1, \psi_2, \mathbf{A} + \mathbf{A}_c) \in \mathcal{V} \times \mathcal{V} \times \mathbf{V}$ to the problem IBVP1 and IBVP2. Let $(\psi_1^1, \psi_2^1, \overline{\mathbf{A}}^1)$ and $(\psi_1^2, \psi_2^2, \overline{\mathbf{A}}^2)$ be any two weak solutions of the problem IBVP1 and IBVP2 with initial conditions $(\psi_{10}^1, \psi_{20}^1, \mathbf{A}_0^1)$ and $(\psi_{10}^2, \psi_{20}^2, \mathbf{A}_0^2)$ respectively, where for $i = 1, 2$, in the case of IBVP1, $\mathbf{A}_0^i = \overline{\mathbf{A}}^i(\mathbf{x}, 0) - \mathbf{A}_c(\mathbf{x}, 0)$, and in the case of IBVP2, $\mathbf{A}_0^i = \overline{\mathbf{A}}^i(\mathbf{x}, 0)$. Then for $t \in [0, T]$,*

$$
\begin{aligned}
&||\psi_1^1(t) - \psi_1^2(t)||_0^2 + ||\psi_2^1(t) - \psi_2^2(t)||_0^2 + ||\overline{\mathbf{A}}^1(t) - \overline{\mathbf{A}}^2(t)||_{\mathbf{L}^2(\Omega)}^2 \\
&\qquad \le C \left(||\psi_{10}^1 - \psi_{10}^2||_0^2 + ||\psi_{20}^1 - \psi_{20}^2||_0^2 + ||\mathbf{A}_0^1 - \mathbf{A}_0^2||_0^2 \right). \qquad (3.115)
\end{aligned}
$$

Hence, the total solution $(\psi_1, \psi_2, \overline{\mathbf{A}})$ is unique, independent of the choice of $\mathbf{A}_c$.

Proof The existence of the weak total solution of the problem IBVP1 is obvious. We only need to show the uniqueness of the total solution. As mentioned in the previous remark, the solution $(\psi_1, \psi_2, \mathbf{A})$ depends on the choice of $\mathbf{A}_c$, so our question is whether a different choice of $\mathbf{A}_c$ would give the same total solution. First recall that ϕ_c is uniquely determined by a Type-A applied current $\mathbf{j}_a$ in the form $\mathbf{j}_a \cdot \mathbf{n}|_{\partial\Omega}$ (see 2.50)-(2.51)). In view of equation (2.70), repeated below

$$
-\frac{\sigma}{\kappa_1}\nabla\phi_c \;=\; \mathrm{curl}^2\mathbf{A}_c + \sigma\frac{x_0^2}{\lambda_1^2}\frac{\partial \mathbf{A}_c}{\partial t} \qquad \text{in } \Omega,
$$

and also by the construction of $\mathbf{A}_c$ that we have $\mathrm{div}\mathbf{A}_c = 0$ in Ω, $\mathbf{A}_c \cdot \mathbf{n} = 0$ and $\mathrm{curl}\mathbf{A}_c \times \mathbf{n} = 0$ on $\partial\Omega$, we can see that by setting $\phi_c = \phi_a$ the first approach problem IBVP1 as in (2.71)-(2.73) and (2.75)-(2.95) are equivalent to the second approach problem IBVP2 as in (2.85)-(2.87) and (2.88)-(2.95). By this equivalent relation, it is clear that there exists the same weak total solution for the problem IBVP2. Following exactly the same proof of lemma 3.2.14 but with the superscript ϵ and $\mathbf{A}_c$ removed everywhere and $\mathbf{A}^j$ replaced by $\overline{\mathbf{A}}^j$, $j = 1, 2$, and notice that the terms involving $\nabla\phi_a$ are cancelled out in the proof, we obtain the data-dependency inequality (3.115). From this, we see that given a unique ϕ_a, the weak solution $(\psi_1, \psi_2, \overline{\mathbf{A}})$ for IBVP2 is unique. In other words, given $\mathbf{j}_a \cdot \mathbf{n}$, there exists a unique weak solution $(\psi_1, \psi_2, \overline{\mathbf{A}})$ to the problem IBVP2. By the equivalence of IBVP1 and IBVP2, the same data-dependency equation holds for the same weak total solution for the problem IBVP1. This completes the proof. ∎

The exact result in theorem 3.2.25 hold for $(\psi_1, \psi_2, \overline{\mathbf{A}})$, as shown in the next corollary.

Corollary 3.2.30 *For any $\epsilon > 0$ and $T > 0$, let $(\psi_1, \psi_2, \overline{\mathbf{A}})$ be the weak total solution of the problem IBVP1 and IBVP2, and let $\Upsilon = \max\{|\mathcal{T}_1|, |\mathcal{T}_2|\}$, $\eta^* = \max\{|\eta|, |\eta|\nu^2\}$ and $a = \sqrt{4\eta^* + \Upsilon}$. Suppose $|\psi_1(\mathbf{x}, 0)| = |\psi_{10}| \leq a$ and $|\psi_2(\mathbf{x}, 0)| = |\psi_{20}| \leq a$ a.e. in Ω, then $|\psi_1(\mathbf{x}, t)| \leq a$ and $|\psi_2(\mathbf{x}, t)| \leq a$ a.e. in $\Omega \times [0, T]$.*

However, due to the fact that $\mathbf{A}_c$ only has spatial regularity of up to $\mathbf{H}^1$, we don't have the same higher regularity for $\overline{\mathbf{A}}$ as in theorem 3.2.26 for $\mathbf{A}$.

Corollary 3.2.31 *Given $T > 0$,*

$$\begin{array}{rcl}
\psi_1,\, \psi_2 & \in & \mathcal{L}^2(0, T; \mathcal{H}^2(\Omega)), \\
\dfrac{\partial \psi_1}{\partial t},\, \dfrac{\partial \psi_2}{\partial t} & \in & \mathcal{L}^2(0, T; \mathcal{H}^1(\Omega)) \cap \mathcal{H}^1(0, T; (\mathcal{H}^1(\Omega))'), \\
\mathrm{div}\overline{\mathbf{A}} & \in & L^\infty(0, T; L^2(\Omega)), \\
\dfrac{\partial \mathrm{div}\overline{\mathbf{A}}}{\partial t} & \in & L^2(0, T; L^2(\Omega)).
\end{array}$$

Moreover, the bounds of the norms are independent of ϵ.

CHAPTER 4

Finite Element Approximations

4.1 Backward Euler Finite Element Approximations

Throughout this work, we will use $\|.\|_0$ to denote the norms of any L^2 Lebesgue spaces, be it a real, complex or vector valued space; and $\|.\|_{r,q}$ for any $W^{r,q}(\Omega)$ spaces, for $1 \leq q \leq \infty$, $q \neq 2$. Also, we use $\|.\|_s$ for any $H^s(\Omega)$ spaces, and $\|.\|_{s,\partial\Omega}$ for any norms defined on the boundary $\partial\Omega$.

We assume $\Omega \subset \mathbb{R}^2$ is at least a simply connected bounded domain with a boundary $\partial\Omega$ of $C^{1,1}$ class or is piecewise smooth with no reentrant corners. When higher regularities are needed for the solution ψ_1, ψ_2 and $\mathbf{A}$ as are required in the error estimates that we will present later, we may need a smoother domain. However we will not go into details to elaborate the necessary requirements in order to attain the required regularities of the solution. With the first assumption on Ω, we have [44]

$$||\mathbf{A}||_1 \leq C(||\mathrm{div}\mathbf{A}||_0 + ||\mathrm{curl}\mathbf{A}||_0) \qquad \forall \mathbf{A} \in \mathbf{H}^1_{\mathrm{n}}(\Omega). \tag{4.1}$$

We will discretize the problem ($\mathbf{WP}^\epsilon$) in time by using backward Euler scheme, and in space by finite element methods. In time, let $\Delta t = T/N$, where N is a positive integer, be the time step. Let $t_0 = 0$, $t_n = t_{n-1} + \Delta t$, for $n = 1, ..., N$. Also let $\psi^n = \psi^\epsilon(\cdot, t_n)$, $\mathbf{A}^n = \mathbf{A}^\epsilon(\cdot, t_n)$, and define $\delta_t \psi^n = (\psi^n - \psi^{n-1})/\Delta t$ and $\delta_t \mathbf{A}^n = (\mathbf{A}^n - \mathbf{A}^{n-1})/\Delta t$. In space, let $\{\mathcal{T}_h\}$ be a family of regular, quasi-uniform triangulations over Ω with h defined as $h = \max_{K \in \mathcal{T}_h}\{\mathrm{diam}(K)\}$. Let $\mathcal{Z}_h \subset \mathcal{H}^1(\Omega)$ and $\mathbf{\Lambda}_h \subset \mathbf{H}^1_n(\Omega)$ be the H^1 conforming finite element spaces under the $\mathcal{T}_h$ triangulation with

$$\begin{aligned} \mathcal{Z}_h &= \{\psi \in \mathcal{C}(\overline{\Omega}) : \psi|_K \in \mathcal{P}_k, \text{for each } K \in \mathcal{T}_h\}, \\ \mathbf{\Lambda}_h &= \{\mathbf{A} \in \mathbf{C}(\overline{\Omega}) : \mathbf{A}|_K \in \mathbf{P}_k, \text{for each } K \in \mathcal{T}_h\} \cap \mathbf{H}^1_n(\Omega), \end{aligned}$$

where $\mathcal{P}_k$ and $\mathbf{P}_k$ are polynomial spaces of degree up to k.

We now state some of the approximation properties of our underlying finite element spaces. Let $I_h : \mathcal{C}(\overline{\Omega}) \to \mathcal{Z}_h$ and $\mathbf{I}_h : \mathbf{C}(\overline{\Omega}) \to \mathbf{\Lambda}_h$ be finite element global interpolation operators. So for $\psi \in \mathcal{C}(\overline{\Omega})$ and $\mathbf{A} \in \mathbf{C}(\overline{\Omega})$, we have

$$I_h\psi = \sum_{j=1}^{N} \psi(a_j)\varphi_j, \qquad \mathbf{I}_h\mathbf{A} = \sum_{j=1}^{N} \mathbf{A}(a_j)\Phi_j,$$

where $\{\varphi_j\}_{j=1}^N$ and $\{\Phi_j\}_{j=1}^N$ are the nodal bases of the spaces $\mathcal{Z}_h$ and $\mathbf{\Lambda}_h$, respectively, with the associated global nodes $\{a_j\}_{j=1}^N$. Then for $0 \le m \le s+1$ and $1 \le s \le k$, the interpolation errors in Ω satisfy

$$|\psi - I_h\psi|_m \le Ch^{s+1-m}|\psi|_{s+1} \qquad \forall\psi \in \mathcal{H}^{s+1}(\Omega), \tag{4.2}$$

$$|\mathbf{A} - \mathbf{I}_h\mathbf{A}|_m \le Ch^{s+1-m}|\mathbf{A}|_{s+1} \qquad \forall\psi \in \mathbf{H}^{s+1}(\Omega). \tag{4.3}$$

Let γ_0 be the trace operator. Let $I_h^{\gamma_0}$ and $\mathbf{I}_h^{\gamma_0}$ be the interpolants of continuous functions on the boundary, i.e., for $g \in \mathcal{C}(\partial\Omega)$ and $\mathbf{g} \in \mathbf{C}(\partial\Omega)$,

$$I_h^{\gamma_0}g = \sum_{\{j|a_j\in\partial\Omega\}} g(a_j)\gamma_0(\varphi_j), \qquad \mathbf{I}_h^{\gamma_0}\mathbf{g} = \sum_{\{j|a_j\in\partial\Omega\}} \mathbf{g}(a_j)\gamma_0(\Phi_j).$$

Then for $\frac{1}{2} \le m \le r$ and $1 < r \le k$, the interpolation errors on the boundary $\partial\Omega$ satisfy

$$|\gamma_0\psi - I_h^{\gamma_0}(\gamma_0\psi)|_{m-\frac{1}{2},\partial\Omega} \le Ch^{r-m}|\gamma_0\psi|_{r-\frac{1}{2},\partial\Omega} \qquad \forall\psi \in \mathcal{H}^{r-\frac{1}{2}}(\partial\Omega), \tag{4.4}$$

$$|\gamma_0\mathbf{A} - \mathbf{I}_h^{\gamma_0}(\gamma_0\mathbf{A})|_{m-\frac{1}{2},\partial\Omega} \le Ch^{r-m}|\gamma_0\mathbf{A}|_{r-\frac{1}{2},\partial\Omega} \qquad \forall\psi \in \mathbf{H}^{r-\frac{1}{2}}(\partial\Omega), \tag{4.5}$$

Note that for $u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{H}^1(\Omega)$, we have $\gamma_0(I_hu) = I_h^{\gamma_0}(\gamma_0u)$. Similarly, for $\mathbf{u} \in \mathbf{C}(\overline{\Omega}) \cap \mathbf{H}^1(\Omega)$, we have $\gamma_0(\mathbf{I}_hu) = \mathbf{I}_h^{\gamma_0}(\gamma_0\mathbf{u})$.

The spaces $\mathcal{Z}_h$ and $\mathbf{\Lambda}_h$ also satisfy the inverse inequalities:,

$$||\psi_h||_1 \le Ch^{-1}||\psi_h||_0 \qquad \forall\psi_h \in \mathcal{Z}_h, \tag{4.6}$$

$$||\mathbf{A}_h||_1 \le Ch^{-1}||\mathbf{A}_h||_0 \qquad \forall\mathbf{A} \in \mathbf{\Lambda}_h, \tag{4.7}$$

$$||\psi_h||_{0,\infty} \le Ch^{-\frac{d}{2}}||\psi_h||_0 \qquad \forall\psi_h \in \mathcal{Z}_h, \tag{4.8}$$

$$||\mathbf{A}_h||_{0,\infty} \le Ch^{-\frac{d}{2}}||\mathbf{A}_h||_0 \qquad \forall\mathbf{A}_h \in \mathbf{\Lambda}_h, \tag{4.9}$$

where d is the dimension of the domain Ω.

We also assume that for sufficiently smooth ψ and $\mathbf{A}$, the finite element spaces $\mathcal{Z}_h$ and $\mathbf{\Lambda}_h$ satisfy the approximability properties, see [19],

$$\lim_{h\to 0}\left[\sup_{t\in[0,T]}\inf_{\varphi_h\in\mathcal{Z}_h}\left(||\psi(t)-\varphi_h||_\infty + h^{-\frac{d}{2}}||\psi(t)-\varphi_h||_0\right)\right]=0, \tag{4.10}$$

$$\lim_{h\to 0}\left[\sup_{t\in[0,T]}\inf_{\mathbf{B}_h\in\mathbf{\Lambda}_h}\left(||\mathbf{A}(t)-\mathbf{B}_h||_\infty + h^{-\frac{d}{2}}||\mathbf{A}(t)-\mathbf{B}_h||_0\right)\right]=0. \tag{4.11}$$

Hereafter, we will use C to denote a generic constant. The context should make clear that the same constant C appearing at different places refers to the same or different values.

Throughout the rest of the work, we assume that ϕ_c, $\mathbf{A}_c$ and $\mathbf{H}_e$ are given, and that ϕ_c, $\mathbf{A}_c$, $\mathbf{H}_e$, ψ_1, ψ_2, and $\mathbf{A}$ satisfy the following regularity assumptions:

RA3: For $1 \leq k$ and $s = 2$ if $k = 1$, otherwise $s = k$ if $k \geq 2$, assume that

$$\begin{aligned}
\psi_1^\epsilon,\, \psi_2^\epsilon &\in \mathcal{C}([0,T];\mathcal{H}^{k+1}(\Omega)), && (4.12)\\
\frac{\partial\psi_1^\epsilon}{\partial t},\, \frac{\partial\psi_2^\epsilon}{\partial t} &\in \mathcal{L}^2(0,T;\mathcal{H}^s(\Omega)), && (4.13)\\
\mathbf{A}^\epsilon &\in \mathbf{C}([0,T];\mathbf{H}^{k+1}(\Omega)\cap\mathbf{H}_n^1(\Omega)), && (4.14)\\
\frac{\partial\mathbf{A}^\epsilon}{\partial t} &\in \mathbf{L}^2(0,T;\mathbf{H}^s(\Omega)\cap\mathbf{H}_n^1(\Omega)), && (4.15)\\
\phi_c &\in C([0,T];L^4(\Omega)), && (4.16)\\
\frac{\partial\phi_c}{\partial t} &\in L^2(0,T;L^4(\Omega)), && (4.17)\\
\mathbf{A}_c &\in \mathbf{C}([0,T];\mathbf{H}_n^1(\Omega)), && (4.18)\\
\frac{\partial\mathbf{A}_c}{\partial t} &\in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega)), && (4.19)\\
\mathbf{H}_e &\in \mathbf{C}([0,T];\mathbf{L}^2(\Omega)), && (4.20)\\
\frac{\partial\mathbf{H}_e}{\partial t} &\in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega)), && (4.21)\\
\psi_{10},\, \psi_{20} &\in \mathcal{H}^{k+1}(\Omega), && (4.22)\\
\mathbf{A}_0 &\in \mathbf{H}^{k+1}(\Omega)\cap\mathbf{H}_n^1(\mathrm{div};\Omega)). && (4.23)
\end{aligned}$$

The fully discretized approximation of the problem ($\mathbf{WP}^\epsilon$) is defined as the following problem.

Problem ($\mathbf{DP}^\epsilon$): For each fixed ϵ, where $0 < \epsilon \le 1$, and $n = 0, 1, \dots N$, find $(\psi^n_{h1}, \psi^n_{h2}, \mathbf{A}^n_h) \in \mathcal{Z}_h \times \mathcal{Z}_h \times \mathbf{\Lambda}_h$ such that for $t \in [0, T]$,

$$
\begin{aligned}
&(\delta_t \psi^n_{1h}, \tilde{\psi}_h) + (i\phi^n_c \psi^n_{1h}, \tilde{\psi}_h) + \Big((|\psi^n_{1h}|^2 - \mathcal{T}_1)\psi^n_{1h}, \tilde{\psi}_h\Big) \\
&\quad + \left(-i\frac{\xi_1}{x_0}\nabla\psi^n_{1h} - \frac{x_0}{\lambda_1}(\mathbf{A}^n_h + \mathbf{A}^n_c)\psi^n_{1h},\ -i\frac{\xi_1}{x_0}\nabla\tilde{\psi}_h - \frac{x_0}{\lambda_1}(\mathbf{A}^n_h + \mathbf{A}^n_c)\tilde{\psi}_h\right) \\
&\quad + \eta(\psi^n_{2h}, \tilde{\psi}_h) + \left(\gamma_1 \frac{\xi_1^2}{x_0^2}\psi^n_{1h}, \tilde{\psi}_h\right)_{\partial\Omega} = 0 \qquad \forall\, \tilde{\psi}_h \in \mathcal{Z}_h,
\end{aligned} \tag{4.24}
$$

$$
\begin{aligned}
&\Gamma(\delta_t \psi^n_{2h}, \tilde{\psi}_h) + \Gamma(i\phi^n_c \psi^n_{2h}, \tilde{\psi}_h) + \Big((|\psi^n_{2h}|^2 - \mathcal{T}_2)\psi^n_{2h}, \tilde{\psi}_h\Big) \\
&\quad + \left(-i\frac{\xi_2}{x_0}\nabla\psi^n_{2h} - \nu\frac{x_0}{\lambda_2}(\mathbf{A}^n_h + \mathbf{A}^n_c)\psi^n_{2h},\ -i\frac{\xi_2}{x_0}\nabla\tilde{\psi}_h - \nu\frac{x_0}{\lambda_2}(\mathbf{A}^n_h + \mathbf{A}^n_c)\tilde{\psi}_h\right) \\
&\quad + \eta\nu^2(\psi^n_{1h}, \tilde{\psi}_h) + \left(\gamma_2 \frac{\xi_2^2}{x_0^2}\psi^n_{2h}, \tilde{\psi}_h\right)_{\partial\Omega} = 0 \qquad \forall\, \tilde{\psi}_h \in \mathcal{Z}_h,
\end{aligned} \tag{4.25}
$$

$$
\begin{aligned}
&\sigma\frac{x_0^2}{\lambda_1^2}\Big(\delta_t \mathbf{A}^n_h,\ \tilde{\mathbf{A}}_h\Big) + (\mathrm{curl}\mathbf{A}^n_h,\ \mathrm{curl}\tilde{\mathbf{A}}_h) + \epsilon(\mathrm{div}\mathbf{A}^n_h,\ \mathrm{div}\tilde{\mathbf{A}}^n_h) \\
&\quad - \Re\left(\Big(-i\frac{\xi_1}{x_0}\nabla\psi^n_{1h} - \frac{x_0}{\lambda_1}(\mathbf{A}^n_h + \mathbf{A}^n_c)\psi^n_{1h}\Big),\ \frac{x_0}{\lambda_1}\psi^n_{1h}\tilde{\mathbf{A}}_h\right) \\
&\quad - \Re\left(\Big(-i\frac{\xi_2}{x_0}\nabla\psi^n_{2h} - \nu\frac{x_0}{\lambda_2}(\mathbf{A}^n_h + \mathbf{A}^n_c)\psi^n_{2h}\Big),\ \frac{1}{\nu}\frac{x_0}{\lambda_2}\psi^n_{2h}\tilde{\mathbf{A}}_h\right) \\
&\quad = (\mathbf{H}^n_e,\ \mathrm{curl}\tilde{\mathbf{A}}_h) \qquad \forall\, \tilde{\mathbf{A}}_h \in \mathbf{\Lambda}_h.
\end{aligned} \tag{4.26}
$$

with the initial conditions $\psi^0_{1h} = I_h\psi_{10} = I_h\psi^0_1 \in \mathcal{Z}_h$, $\psi^0_{2h} = I_h\psi_{20} = I_h\psi^0_2 \in \mathcal{Z}_h$, and $\mathbf{A}^0_h = \mathbf{I}_h\mathbf{A}_0 = \mathbf{I}_h\mathbf{A}^0 \in \mathbf{\Lambda}_h$.

4.2 Existence and Uniqueness of the Problem ($\mathbf{DP}^\epsilon$)

Theorem 4.2.1 *Assume $\mathbf{A}_c$, $\mathbf{H}_e$, ψ_{10}, ψ_{20} and $\mathbf{A}_0$ satisfy the regularity assumptions (4.18), (4.20), (4.22) and (4.23), respectively, then for small enough $\Delta t^{\frac{1}{2}}/h$ and for $1 \le n \le N$, there exists a unique solution $(\psi^n_{1h}, \psi^n_{2h}, \mathbf{A}^n_h) \in \mathcal{Z}_h \times \mathcal{Z}_h \times \mathbf{\Lambda}_h$ to the problem ($\mathbf{DP}^\epsilon$).*

Proof We will use Banach's fixed-point theorem to prove the existence and uniqueness of the solution to the problem ($\mathbf{DP}^\epsilon$).

Define a finite-dimensional Hilbert space

$$
\mathbf{B} = \mathcal{Z}_h \times \mathcal{Z}_h \times \mathbf{\Lambda}_h,
$$

with norm

$$||(\psi_{1h},\,\psi_{2h},\,\mathbf{A}_h)||_{\mathbf{B}} = ||\psi_{1h}||_1 + ||\psi_{2h}||_1 + ||\mathbf{A}_h||_1.$$

Let $\epsilon > 0$ be given. For each fixed n, given a unique $(\psi_{1h}^{n-1}, \psi_{2h}^{n-1}, \mathbf{A}_h^{n-1}) \in \mathbf{B}$ with $(\psi_{1h}^0, \psi_{2h}^0, \mathbf{A}_h^0) = (I_h\psi_{10}, I_h\psi_{20}, \mathbf{I}_h\mathbf{A}_0)$, define an operator $\mathbf{G}_h : \mathbf{B} \to \mathbf{B}$ as follows: for $(\tilde{\psi}_{1h}, \tilde{\psi}_{2h}, \tilde{\mathbf{A}}_h) \in \mathbf{B}$, $\mathbf{G}_h(\tilde{\psi}_{1h}, \tilde{\psi}_{2h}, \tilde{\mathbf{A}}_h) = (\psi_{1h}, \psi_{2h}, \mathbf{A}_h)$ is the solution to the following problem:

$$\begin{aligned}
a_1(\psi_{1h},\varphi) &= \langle f_1(\tilde{\psi}_{1h}, \tilde{\psi}_{2h}, \tilde{\mathbf{A}}_h),\,\varphi\rangle \qquad \forall\varphi \in \mathcal{Z}_h && (4.27)\\
a_2(\psi_{2h},\varphi) &= \langle f_2(\tilde{\psi}_{1h}, \tilde{\psi}_{2h}, \tilde{\mathbf{A}}_h),\,\varphi\rangle \qquad \forall\varphi \in \mathcal{Z}_h && (4.28)\\
b(\mathbf{A}_h,\,\Phi) &= \langle \mathbf{g}(\tilde{\psi}_{1h}, \tilde{\psi}_{2h}, \tilde{\mathbf{A}}_h),\,\Phi\rangle \qquad \forall\Phi \in \mathbf{\Lambda}_h && (4.29)
\end{aligned}$$

where $\langle\cdot,\,\cdot\rangle$ is the duality pairing in Ω; and for $i = 1, 2$, $a_i : \mathcal{Z}_h \times \mathcal{Z}_h \to \mathbb{C}$ is a sesquilinear form, and $b : \mathbf{\Lambda}_h \times \mathbf{\Lambda}_h \to \mathbb{R}^d$ is a bilinear form defined as

$$a_1(\psi_{1h},\,\varphi) = \Delta t\frac{\xi_{1h}^2}{x_0^2}(\nabla\psi_{1h},\,\nabla\,\varphi) + (1 + i\,\Delta t\phi_c^n)(\psi_{1h},\ \varphi) + \Delta t\gamma_1(\psi_{1h},\ \varphi)_{\partial\Omega}, \qquad (4.30)$$

$$a_2(\psi_{2h},\,\varphi) = \Delta t\frac{\xi_{2h}^2}{x_0^2}(\nabla\psi_{2h},\,\nabla\,\varphi) + \Gamma(1 + i\,\Delta t\phi_c^n)(\psi_{2h},\ \varphi) + \Delta t\gamma_2(\psi_{2h},\ \varphi)_{\partial\Omega}, \qquad (4.31)$$

$$b(\mathbf{A}_h,\,\Phi) = \epsilon\Delta t(\mathrm{div}\mathbf{A}_h,\,\mathrm{div}\Phi) + \Delta t(\mathrm{curl}\mathbf{A}_h,\,\mathrm{curl}\Phi) + \sigma\frac{x_0^2}{\lambda_1^2}(\mathbf{A}_h,\,\Phi), \qquad (4.32)$$

and

$$\begin{aligned}
\langle f_1(\tilde{\psi}_{1h}, \tilde{\psi}_{2h}, \tilde{\mathbf{A}}_h),\,\varphi\rangle &= \Big\langle \big[\,\psi_{1h}^{n-1} - \Delta t(|\tilde{\psi}_{1h}|^2 - \mathcal{T}_1)\tilde{\psi}_{1h}\\
&\quad -i\Delta t\frac{\xi_1}{\lambda_1}\big((\tilde{\mathbf{A}}_h + \mathbf{A}_c)\cdot\nabla\tilde{\psi}_{1h} + \nabla\cdot\big(\tilde{\psi}_{1h}(\tilde{\mathbf{A}}_h + \mathbf{A}_c)\big)\big)\\
&\quad -\Delta t\frac{x_0^2}{\lambda_1^2}|\tilde{\mathbf{A}}_h + \mathbf{A}_c|^2\tilde{\psi}_{1h} - \eta\Delta t\tilde{\psi}_{2h}\big],\,\varphi\Big\rangle, && (4.33)
\end{aligned}$$

$$
\begin{aligned}
\langle f_2(\tilde{\psi}_{1h}, \tilde{\psi}_{2h}, \tilde{\mathbf{A}}_h),\, \varphi\rangle &= \Big\langle \big[\, \Gamma\psi_{2h}^{n-1} - \Delta t(|\tilde{\psi}_{2h}|^2 - \mathcal{T}_2)\tilde{\psi}_{2h} \\
&\quad -i\Delta t\frac{\xi_2}{\lambda_2}\big((\tilde{\mathbf{A}}_h + \mathbf{A}_c)\cdot\nabla\tilde{\psi}_{2h} + \nabla\cdot\big(\tilde{\psi}_{2h}(\tilde{\mathbf{A}}_h + \mathbf{A}_c)\big)\big) \\
&\quad -\Delta t\frac{x_0^2}{\lambda_2^2}|\tilde{\mathbf{A}}_h + \mathbf{A}_c|^2\tilde{\psi}_{2h} - \eta\Delta t\tilde{\psi}_{1h}\big],\, \varphi\Big\rangle, \qquad (4.34)\\
\langle \mathbf{g}(\tilde{\psi}_{1h}, \tilde{\psi}_{2h}, \tilde{\mathbf{A}}_h),\, \Phi\rangle &= \langle\, \mathbf{H}_e,\, \mathrm{curl}\Phi\rangle + \Big\langle \big[\, \sigma\frac{x_0^2}{\lambda_1^2}\mathbf{A}_h^{n-1} \\
&\quad - \Delta t\frac{1}{\kappa_1}\Re(i\tilde{\psi}_{1h}^*\nabla\tilde{\psi}_{1h}) - \Delta t\frac{x_0^2}{\lambda_1^2}|\tilde{\psi}_{1h}|^2(\tilde{\mathbf{A}}_h + \mathbf{A}_c) \\
&\quad - \Delta t\frac{1}{\nu}\frac{1}{\kappa_2}\Re(i\tilde{\psi}_{2h}^*\nabla\tilde{\psi}_{2h}) - \Delta t\frac{x_0^2}{\lambda_2^2}|\tilde{\psi}_{2h}|^2(\tilde{\mathbf{A}}_h + \mathbf{A}_c)\big],\, \Phi\Big\rangle \qquad (4.35)\\
&:= \langle\, \mathbf{H}_e,\, \mathrm{curl}\Phi\rangle + \langle \mathbf{h}(\tilde{\psi}_{1h}, \tilde{\psi}_{2h}, \tilde{\mathbf{A}}_h),\, \Phi\rangle. \qquad (4.36)
\end{aligned}
$$

Note that the above problem is equivalent to the following distributional problem:

$$
\begin{aligned}
-\Delta t\frac{\xi_{1h}^2}{x_0^2}\triangle\psi_{1h} + (1 + i\,\Delta t\phi_c^n)\psi_{1h} &= f_1(\tilde{\psi}_{1h}, \tilde{\psi}_{2h}, \tilde{\mathbf{A}}_h) \quad \text{in } \Omega, \times[0, T], \\
\frac{\partial\psi_{1h}}{\partial n} &= -\gamma_1\psi_{1h} \quad \text{on } \partial\Omega\times[0, T], \\
-\Delta t\frac{\xi_{2h}^2}{x_0^2}\triangle\psi_{2h} + \Gamma(1 + i\,\Delta t\phi_c^n)\psi_{2h} &= f_2(\tilde{\psi}_{2h}, \tilde{\psi}_{2h}, \tilde{\mathbf{A}}_h) \quad \text{in } \Omega, \times[0, T], \\
\frac{\partial\psi_{2h}}{\partial n} &= -\gamma_2\psi_{2h} \quad \text{on } \partial\Omega\times[0, T], \\
-\epsilon\Delta t\nabla\mathrm{div}\mathbf{A}_h + \Delta t\mathrm{curl}^2\mathbf{A}_h + \sigma\frac{x_0^2}{\lambda_1^2}\mathbf{A}_h &= \mathrm{curl}\mathbf{H}_e + \mathbf{h}(\tilde{\psi}_{1h}, \tilde{\psi}_{2h}, \tilde{\mathbf{A}}_h) \quad \Omega\times[0, T], \\
(\mathrm{curl}\mathbf{A}_h - \mathbf{H}_e)\times\mathbf{n} &= \mathbf{0} \quad \text{on } \partial\Omega\times[0, T], \\
\mathbf{A}_h\cdot\mathbf{n} &= 0 \quad \text{on } \partial\Omega, \times[0, T],
\end{aligned}
$$

where $\mathbf{h}(\tilde{\psi}_{1h}, \tilde{\psi}_{2h}, \tilde{\mathbf{A}}_h)$ is defined in the last term in (4.36).

We first show that the operator is well-defined. We note that the sesquilinear form a_1 and a_2 are continuous and strongly coercive. Indeed, we have

$$
\begin{aligned}
|a_i(\psi_h, \psi_h)| &= \Big|\, \alpha\Delta t||\nabla\psi_h||_0^2 + \beta\int_\Omega(1 + i\Delta t\phi_c^n)\,|\psi_h|^2 + \gamma\Delta t||\psi_h||_{0,\partial\Omega}^2\,\Big| \\
&\geq \alpha\Delta t||\nabla\psi_h||_0^2 + \beta\,||\psi_h||_0^2 \\
&\geq \min(\alpha\Delta t, \beta)||\psi_h||_1^2,
\end{aligned}
$$

where α, β and γ are positive real constants.

Similarly, the bilinear form b is continuous and coercive.

It is easy to show that $f_1,\ f_2 \in (\mathcal{H}^1)' \subset \mathcal{Z}_h'$ and $\mathbf{g} \in (\mathbf{H}_n^1)' \subset \mathbf{\Lambda}_h'$. So by the Lax-Milgram theorem, given $(\tilde\psi_{1h},\ \tilde\psi_{2h},\ \tilde{\mathbf{A}}_h)$, there is a unique solution $(\psi_{1h}, \psi_{2h}, \mathbf{A}_h) = \mathbf{G}_h(\tilde\psi_{1h}, \tilde\psi_{2h}, \tilde{\mathbf{A}}_h) \in \mathbf{B}$. Therefore the operator $\mathbf{G}_h$ is well-defined.

Next we show that the operator $\mathbf{G}_h$ is strictly contractive when $\Delta t^{\frac{1}{2}}/h$ is chosen sufficiently small. Let $(\tilde\psi_{1h}^1, \tilde\psi_{2h}^1, \tilde{\mathbf{A}}_h^1)$ and $(\tilde\psi_{1h}^2, \tilde\psi_{2h}^2, \tilde{\mathbf{A}}_h^2)$ be two elements in the space $\mathbf{B}$. Also let $\tilde\psi_1 = \tilde\psi_{1h}^1 - \tilde\psi_{1h}^2$, $\tilde\psi_2 = \tilde\psi_{2h}^1 - \tilde\psi_{2h}^2$ and $\tilde{\mathbf{A}} = \tilde{\mathbf{A}}_h^1 - \tilde{\mathbf{A}}_h^2$. Then for any $\varphi \in \mathcal{Z}_h$,

$$
\begin{aligned}
&\langle f_1(\tilde\psi_{1h}^1, \tilde\psi_{2h}^1, \tilde{\mathbf{A}}_h^1) - f_1(\tilde\psi_{1h}^2, \tilde\psi_{2h}^2, \tilde{\mathbf{A}}_h^2),\ \varphi\rangle \\
&\quad= -\Delta t \int_\Omega \big[(|\tilde\psi_{1h}^1|^2 - \mathcal{T}_1)\tilde\psi_{1h}^1 - (|\tilde\psi_{1h}^2|^2 - \mathcal{T}_1)\tilde\psi_{1h}^2\big]\varphi dt \\
&\qquad - i\Delta t \frac{2\xi_1}{\lambda_1} \int_\Omega \big[(\tilde{\mathbf{A}}_h^1 + \mathbf{A}_c)\cdot\nabla\tilde\psi_{1h}^1 - (\tilde{\mathbf{A}}_h^2 + \mathbf{A}_c)\cdot\nabla\tilde\psi_{1h}^2\big]\varphi dt \\
&\qquad - i\Delta t \frac{\xi_1}{\lambda_1} \int_\Omega \big[\tilde\psi_{1h}^1 \nabla\cdot(\tilde{\mathbf{A}}_h^1 + \mathbf{A}_c) - \tilde\psi_{1h}^2 \nabla\cdot(\tilde{\mathbf{A}}_h^2 + \mathbf{A}_c)\big]\varphi dt \\
&\qquad - \Delta t \frac{x_0^2}{\lambda_1^2} \int_\Omega \big[|\tilde{\mathbf{A}}_h^1 + \mathbf{A}_c|^2 \tilde\psi_{1h}^1 - |\tilde{\mathbf{A}}_h^2 + \mathbf{A}_c|^2 \tilde\psi_{1h}^2\big]\varphi dt - \eta\Delta t \int_\Omega \tilde\psi_2 \varphi dt \\
&\quad\le -\Delta t \int_\Omega \big[(|\tilde\psi_{1h}^1|^2 - \mathcal{T}_1)\tilde\psi_1 + (\tilde\psi_{1h}^{1*}\tilde\psi_1 + \tilde\psi_{1h}^2 \tilde\psi_1^*)\tilde\psi_{1h}^2\big]\varphi dt \\
&\qquad - i\Delta t \frac{2\xi_1}{\lambda_1} \int_\Omega \big[\tilde{\mathbf{A}}\cdot\nabla\tilde\psi_{1h}^1 + (\tilde{\mathbf{A}}_h^2 + \mathbf{A}_c)\cdot\nabla\tilde\psi_1\big]\varphi dt \\
&\qquad - i\Delta t \frac{\xi_1}{\lambda_1} \int_\Omega \big[\tilde\psi_1 \nabla\cdot(\tilde{\mathbf{A}}_h^1 + \mathbf{A}_c) + \tilde\psi_{1h}^2 \nabla\cdot\tilde{\mathbf{A}}\big]\varphi dt \\
&\qquad - \Delta t \frac{x_0^2}{\lambda_1^2} \int_\Omega \big[\tilde{\mathbf{A}}\cdot(\tilde{\mathbf{A}}_h^1 + \tilde{\mathbf{A}}_h^2 + 2\mathbf{A}_c)\tilde\psi_{1h}^1 + |\tilde{\mathbf{A}}_h^2 + \mathbf{A}_c|^2 \tilde\psi_1\big]\varphi dt - \eta\Delta t \int_\Omega \tilde\psi_2 \varphi dt \\
&\quad\le C\Delta t \Bigg[(||\tilde\psi_{1h}^1||_{0,4}^2 + |\mathcal{T}_1|)\,||\tilde\psi_1||_{0,4} \\
&\qquad + \big(||\tilde\psi_{1h}^1||_{0,4}\,||\tilde\psi_1||_{0,4} + ||\tilde\psi_{1h}^2||_{0,4}\,||\tilde\psi_1||_{0,4}\big)||\tilde\psi_{1h}^2||_{0,4} \\
&\qquad + ||\tilde{\mathbf{A}}||_{0,4}\,||\nabla\tilde\psi_{1h}^1||_0 + ||(\tilde{\mathbf{A}}_h^2 + \mathbf{A}_c)||_{0,4}\,||\nabla\tilde\psi_1||_0 \\
&\qquad + ||\tilde\psi_1||_{0,4}\,||\nabla\cdot(\tilde{\mathbf{A}}_h^1 + \mathbf{A}_c)||_0 + ||\tilde\psi_{1h}^2||_{0,4}\,||\nabla\cdot\tilde{\mathbf{A}}||_0 \\
&\qquad + ||\tilde{\mathbf{A}}||_{0,4}\,||(\tilde{\mathbf{A}}_h^1 + \tilde{\mathbf{A}}_h^2 + 2\mathbf{A}_c)||_{0,4}\,||\tilde\psi_{1h}^1||_{0,4} + ||\tilde{\mathbf{A}}_h^2 + \mathbf{A}_c||_{0,4}^2\,|||\tilde\psi_1||_{0,4} \\
&\qquad + ||\tilde\psi_2||_{0,4}\Bigg]||\varphi||_{0,4} \\
&\quad\le C\Delta t\big[||\tilde\psi_1||_1 + ||\tilde\psi_2||_1 + ||\tilde{\mathbf{A}}||_1\big]||\varphi||_{0,4}.
\end{aligned} \tag{4.37}
$$

In the above last two inequalities, we have used the embedding relation $H^1(\Omega) \hookrightarrow L^4(\Omega) \hookrightarrow L^2(\Omega)$. We will use this relation repeatedly later.

Likewise, we have

$$\langle f_2(\tilde{\psi}_{1h}^1, \tilde{\psi}_{2h}^1, \tilde{\mathbf{A}}_h^1) - f_2(\tilde{\psi}_{1h}^2, \tilde{\psi}_{2h}^2, \tilde{\mathbf{A}}_h^2),\, \varphi\rangle \le C\Delta t\big[||\tilde{\psi}_1||_1 + ||\tilde{\psi}_2||_1 + ||\tilde{\mathbf{A}}||_1\big]||\varphi||_{0,4}. \quad (4.38)$$

As for the functional $\mathbf{g}$, for any $\Phi \in \mathbf{\Lambda}_h$ we have

$$\begin{aligned}
&\langle \mathbf{g}(\tilde{\psi}_{1h}^1, \tilde{\psi}_{2h}^1, \tilde{\mathbf{A}}_h^1) - \mathbf{g}(\tilde{\psi}_{1h}^2, \tilde{\psi}_{2h}^2, \tilde{\mathbf{A}}_h^2),\, \Phi\rangle \\
&= -\Delta t \frac{1}{\kappa_1}\int_\Omega \Re\big(i\tilde{\psi}_{1h}^{1*}\nabla\tilde{\psi}_{1h}^1 - i\tilde{\psi}_{1h}^{2*}\nabla\tilde{\psi}_{1h}^2\big)\cdot\Phi d\Omega \\
&\quad -\Delta t\frac{x_0^2}{\lambda_1^2}\int_\Omega \big[\,|\tilde{\psi}_{1h}^1|^2(\tilde{\mathbf{A}}_h^1 + \mathbf{A}_c) - |\tilde{\psi}_{1h}^2|^2(\tilde{\mathbf{A}}_h^2 + \mathbf{A}_c)\big]\cdot\Phi d\Omega \\
&\quad - \Delta t\frac{1}{\nu}\frac{1}{\kappa_2}\int_\Omega \Re\big(i\tilde{\psi}_{2h}^{1*}\nabla\tilde{\psi}_{2h}^1 - i\tilde{\psi}_{2h}^{2*}\nabla\tilde{\psi}_{2h}^2\big)\cdot\Phi d\Omega \\
&\quad -\Delta t\frac{x_0^2}{\lambda_2^2}\int_\Omega \big[\,|\tilde{\psi}_{2h}^1|^2(\tilde{\mathbf{A}}_h^1 + \mathbf{A}_c) - |\tilde{\psi}_{2h}^2|^2(\tilde{\mathbf{A}}_h^2 + \mathbf{A}_c)\big]\cdot\Phi d\Omega \\
&\le -\Delta t\frac{1}{\kappa_1}\int_\Omega \Re\big(i\tilde{\psi}_1^*\nabla\tilde{\psi}_{1h}^1 + i\tilde{\psi}_{1h}^{2*}\nabla\tilde{\psi}_1\big)\cdot\Phi d\Omega \\
&\quad -\Delta t\frac{x_0^2}{\lambda_1^2}\int_\Omega \big[\,|\tilde{\psi}_{1h}^1|^2\tilde{\mathbf{A}} + (|\tilde{\psi}_{1h}^1|^2 - |\tilde{\psi}_{1h}^2|^2)(\tilde{\mathbf{A}}_h^2 + \mathbf{A}_c)\big]\cdot\Phi d\Omega \\
&\quad - \Delta t\frac{1}{\nu}\frac{1}{\kappa_2}\int_\Omega \Re\big(i\tilde{\psi}_2^*\nabla\tilde{\psi}_{2h}^1 + i\tilde{\psi}_{2h}^{2*}\nabla\tilde{\psi}_2\big)\cdot\Phi d\Omega \\
&\quad -\Delta t\frac{x_0^2}{\lambda_2^2}\int_\Omega \big[\,|\tilde{\psi}_{2h}^1|^2\tilde{\mathbf{A}} + (|\tilde{\psi}_{2h}^1|^2 - |\tilde{\psi}_{2h}^2|^2)(\tilde{\mathbf{A}}_h^2 + \mathbf{A}_c)\big]\cdot\Phi d\Omega \\
&\le C\Delta t\Big[||\tilde{\psi}_1||_{0,4}\,||\nabla\tilde{\psi}_{1h}^1||_0 + ||\tilde{\psi}_{1h}^2||_{0,4}\,||\nabla\tilde{\psi}_1||_0 \\
&\quad +||\tilde{\psi}_{1h}^1||_{0,4}^2\,||\tilde{\mathbf{A}}||_{0,4} + ||\tilde{\psi}_1||_{0,4}\big(||\tilde{\psi}_{1h}^1||_{0,4} + ||\tilde{\psi}_{1h}^2||_{0,4}\big)||\tilde{\mathbf{A}}_h^2 + \mathbf{A}_c||_{0,4} \\
&\quad + ||\tilde{\psi}_2||_{0,4}\,||\nabla\tilde{\psi}_{2h}^1||_0 + ||\tilde{\psi}_{2h}^2||_{0,4}\,||\nabla\tilde{\psi}_2||_0 \\
&\quad +||\tilde{\psi}_{2h}^1||_{0,4}^2\,||\tilde{\mathbf{A}}||_{0,4} + ||\tilde{\psi}_2||_{0,4}\big(||\tilde{\psi}_{2h}^1||_{0,4} + ||\tilde{\psi}_{2h}^2||_{0,4}\big)||\tilde{\mathbf{A}}_h^2 + \mathbf{A}_c||_{0,4}\Big]||\Phi||_{0,4}. \\
&\le C\Delta t\big[||\tilde{\psi}_1||_1 + ||\tilde{\psi}_2||_1 + ||\tilde{\mathbf{A}}||_1\big]||\Phi||_{0,4}. \qquad (4.39)
\end{aligned}$$

Now Let $(\psi_{1h}^1, \psi_{2h}^1, \mathbf{A}_h^1) = \mathbf{G}_h(\tilde{\psi}_{1h}^1, \tilde{\psi}_{2h}^1, \tilde{\mathbf{A}}_h^1)$ and $(\psi_{1h}^2, \psi_{2h}^2, \mathbf{A}_h^2) = \mathbf{G}_h(\tilde{\psi}_{1h}^2, \tilde{\psi}_{2h}^2, \tilde{\mathbf{A}}_h^2)$. Also let $\psi_1 = \psi_{1h}^1 - \psi_{1h}^2$, $\psi_2 = \psi_{2h}^1 - \psi_{2h}^2$ and $\mathbf{A} = \mathbf{A}_h^1 - \mathbf{A}_h^2$.

Then by using (4.30) and (4.37),

$$\Re\, a_1(\psi_1,\, \psi_1) = \Re\,\langle f_1(\tilde{\psi}_{1h}^1, \tilde{\psi}_{2h}^1, \tilde{\mathbf{A}}_h^1) - f_1(\tilde{\psi}_{1h}^2, \tilde{\psi}_{2h}^2, \tilde{\mathbf{A}}_h^2),\, \psi_1\rangle$$

gives

$$
\begin{aligned}
&||\psi_1||_0^2 + \Delta t \frac{\xi_{1h}^2}{x_0^2}||\nabla\psi_1||_0^2 + \gamma_1||\psi_1||_{\partial\Omega}^2 \\
&\quad \le C\Delta t\big[\,||\tilde{\psi}_1||_1 + ||\tilde{\psi}_2||_1 + ||\tilde{\mathbf{A}}||_1\big]\,||\psi_1||_{0,4} \\
&\quad \le C\Delta t\big[\,||\tilde{\psi}_1||_1^2 + ||\tilde{\psi}_2||_1^2 + ||\tilde{\mathbf{A}}||_1^2\big] + \Delta t\big[C_\varepsilon||\psi_1||_0^2 + \varepsilon||\nabla\psi_1||_0^2\big],
\end{aligned}
$$

here the last term enclosed in brackets is obtained from the following inequality which is a result of the Nirenberg-Gagliardo inequality

$$
\begin{aligned}
||\psi_1||_{0,4}^2 &\le C||\psi_1||_0\,||\psi_1||_1 \\
&\le \frac{C}{4\varepsilon}||\psi_1||_0^2 + \varepsilon||\psi_1||_1^2 \\
&\le \frac{C}{4\varepsilon}||\psi_1||_0^2 + \varepsilon\big[||\psi_1||_0^2 + ||\nabla\psi_1||_0^2\big] \\
&= C_\varepsilon||\psi_1||_0^2 + \varepsilon||\nabla\psi_1||_0^2. \qquad (4.40)
\end{aligned}
$$

This gives

$$
(1 - C_\varepsilon\Delta t)||\psi_1||_0^2 + \Delta t\big(\frac{\xi_{1h}^2}{x_0^2} - \varepsilon\big)||\nabla\psi_1||_0^2 \le C\Delta t\big[\,||\tilde{\psi}_1||_1^2 + ||\tilde{\psi}_2||_1^2 + ||\tilde{\mathbf{A}}||_1^2\big].
$$

Therefore by choosing ε small enough such that $\xi_{1h}^2/x_0^2 - \varepsilon > 0$, and Δt small enough such that $1 - C_\varepsilon\Delta t > 0$, we get

$$
||\psi_1||_0^2 \le C\Delta t\big[\,||\tilde{\psi}_1||_1^2 + ||\tilde{\psi}_2||_1^2 + ||\tilde{\mathbf{A}}||_1^2\,\big]. \qquad (4.41)
$$

Similarly, by using (4.31) and (4.38), we get

$$
||\psi_2||_0^2 \le C\Delta t\big[\,||\tilde{\psi}_1||_1^2 + ||\tilde{\psi}_2||_1^2 + ||\tilde{\mathbf{A}}||_1^2\,\big]. \qquad (4.42)
$$

By using (4.32) and (4.39),

$$
b(\mathbf{A},\,\mathbf{A}) = \langle \mathbf{g}(\tilde{\psi}_{1h}^1, \tilde{\psi}_{2h}^1, \tilde{\mathbf{A}}_h^1) - \mathbf{g}(\tilde{\psi}_{1h}^2, \tilde{\psi}_{2h}^2, \tilde{\mathbf{A}}_h^2),\,\mathbf{A}\rangle,
$$

gives

$$
\begin{aligned}
&\sigma\frac{x_0^2}{\lambda_1^2}||\mathbf{A}||_0^2 + \epsilon\Delta t||\mathrm{div}\mathbf{A}||_0^2 + \Delta t||\mathrm{curl}\mathbf{A}||_0^2 \\
&\quad \le C\Delta t\big[\,||\tilde{\psi}_1||_1 + ||\tilde{\psi}_2||_1 + ||\tilde{\mathbf{A}}||_1\big]\,||\mathbf{A}||_{0,4} \\
&\quad \le C\Delta t\big[\,||\tilde{\psi}_1||_1^2 + ||\tilde{\psi}_2||_1^2 + ||\tilde{\mathbf{A}}||_1^2\big] + \Delta t\big[\,C_\varepsilon||\mathbf{A}||_0^2 + \varepsilon C(\,||\mathrm{div}\mathbf{A}||_0^2 + ||\mathrm{curl}\mathbf{A}||_0^2)\,\big],
\end{aligned}
$$

here the last term enclosed in brackets is obtained from the following inequality in a similar way as in (4.40) and by using inequality (4.1)

$$\begin{aligned} ||\mathbf{A}||_{0,4}^2 &\leq \frac{C}{4\varepsilon}||\mathbf{A}||_0^2 + \varepsilon||\mathbf{A}||_1^2 \\ &\leq C_\varepsilon||\mathbf{A}||_0^2 + \varepsilon\, C\big[||\mathrm{div}\mathbf{A}||_0^2 + ||\mathrm{curl}\mathbf{A}||_0^2\big]. \end{aligned} \tag{4.43}$$

This gives

$$\begin{aligned} &(\sigma\frac{x_0^2}{\lambda_1^2} - C_\varepsilon\Delta t)||\mathbf{A}||_0^2 + \Delta t(\epsilon - \varepsilon C))||\mathrm{div}\mathbf{A}||_0^2 + \Delta t(1-\varepsilon C)||\mathrm{curl}\mathbf{A}||_0^2 \\ &\quad\leq C\Delta t\big[\,||\tilde{\psi}_1||_1^2 + ||\tilde{\psi}_2||_1^2 + ||\tilde{\mathbf{A}}||_1^2\big]. \end{aligned}$$

Therefore by choosing ε small enough such that $\epsilon - \varepsilon C > 0$, and Δt small enough such that $\sigma x_0^2/\lambda_1^2 - C_\varepsilon\Delta t > 0$, we get

$$||\mathbf{A}||_0^2 \leq C\Delta t\big[\,||\tilde{\psi}_1||_1^2 + ||\tilde{\psi}_2||_1^2 + ||\tilde{\mathbf{A}}||_1^2\big]. \tag{4.44}$$

By applying the inverse inequalities (4.6) and (4.7) on the inequalities (4.41), (4.42) and (4.44), the resulted inequalities together give

$$||\psi_2||_1 + ||\psi_2||_1 + ||\mathbf{A}||_1 \leq Ch^{-1}\Delta t^{\frac{1}{2}}\big[\,||\tilde{\psi}_1||_1 + ||\tilde{\psi}_2||_1 + ||\tilde{\mathbf{A}}||_1\big],$$

where the constant C is independent of n, N, h and Δt. Therefore, by choosing $\Delta t^{\frac{1}{2}}/h$ sufficient small such that $C\Delta t^{\frac{1}{2}}/h = \alpha < 1$, independent of n and N, we get

$$||(\psi_1, \psi_2, \mathbf{A})||_{\mathbf{B}} \leq \alpha||(\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\mathbf{A}})||_{\mathbf{B}}$$

which is equivalent to

$$||\mathbf{G}_h(\tilde{\psi}_{1h}^1, \tilde{\psi}_{2h}^1, \tilde{\mathbf{A}}_h^1) - \mathbf{G}_h(\tilde{\psi}_{1h}^2, \tilde{\psi}_{2h}^2, \tilde{\mathbf{A}}_h^2)||_{\mathbf{B}} \leq \alpha||(\tilde{\psi}_{1h}^1, \tilde{\psi}_{2h}^1, \tilde{\mathbf{A}}_h^1) - (\tilde{\psi}_{1h}^2, \tilde{\psi}_{2h}^2, \tilde{\mathbf{A}}_h^2)||_{\mathbf{B}}.$$

This shows that the operator $\mathbf{G}_h$ is strictly contractive. Whence Banach's fixed point theorem implies that there exists a unique fixed point $(\psi_{1h}, \psi_{2h}, \mathbf{A}_h) = (\psi_{1h}^n, \psi_{2h}^n, \mathbf{A}_h^n)$ as a solution to the problem $(\mathbf{DP}^\epsilon)$ at time step n. Finally, an induction starting from the initial time step with $(\psi_{1h}^0, \psi_{2h}^0, \mathbf{A}_h^0) = (I_h\psi_{10}, I_h\psi_{20}, \mathbf{I}_h\mathbf{A}_0)$ shows the existence and uniqueness of the solution $(\psi_{1h}^n, \psi_{2h}^n, \mathbf{A}_h^n)$ to the problem $(\mathbf{DP}^\epsilon)$ for all time step n, $1 \leq n \leq N$. ∎

4.3 Stability Estimates of the Problem $(\mathbf{DP}^\epsilon)$

Theorem 4.3.1 *Assume $\mathbf{A}_c$, $\mathbf{H}_e$, ψ_{10}, ψ_{20} and $\mathbf{A}_0$ satisfy the regularity assumptions (4.18), (4.20), (4.22) and (4.23), respectively, then for $\Delta t < \min(1,\Gamma)/(|\eta|(1+\nu^2)+\max(|\mathcal{T}_1|, |\mathcal{T}_2|))$, the solution of the problem $(\mathbf{DP}^\epsilon)$ satisfies the following estimates:*

$$\max_{1\le n\le N}\left[||\psi_{1h}^n||_0^2+||\psi_{2h}^n||_0^2+||\mathbf{A}_h^n||_0^2\right]\le C, \tag{4.45}$$

$$\sum_{n=1}^{N}\left[||\psi_{1h}^n-\psi_{1h}^{n-1}||_0^2+||\psi_{2h}^n-\psi_{2h}^{n-1}||_0^2+||\mathbf{A}_h^n-\mathbf{A}_h^{n-1}||_0^2\right]\le C, \tag{4.46}$$

$$\sum_{n=1}^{N}\Delta t\left[||\nabla\psi_{1h}^n||_0^2+||\nabla\psi_{2h}^n||_0^2\right]\le C, \tag{4.47}$$

$$\sum_{n=1}^{N}\Delta t\left[\epsilon||\mathrm{div}\mathbf{A}_h^n||_0^2+||\mathrm{curl}\mathbf{A}_h^n||_0^2\right]\le C, \tag{4.48}$$

$$\sum_{n=1}^{N}\Delta t\left[||\psi_{1h}^n||_{0,4}^4+||\psi_{2h}^n||_{0,4}^4+||\mathbf{A}_h^n||_{0,4}^4\right]\le C, \tag{4.49}$$

$$\sum_{n=1}^{N}\Delta t\left[||\psi_{1h}^n||_{0,\partial\Omega}^2+||\psi_{2h}^n||_{0,\partial\Omega}^2\right]\le C. \tag{4.50}$$

where the constant $C>0$ is independent of h, Δt and N but dependent on ϵ.

Proof By choosing the test function $\tilde{\psi}_h=\Delta t\psi_{1h}^n\in\mathcal{Z}_h$ in (4.24) and taking the real part of the resulted equation, we obtain by using the identity $2(a-b,a)=|a|^2-|b|^2+|a-b|^2$ that (after the superscript ϵ is dropped)

$$\begin{aligned}
&\frac{1}{2}\left[\,||\psi_{1h}^n||_0^2-||\psi_{1h}^{n-1}||_0^2+||\psi_{1h}^n-\psi_{1h}^{n-1}||_0^2\right]+\ \Delta t||\psi_{1h}^n||_{0,4}^4\\
&\quad+\Delta t\,||-i\frac{\xi_1}{x_0}\nabla\psi_{1h}^n-\frac{x_0}{\lambda_1}(\mathbf{A}_h^n+\mathbf{A}_c^n)\psi_{1h}^n||_0^2+\gamma_1\frac{\xi_1^2}{x_0^2}\Delta t||\psi_{1h}^n||_{0,\partial\Omega}^2\\
&\le-\Delta t\eta\int_\Omega\psi_{2h}^n\psi_{1h}^{n*}d\Omega+\Delta t\mathcal{T}_1||\psi_{1h}^n||_0^2\\
&\le\Delta t\frac{|\eta|}{2}\left[\,||\psi_{2h}^n||_0^2+||\psi_{1h}^n||_0^2\,\right]+\Delta t|\mathcal{T}_1|\,||\psi_{1h}^n||_0^2.
\end{aligned}\tag{4.51}$$

Likewise, equation (4.25) becomes

$$\frac{\Gamma}{2}\big[\,||\psi_{2h}^n||_0^2-||\psi_{2h}^{n-1}||_0^2+||\psi_{2h}^n-\psi_{2h}^{n-1}||_0^2\big]+\ \Delta t||\psi_{2h}^n||_{0,4}^4$$
$$+\Delta t\,||-i\frac{\xi_2}{x_0}\nabla\psi_{2h}^n-\nu\frac{x_0}{\lambda_2}(\mathbf{A}_h^n+\mathbf{A}_c^n)\psi_{2h}^n||_0^2+\gamma_2\frac{\xi_2^2}{x_0^2}\Delta t||\psi_{2h}^n||_{0,\partial\Omega}^2$$
$$\le\Delta t\frac{|\eta|\nu^2}{2}\big[\,||\psi_{1h}^n||_0^2+||\psi_{2h}^n||_0^2\,\big]+\Delta t|\mathcal{T}_2|\,||\psi_{2h}^n||_0^2. \tag{4.52}$$

Combining the two inequalities (4.51) and (4.52) and choosing Δt small enough such that

$$\Delta t<\frac{1}{|\eta|(1+\nu^2)+|\mathcal{T}_1|},\qquad \Delta t<\frac{\Gamma}{|\eta|(1+\nu^2)+|\mathcal{T}_2|}. \tag{4.53}$$

Then the discrete Gronwall inequality gives

$$\max_{1\le n\le N}\big[\,||\psi_{1h}^n||_0^2+||\psi_{2h}^n||_0^2\,\big]$$
$$+\sum_{n=1}^N\big[\,||\psi_{1h}^n-\psi_{1h}^{n-1}||_0^2+||\psi_{2h}^n-\psi_{2h}^{n-1}||_0^2\big]$$
$$+\sum_{n=1}^N\Delta t\big[\,||\psi_{1h}^n||_{0,4}^4+\Delta t||\psi_{2h}^n||_{0,4}^4\,\big]$$
$$+\sum_{n=1}^N\Delta t\big[\,||-i\frac{\xi_1}{x_0}\nabla\psi_{1h}^n-\frac{x_0}{\lambda_1}(\mathbf{A}_h^n+\mathbf{A}_c^n)\psi_{1h}^n||_0^2$$
$$+||-i\frac{\xi_2}{x_0}\nabla\psi_{2h}^n-\nu\frac{x_0}{\lambda_2}(\mathbf{A}_h^n+\mathbf{A}_c^n)\psi_{2h}^n||_0^2\,\big]$$
$$+\sum_{n=1}^N\Delta t\big[\,||\psi_{1h}^n||_{0,\partial\Omega}^2+||\psi_{2h}^n||_{0,\partial\Omega}^2\,\big]$$
$$\le A, \tag{4.54}$$

where the constant A depends only on the norms of the initial conditions, namely $||\psi_{1h}^0||_0$ and $||\psi_{2h}^0||_0$; and is independent of Δt, h, N and ϵ.

Now by choosing the test function $\tilde{\mathbf{A}}_h=\Delta t\mathbf{A}_h^n\in\Lambda_h$ in (4.26), then we obtain by using the identity $2(a-b,a)=|a|^2-|b|^2+|a-b|^2$ that (after the superscript ϵ is dropped)

$$\sigma\frac{x_0^2}{2\lambda_1^2}\big[\,||\mathbf{A}_h^n||_0^2-||\mathbf{A}_h^{n-1}||_0^2+||\mathbf{A}_h^n-\mathbf{A}_h^{n-1}||_0^2\,\big]+\epsilon\Delta t||\mathrm{div}\,\mathbf{A}_h^n||_0^2+\Delta t||\mathrm{curl}\,\mathbf{A}_h^n||_0^2$$
$$-\Delta t\int_\Omega\Re\big[\big(-i\frac{\xi_1}{x_0}\nabla\psi_{1h}^n-\frac{x_0}{\lambda_1}(\mathbf{A}_h^n+\mathbf{A}_c^n)\psi_{1h}^n\big)\frac{x_0}{\lambda_1}\psi_{1h}^{n*}\big]\cdot\mathbf{A}_h^n d\Omega$$
$$-\Delta t\int_\Omega\Re\big[\big(-i\frac{\xi_2}{x_0}\nabla\psi_{2h}^n-\nu\frac{x_0}{\lambda_2}(\mathbf{A}_h^n+\mathbf{A}_c^n)\psi_{2h}^n\big)\frac{1}{\nu}\frac{x_0}{\lambda_2}\psi_{2h}^{n*}\big]\cdot\mathbf{A}_h^n\,d\Omega$$
$$=\Delta t\int_\Omega\mathbf{H}_e^n\cdot\mathrm{curl}\mathbf{A}_h^n d\Omega.$$

This gives

$$\sigma \frac{x_0^2}{2\lambda_1^2}\big[\,||\mathbf{A}_h^n||_0^2 - ||\mathbf{A}_h^{n-1}||_0^2 + ||\mathbf{A}_h^n - \mathbf{A}_h^{n-1}||_0^2\,\big] + \epsilon\Delta t||\text{div}\,\mathbf{A}_h^n||_0^2 + \Delta t||\text{curl}\,\mathbf{A}_h^n||_0^2$$
$$\le C_\varepsilon \Delta t|| - i\frac{\xi_1}{x_0}\nabla\psi_{1h}^n - \frac{x_0}{\lambda_1}(\mathbf{A}_h^n + \mathbf{A}_c^n)\psi_{1h}^n||_0^2 + \varepsilon\Delta t||\psi_{1h}^n||_{0,4}^2\,||\mathbf{A}_h^n||_{0,4}^2$$
$$+C_\varepsilon \Delta t|| - i\frac{\xi_2}{x_0}\nabla\psi_{2h}^n - \nu\frac{x_0}{\lambda_2}(\mathbf{A}_h^n + \mathbf{A}_c^n)\psi_{2h}^n||_0^2 + \varepsilon\Delta t||\psi_{2h}^n||_{0,4}^2\,||\mathbf{A}_h^n||_{0,4}^2$$
$$+\Delta t\big[\,C_\varepsilon||\mathbf{H}_e^n||_0^2 + \varepsilon||\text{curl}\mathbf{A}_h^n||_0^2\big]. \tag{4.55}$$

Since by the inequality (4.1), $||\mathbf{A}_h^n||_1^2 \le C'(||\text{div}\,\mathbf{A}_h^n||_0^2 + ||\text{curl}\,\mathbf{A}_h^n||_0^2)$, we have

$$\begin{aligned} ||\psi_{1h}^n||_{0,4}^2\,||\mathbf{A}_h^n||_{0,4}^2 &\le C||\psi_{1h}^n||_{0,4}^2\,||\mathbf{A}_h^n||_0\,||\mathbf{A}_h^n||_1 \\ &\le C||\psi_{1h}^n||_{0,4}^4\,||\mathbf{A}_h^n||_0^2 + \big[\,||\text{curl}\,\mathbf{A}_h^n||_0^2 + ||\text{div}\,\mathbf{A}_h^n||_0^2\,\big]. \end{aligned}$$

So the inequality (4.55) becomes

$$\sigma \frac{x_0^2}{2\lambda_1^2}\big[||\mathbf{A}_h^n||_0^2 - ||\mathbf{A}_h^{n-1}||_0^2 + ||\mathbf{A}_h^n - \mathbf{A}_h^{n-1}||_0^2\big] + \epsilon\Delta t||\text{div}\,\mathbf{A}_h^n||_0^2 + \Delta t||\text{curl}\,\mathbf{A}_h^n||_0^2$$
$$\le C_\varepsilon \Delta t|| - i\frac{\xi_1}{x_0}\nabla\psi_{1h}^n - \frac{x_0}{\lambda_1}(\mathbf{A}_h^n + \mathbf{A}_c^n)\psi_{1h}^n||_0^2 + \varepsilon C\Delta t||\psi_{1h}^n||_{0,4}^4\,||\mathbf{A}_h^n||_0^2$$
$$+C_\varepsilon \Delta t|| - i\frac{\xi_2}{x_0}\nabla\psi_{2h}^n - \nu\frac{x_0}{\lambda_2}(\mathbf{A}_h^n + \mathbf{A}_c^n)\psi_{2h}^n||_0^2 + \varepsilon C\Delta t||\psi_{2h}^n||_{0,4}^4\,||\mathbf{A}_h^n||_0^2$$
$$+2\varepsilon\Delta t\big[\,||\text{curl}\,\mathbf{A}_h^n||_0^2 + ||\text{div}\,\mathbf{A}_h^n||_0^2\,\big] + \Delta t\big[\,C_\varepsilon||\mathbf{H}_e^n||_0^2 + \varepsilon||\text{curl}\mathbf{A}_h^n||_0^2\big]$$
$$\le 2C_\varepsilon A + 2\varepsilon CA||\mathbf{A}_h^n||_0^2$$
$$+2\varepsilon\Delta t\big[\,||\text{curl}\,\mathbf{A}_h^n||_0^2 + ||\text{div}\,\mathbf{A}_h^n||_0^2\,\big] + \Delta t\big[\,C\varepsilon||\mathbf{H}_e^n||_0^2 + \varepsilon||\text{curl}\mathbf{A}_h^n||_0^2\big], \tag{4.56}$$

where the constant A is derived from the inequality (4.54).

Now choose ε small enough such that

$$\varepsilon < \min\left[\frac{\epsilon}{3}, \frac{\sigma x_0^2}{2\lambda_1^2}\frac{1}{2CA}\right],$$

then by the discrete Gronwall inequality and by the regularity assumption of $\mathbf{H}_e$, we have

$$\max_{1\le n\le N}||\mathbf{A}_h^n||_0^2 + \sum_{n=1}^{N}||\mathbf{A}_h^n - \mathbf{A}_h^{n-1}||_0^2$$
$$+\sum_{n=1}^{N}\Delta t\big[\epsilon||\text{div}\,\mathbf{A}_h^n||_0^2 + ||\text{curl}\,\mathbf{A}_h^n||_0^2\,\big]$$
$$\le B, \tag{4.57}$$

where the constant B depends on ϵ and $||\mathbf{A}_h^0||_0$; but is independent of Δt, h and N.

Lastly, by the above estimates we have

$$
\begin{aligned}
&\sum_{n=1}^{N} \Delta t \big[\, ||\mathbf{A}_h^n||_{0,4}^4 + ||\nabla \psi_{1h}^n||_0^2 + ||\nabla \psi_{2h}^n||_0^2 \,\big] \\
&\quad \le \sum_{n=1}^{N} \Delta t \Bigg[\, ||\mathbf{A}_h^n||_{0,4}^4 \\
&\qquad + C'|| - i\frac{\xi_1}{x_0}\nabla \psi_{1h}^n - \frac{x_0}{\lambda_1}(\mathbf{A}_h^n + \mathbf{A}_c^n)\psi_{1h}^n||_0^2 + \frac{x_0}{\lambda_1}||(\mathbf{A}_h^n + \mathbf{A}_c^n)\psi_{1h}^n||_0^2 \\
&\qquad + C'|| - i\frac{\xi_2}{x_0}\nabla \psi_{2h}^n - \nu\frac{x_0}{\lambda_2}(\mathbf{A}_h^n + \mathbf{A}_c^n)\psi_{2h}^n||_0^2 + \nu\frac{x_0}{\lambda_2}||(\mathbf{A}_h^n + \mathbf{A}_c^n)\psi_{2h}^n||_0^2 \Bigg] \\
&\quad \le C'' \sum_{n=1}^{N} 2A + \Bigg[1 + \Delta t\Bigg(||\psi_{1h}^n||_{0,4}^4 + ||\psi_{2h}^n||_{0,4}^4 \Bigg)\Bigg] \\
&\qquad\qquad \times \Delta t \Bigg[\, ||\mathbf{A}_h^n||_0^2 \left(||\mathrm{div}\, \mathbf{A}_h^n||_0^2 + ||\mathrm{curl}\, \mathbf{A}_h^n||_0^2 \right) + ||\mathbf{A}_c^n||_{0,4}^4 \Bigg] \\
&\quad \le C,
\end{aligned}
$$

here we have used the regularity assumption of $\mathbf{A}_c$, and the constant C depends on ϵ but is independent of Δt, h and N. This completes the proof. ∎

4.4 Error Estimates of the Problem $(\mathbf{DP}^\epsilon)$

We follow the methods used in [16] to deal with the cubic nonlinear term and to estimate the errors, also see [17].

Lemma 4.4.1 *Under the regularity assumptions (4.12) and for sufficiently small h, we have for $i = 1, 2$ that*

$$
\begin{aligned}
\max_{1 \le n \le N} ||I_h \psi_i^n||_{0,\infty} &\le C_{\psi_i}, && (4.58) \\
\max_{1 \le n \le N} ||I_h \psi_i^n - \psi_i^n||_{0,\infty} &\le C_{\psi_i}, && (4.59)
\end{aligned}
$$

where C_{ψ_i} is a constant which depends on the magnitude of $||\psi_i||_{k+1}$.

Proof For $n = 1, \cdots, N$, $k \geq 1$ and any $\varphi_h \in \mathcal{Z}_h$,

$$\begin{aligned}
||I_h\psi_i^n||_{0,\infty} &\leq ||I_h\psi_i^n - \psi_i^n||_{0,\infty} + ||\psi_i^n||_{0,\infty} \\
&\leq ||I_h\psi_i^n - \varphi_h||_{0,\infty} + ||\varphi_h - \psi_i^n||_{0,\infty} + ||\psi_i^n||_{0,\infty} \\
&\leq Ch^{-\frac{d}{2}}||I_h\psi_i^n - \varphi_h||_0 + ||\varphi_h - \psi_i^n||_{0,\infty} + ||\psi_i^n||_{0,\infty} \\
&\leq Ch^{-\frac{d}{2}}\big[\,||I_h\psi_i^n - \psi_i^n||_0 + ||\psi_i^n - \varphi_h||_0\big] + ||\varphi_h - \psi_i^n||_{0,\infty} + ||\psi_i^n||_{0,\infty} \\
&\leq Ch^{k+1-\frac{d}{2}}||\psi_i^n||_{k+1} + \big[Ch^{-\frac{d}{2}}||\psi_i^n - \varphi_h||_0 + ||\varphi_h - \psi_i^n||_{0,\infty}\big] + ||\psi_i^n||_{0,\infty},
\end{aligned}$$

where the third inequality is a result of the inverse inequality (4.8) and the last inequality is a result of the interpolation error property (4.2).

Now by virtue of the approximation property (4.10), given a constant δ, for sufficiently small h and $k+1 > d/2$, and by the regularity assumptions (4.12) which gives $\psi_i^n \in \mathcal{L}^\infty(\Omega)$, we get

$$\begin{aligned}
||I_h\psi_i^n||_{0,\infty} &\leq Ch^{k+1-\frac{d}{2}}||\psi_i^n||_{k+1} + \delta + ||\psi_i^n||_{0,\infty}, \\
&\leq C_{\psi_i}. \qquad \blacksquare
\end{aligned}$$

Lemma 4.4.2 *For $i = 1, 2$, define function $g_i : \mathbb{C} \to \mathbb{C}$ by setting $g(\psi) = (|\psi|^2 - \mathcal{T}_i)\psi$, where $\mathcal{T}_i \in \mathbb{R}$ is defined in (2.17). Then we have*

$$\Re\left\{(g_i(\psi) - g_i(\varphi))(\varphi^* - \psi^*)\right\} \leq |\psi - \varphi|^2 \qquad \forall \psi,\, \varphi \in \mathbb{C}, \tag{4.60}$$

$$|g_i(\psi) - g_i(\varphi)| \leq C|\psi - \varphi| \quad \forall \psi,\, \varphi \in \mathcal{C} \text{ s.t. } |\psi|,\, |\varphi| \leq K \text{ for } K \in (0, \infty). \tag{4.61}$$

Proof For the first inequality,

$$\begin{aligned}
&\Re\{((|\psi|^2 - \mathcal{T}_i)\psi - (|\varphi|^2 - \mathcal{T}_i)\varphi)(\varphi^* - \phi^*)\} \\
&\quad = \mathcal{T}_i|\psi|^2 + \mathcal{T}_i|\varphi|^2 - 2\mathcal{T}_i\Re(\psi\varphi^*) + |\psi|^2\psi\varphi^* + |\varphi|^2\varphi\psi^* - |\psi|^4 - |\varphi|^4 \\
&\quad \leq \mathcal{T}_i|\psi - \varphi|^2 + |\psi|^3|\varphi| + |\varphi|^3|\psi| - |\psi|^4 - |\varphi|^4 \\
&\quad = \mathcal{T}_i|\psi - \varphi|^2 - (|\varphi| - |\psi|)(|\varphi|^3 - |\psi|^3) \\
&\quad \leq |\mathcal{T}_i||\psi - \varphi|^2.
\end{aligned}$$

For the second inequality, suppose $|\psi|$, $|\varphi| \leq K$, for some $K \in (0, \infty)$, then

$$\big|(|\psi|^2 - \mathcal{T}_i)\psi - (|\varphi|^2 - \mathcal{T}_i)\varphi\big| \;\leq\; |K - \mathcal{T}_i||\psi - \varphi|. \qquad \blacksquare \tag{4.62}$$

Lemma 4.4.3 *For $1 \le s \le k$, assume $\psi \in \mathcal{H}^1(0,T;\mathcal{H}^{s+1}(\Omega))$,*

$$\Delta t||\delta_t(\psi^n - I_h\psi^n)||_0 \le C\Delta t^{\frac{1}{2}} h^{s+1}\left[\int_{I_n} ||\frac{\partial \psi}{\partial t}||_{s+1}^2 dt\right]^{\frac{1}{2}},$$

where $I_n = [t_{n-1}, t_n]$.

Proof By the definition of $\delta_t\psi^n = (\psi^n - \psi^{n-1})/\Delta t$, we get

$$\begin{aligned}
||(\psi^n - I_h\psi^n) - (\psi^{n-1} - I_h\psi^{n-1})||_0 &= \Big|\Big|\int_{I_n} \frac{\partial(\psi(t) - I_h\psi(t))}{\partial t} dt\Big|\Big|_0 \\
&\le \int_{I_n} \Big|\Big|\frac{\partial(\psi(t) - I_h\psi(t))}{\partial t}\Big|\Big|_0 dt \\
&\le Ch^{s+1}\int_{I_n} \Big|\Big|\frac{\partial \psi(t)}{\partial t}\Big|\Big|_{s+1} dt \\
&\le C\Delta t^{\frac{1}{2}} h^{s+1}\left[\int_{I_n} \Big|\Big|\frac{\partial \psi(t)}{\partial t}\Big|\Big|_{s+1}^2 dt\right]^{\frac{1}{2}},
\end{aligned}$$

where the third inequality is obtained from the interpolation error (4.2), and the last inequality is obtained by using the Schwarz inequality. ∎

Lemma 4.4.4 *For $\psi \in \mathcal{H}^1(0,T;\mathcal{L}^2(\Omega))$,*

$$||\int_{I_n} \psi(t)\, dt - \Delta t\, \psi^n\, ||_0 \le \Delta t^{\frac{3}{2}}\left[\int_{I_n} ||\frac{\partial \psi}{\partial t}||_0^2\, dt\right]^{\frac{1}{2}}.$$

Proof

$$\begin{aligned}
||\int_{I_n} \psi(t)dt - \Delta t\psi^n||_0 &= ||\int_{I_n} (\psi(t) - \psi^n)dt||_0 \\
&= \Big|\Big|\int_{I_n}\int_{t_n}^{t} \frac{\partial \psi(s)}{\partial s} ds dt\Big|\Big|_0 \\
&\le \Big|\Big|\int_{I_n}\int_{I_n} \Big|\frac{\partial \psi(s)}{\partial s}\Big| ds dt\Big|\Big|_0 \\
&\le \Delta t\Big|\Big|\int_{I_n} \Big|\frac{\partial \psi(s)}{\partial s}\Big| ds\Big|\Big|_0 \\
&\le \Delta t\int_{I_n} \Big|\Big|\frac{\partial \psi(s)}{\partial s}\Big|\Big|_0\, ds \\
&\le \Delta t\Delta t^{\frac{1}{2}}\left[\int_{I_n} \Big|\Big|\frac{\partial \psi(s)}{\partial s}\Big|\Big|_0^2\, ds\right]^{\frac{1}{2}},
\end{aligned}$$

where the last inequality is obtained by using the Schwarz inequality. ∎

Theorem 4.4.5 *Under the regularity assumptions (4.12)-(4.23), then for small enough h and Δt, the solution of the approximation problem* ($\mathbf{DP}^\epsilon$) *satisfies the following error estimates:*

$$\begin{aligned}
&\max_{1\le n\le N}\left[||\psi_1^\epsilon(\cdot,t_n)-\psi_{1h}^n||_0^2+||\psi_2^\epsilon(\cdot,t_n)-\psi_{2h}^n||_0^2+||\mathbf{A}^\epsilon(\cdot,t_n)-\mathbf{A}_h^n||_0^2\right]\\
&\quad+\sum_{n=1}^{N}\Delta t\left[||\psi_1^\epsilon(\cdot,t_n)-\psi_{1h}^n||_1^2+||\psi_2^\epsilon(\cdot,t_n)-\psi_{2h}^n||_1^2\right]\\
&\quad+\epsilon\sum_{n=1}^{N}\Delta t||\mathbf{A}^\epsilon(\cdot,t_n)-\mathbf{A}_h^n||_1^2\\
&\quad+\sum_{n=1}^{N}\Delta t\left[||\psi_1^\epsilon(\cdot,t_n)-\psi_{1h}^n||_{0,\partial\Omega}^2+||\psi_2^\epsilon(\cdot,t_n)-\psi_{2h}^n||_{0,\partial\Omega}^2\right]\\
&\le C_N^\epsilon(h^{2k}+\Delta t^2),
\end{aligned}$$

where $k\ge 1$ and C_N^ϵ is a constant independent of h and Δt but dependent of ϵ and N.

Proof For $i=1,2$, let the errors be $e_i^n=\psi_{ih}^n-\psi_i^n$ and $\mathbf{E}^n=\mathbf{A}_h^n-\mathbf{A}^n$, recall that $\psi_i^n=\psi_i^\epsilon(\cdot,t_n)$, $\mathbf{A}^n=\mathbf{A}^\epsilon(\cdot,t_n)$. Split the errors as $e_i^n=\theta_i^n+\rho_i^n$ and $\mathbf{E}^n=\Theta^n+\Phi^n$, where $\theta_i^n=\psi_{ih}^n-I_h\psi_i^n$, $\rho_i^n=I_h\psi_i^n-\psi_i^n$, $\Theta^n=\mathbf{A}_h^n-\mathbf{I}_h\mathbf{A}^n$ and $\Phi^n=\mathbf{I}_h\mathbf{A}^n-\mathbf{A}^n$. By the finite element interpolation errors (4.2) and (4.3), we know the errors for ρ_i^n and Φ^n, so we only need to estimate the errors for θ_i^n and Θ^n.

By choosing the test function $\tilde{\psi}_h=\Delta t\theta_1^n\in\mathcal{Z}_h$ in (4.24) and $\tilde{\psi}=\Delta t\theta_1^n$ in the weak form for ψ_1^ϵ of the problem ($\mathbf{WP}^\epsilon$), and subtracting (4.24) by $1/\Delta t\int_{I_n}$ of the weak form for ψ_1^ϵ of the problem ($\mathbf{WP}^\epsilon$), where $I_n=[t^{n-1},t^n]$, then we obtain by using the identity

$2(a-b,a) = |a|^2 - |b|^2 + |a-b|^2$ that (after the superscript ϵ is dropped)

$$
\begin{aligned}
&(\delta_t\theta_1^n,\, \Delta t\theta_1^n) + \Delta t\left(\gamma_1\frac{\xi_1^2}{x_0^2}\theta_1^n,\, \theta_1^n\right)_{\partial\Omega} \\
&\quad + \Delta t\left(-i\frac{\xi_1}{x_0}\nabla\theta_1^n - \frac{x_0}{\lambda_1}(\mathbf{A}_h^n + \mathbf{A}_c^n)\theta_1^n,\ -i\frac{\xi_1}{x_0}\nabla\theta_1^n - \frac{x_0}{\lambda_1}(\mathbf{A}_h^n + \mathbf{A}_c^n)\theta_1^n\right) \\
&\quad = \frac{1}{2}\big[\,||\theta_1^n||_0^2 - ||\theta_1^{n-1}||_0^2 + ||\theta_1^n - \theta_1^{n-1}||_0^2\big] + \Delta t\gamma_1\frac{\xi_1^2}{x_0^2}||\theta_1^n||_{0,\partial\Omega}^2 \\
&\qquad + \Delta t\,||-i\frac{\xi_1}{x_0}\nabla\theta_1^n - \frac{x_0}{\lambda_1}(\mathbf{A}_h^n + \mathbf{A}_c^n)\theta_1^n||_0^2 \\
&\quad = \big(\delta_t(\psi_1^n - I_h\psi_1^n),\, \Delta t\theta_1^n\big) \\
&\qquad + \Bigg[\int_{I_n}\left(-i\frac{\xi_1}{x_0}\nabla\psi_1 - \frac{x_0}{\lambda_1}(\mathbf{A} + \mathbf{A}_c)\psi_1,\ -i\frac{\xi_1}{x_0}\nabla\theta_1^n - \frac{x_0}{\lambda_1}(\mathbf{A} + \mathbf{A}_c)\theta_1^n\right)dt \\
&\qquad - \Delta t\left(-i\frac{\xi_1}{x_0}\nabla I_h\psi_1^n - \frac{x_0}{\lambda_1}(\mathbf{A}_h^n + \mathbf{A}_c^n)I_h\psi_1^n,\ -i\frac{\xi_1}{x_0}\nabla\theta_1^n - \frac{x_0}{\lambda_1}(\mathbf{A}_h^n + \mathbf{A}_c^n)\theta_1^n\right)\Bigg] \\
&\qquad + \left[\int_{I_n}\big(i\phi_c\psi_1,\, \theta_1^n\big)\,dt - \Delta t\big(i\phi_c^n\psi_{1h}^n,\, \theta_1^n\big)\right] \\
&\qquad + \left[\int_{I_n}\big((|\psi_1|^2 - \mathcal{T}_1)\psi_1,\, \theta_1^n\big)\,dt - \Delta t\,\big((|\psi_{1h}^n|^2 - \mathcal{T}_1)\psi_{1h}^n,\, \theta_1^n\big)\right] \\
&\qquad + \left[\int_{I_n}\eta(\psi_2,\, \theta_1^n)dt - \Delta t\,\eta(\psi_{2h}^n,\, \theta_1^n)\right] \\
&\qquad + \left[\int_{I_n}\left(\gamma_1\frac{\xi_1^2}{x_0^2}\psi_1,\, \theta_1^n\right)_{\partial\Omega}dt - \Delta t\left(\gamma_1\frac{\xi_1^2}{x_0^2}\,I_h\psi_1^n,\, \theta_1^n\right)_{\partial\Omega}\right] \\
&\quad = (I) + (II) + (III) + (IV) + (V) + (VI).
\end{aligned}
\tag{4.63}
$$

Since $\left(\frac{\xi_1}{x_0}\right)^2||\nabla\theta_1^n||_0^2 \le ||-i\frac{\xi_1}{x_0}\nabla\theta_1^n - \frac{x_0}{\lambda_1}(\mathbf{A}_h^n + \mathbf{A}_c^n)\theta_1^n||_0^2 + ||\frac{x_0}{\lambda_1}(\mathbf{A}_h^n + \mathbf{A}_c^n)\theta_1^n||_0^2$, we have

$$
\begin{aligned}
&\frac{1}{2}\big[\,||\theta_1^n||_0^2 - ||\theta_1^{n-1}||_0^2\,\big] + \Delta t\,\big(\frac{\xi_1}{x_0}\big)^2||\theta_1^n||_1^2 + \Delta t||\theta_1^n||_{0,\partial\Omega}^2 \\
&\quad \le \Re\big\{(I) + (II) + (III) + (IV) + (V) + (VI)\big\} \\
&\qquad + \Delta t\big(\frac{\xi_1}{x_0}\big)^2||\theta_1^n||_0^2 + \Delta t||\frac{x_0}{\lambda_1}(\mathbf{A}_h^n + \mathbf{A}_c^n)\theta_1^n||_0^2.
\end{aligned}
\tag{4.64}
$$

We now estimate the terms on the R.H.S. of the above inequality. By lemma 4.4.3, we have

$$
\begin{aligned}
|\Re(I)| &\le C\Delta t^{\frac{1}{2}}\,h^k\left[\int_{I_n}||\frac{\partial\psi_1}{\partial t}||_s^2dt\right]^{\frac{1}{2}}||\theta_1^n||_0. \\
&\le Ch^{2k}\int_{I_n}||\frac{\partial\psi_1}{\partial t}||_s^2dt + \Delta t||\theta_1^n||_0^2,
\end{aligned}
$$

where $s = 2$ if $k = 1$, and $s = k$ if $k \ge 2$.

Rewrite (II) as

$$
\begin{aligned}
&(II) = \\
&\int_{I_n}\int_{\Omega} \big[-i\frac{\xi_1}{x_0}\nabla(\psi_1 - \psi_1^n) - \frac{x_0}{\lambda_1}(\mathbf{A}+\mathbf{A}_c)\psi_1 - \frac{x_0}{\lambda_1}(\mathbf{A}^n+\mathbf{A}_c^n)\psi_1^n\big] \\
&\qquad \cdot \big(i\frac{\xi_1}{x_0}\nabla\theta_1^{n*} - \frac{x_0}{\lambda_1}(\mathbf{A}+\mathbf{A}_c)\theta_1^{n*}\big)d\Omega\, dt \\
&+\Delta t\int_{\Omega} \big[-i\frac{\xi_1}{x_0}\nabla(\psi_1^n - I_h\psi_1^n) - \frac{x_0}{\lambda_1}(\mathbf{A}_h^n+\mathbf{A}_c^n)(\psi_1^n - I_h\psi_1^n)\big] \\
&\qquad \cdot \big(i\frac{\xi_1}{x_0}\nabla\theta_1^{n*} - \frac{x_0}{\lambda_1}(\mathbf{A}_h^n+\mathbf{A}_c^n)\theta_1^{n*}\big)d\Omega \\
&+\Delta t\int_{\Omega} \big[-i\frac{\xi_1}{x_0}\nabla\psi_1^n - \frac{x_0}{\lambda_1}(\mathbf{A}^n+\mathbf{A}_c^n)\psi_1^n\big]\cdot\big((\mathbf{A}_h^n+\mathbf{A}_c^n) - (\mathbf{I}_h\mathbf{A}^n+\mathbf{A}_c^n)\big)\,\theta_1^{n*}d\Omega \\
&+\Delta t\int_{\Omega} \big[-i\frac{\xi_1}{x_0}\nabla\psi_1^n - \frac{x_0}{\lambda_1}(\mathbf{A}^n+\mathbf{A}_c^n)\psi_1^n\big]\cdot\big((\mathbf{I}_h\mathbf{A}^n+\mathbf{A}_c^n) - (\mathbf{A}^n+\mathbf{A}_c^n)\big)\,\theta_1^{n*}d\Omega \\
&+\int_{I_n}\int_{\Omega} \big[-i\frac{\xi_1}{x_0}\nabla\psi_1^n - \frac{x_0}{\lambda_1}(\mathbf{A}^n+\mathbf{A}_c^n)\psi_1^n\big]\cdot\big((\mathbf{A}^n+\mathbf{A}_c^n) - (\mathbf{A}+\mathbf{A}_c)\big)\,\theta_1^{n*}d\Omega\, dt \\
&+\Delta t\int_{\Omega} \big((\mathbf{A}_h^n+\mathbf{A}_c^n) - (\mathbf{I}_h\mathbf{A}^n+\mathbf{A}_c^n)\big)\,\psi_1^n\cdot\big[i\frac{\xi_1}{x_0}\nabla\theta_1^{n*} - \frac{x_0}{\lambda_1}(\mathbf{A}_h^n+\mathbf{A}_c^n)\theta_1^{n*}\big]\,d\Omega \\
&+\Delta t\int_{\Omega} \big((\mathbf{I}_h\mathbf{A}^n+\mathbf{A}_c^n) - (\mathbf{A}^n+\mathbf{A}_c^n)\big)\,\psi_1^n\cdot\big[i\frac{\xi_1}{x_0}\nabla\theta_1^{n*} - \frac{x_0}{\lambda_1}(\mathbf{A}_h^n+\mathbf{A}_c^n)\theta_1^{n*}\big]\,d\Omega \\
&= (II)_1 + (II)_2 + (II)_3 + (II)_4 + (II)_5 + (II)_6 + (II)_7.
\end{aligned}
$$

Then by using lemma 4.4.4, we have

$$
\begin{aligned}
|\Re(II)_1| \;&\leq\; \int_{I_n} ||-i\frac{\xi_1}{x_0}\nabla(\psi_1-\psi_1^n) - \frac{x_0}{\lambda_1}(\mathbf{A}+\mathbf{A}_c)\psi_1 + \frac{x_0}{\lambda_1}(\mathbf{A}^n+\mathbf{A}_c^n)\psi_1^n||_0 \\
&\qquad \times ||i\frac{\xi_1}{x_0}\nabla\theta_1^n - \frac{x_0}{\lambda_1}(\mathbf{A}+\mathbf{A}_c)\theta_1^n||_0\, dt \\
&\leq\; C\Delta t^{\frac{3}{2}}\Big[\int_{I_n}||\frac{\partial\nabla\psi_1}{\partial t} + \frac{\partial}{\partial t}\big((\mathbf{A}+\mathbf{A}_c)\psi_1\big)||_0^2 dt\Big]^{\frac{1}{2}} \\
&\qquad \times\big[\,||\nabla\theta_1^n||_0 + \max_{1\leq t\leq T}||\mathbf{A}+\mathbf{A}_c||_{0,4}||\theta_1^n||_{0,4}\big] \\
&\leq\; C_\varepsilon\Delta t^2\int_{I_n}\big[\,||\frac{\partial\psi_1}{\partial t}||_1^2 + ||\frac{\partial}{\partial t}(\mathbf{A}+\mathbf{A}_c)||_0^2\,\big]dt + \varepsilon\Delta t||\theta_1^n||_1^2.
\end{aligned}
$$

For $(II)_2$, first note that the regularity of $\mathbf{A}_c$ (4.18) and the stability result (4.49) give

$||\mathbf{A}_h^n + \mathbf{A}_c^n||_{0,4} \leq C$. Then making use of the interpolation error (4.2), we get

$$
\begin{aligned}
|\Re(II)_2| &\leq \Delta t\big[\, ||-i\frac{\xi_1}{x_0}\nabla(\psi_1^n - I_h\psi_1^n)||_0 + ||\frac{x_0}{\lambda_1}(\mathbf{A}_h^n + \mathbf{A}_c^n)||_{0,4}\, ||(\psi_1^n - I_h\psi_1^n)||_{0,4}\big] \\
&\qquad \times \big[\, \frac{\xi_1}{x_0}||\nabla\theta_1^n||_0 + ||\frac{x_0}{\lambda_1}(\mathbf{A}_h^n + \mathbf{A}_c^n)||_{0,4}||\theta_1^n||_{0,4}\big] \\
&\leq C\Delta t\, h^k\, ||\psi_1^n||_{k+1}\, ||\theta_1^n||_1 \\
&\leq C_\varepsilon \Delta t\, h^{2k}||\psi_1^n||_{k+1}^2 \,+\, \varepsilon\Delta t||\theta_1^n||_1^2,
\end{aligned}
$$

$$
\begin{aligned}
|\Re(II)_3| &\leq C\Delta t\big[\, ||\nabla\psi_1^n||_{0,4} + ||\mathbf{A}^n + \mathbf{A}_c^n||_{0,4}\, ||\psi_1^n||_{0,\infty}\big]\, ||\Theta^n||_0\, ||\theta_1^n||_{0,4} \\
&\leq C_\varepsilon \Delta t||\Theta^n||_0^2 + \varepsilon\Delta t||\theta_1^n||_1^2,
\end{aligned}
$$

By the interpolation error (4.3),

$$
\begin{aligned}
|\Re(II)_4| &\leq C\Delta t\big[\, ||\nabla\psi_1^n||_{0,4} + ||\mathbf{A}^n + \mathbf{A}_c^n||_{0,4}\, ||\psi_1^n||_{0,\infty}\big]\, h^{k+1}||\mathbf{A}^n||_{k+1}\, ||\theta_1^n||_{0,4} \\
&\leq C_\varepsilon \Delta t\, h^{2(k+1)}||\mathbf{A}^n||_{k+1}^2 + \varepsilon\Delta t||\theta_1^n||_1^2,
\end{aligned}
$$

$$
\begin{aligned}
|\Re(II)_5| &\leq C\big[\, ||\nabla\psi_1^n||_{0,4} + ||\mathbf{A}^n + \mathbf{A}_c^n||_{0,4}\, ||\psi_1^n||_{0,\infty}\big] \\
&\qquad \times\Delta t^{\frac{3}{2}}\left[\int_{I_n} ||\frac{\partial}{\partial t}(\mathbf{A} + \mathbf{A}_c)||_0^2\, dt\right]^{\frac{1}{2}} ||\theta_1^n||_{0,4} \\
&\leq C_\varepsilon \Delta t^2 \int_{I_n} ||\frac{\partial}{\partial t}(\mathbf{A} + \mathbf{A}_c)||_0^2\, dt + \varepsilon\Delta t||\theta_1^n||_1^2,
\end{aligned}
$$

$$
\begin{aligned}
|\Re(II)_6| &\leq C\Delta t||\Theta^n||_0\, ||\psi_1^n||_{0,\infty}\, \big[\, ||\nabla\theta_1^n||_0 + ||(\mathbf{A}_h^n + \mathbf{A}_c^n)||_{0,4}||\theta_1^n||_{0,4}\big] \\
&\leq C_\varepsilon \Delta t||\Theta^n||_0^2 + \varepsilon\Delta t||\theta_1^n||_1^2,
\end{aligned}
$$

By the interpolation error (4.3),

$$
\begin{aligned}
|\Re(II)_7| &\leq C\Delta t\, h^{k+1}||\mathbf{A}^n||_{k+1}\, ||\psi_1^n||_{0,\infty}\, \big[\, ||\nabla\theta_1^n||_0 + ||(\mathbf{A}_h^n + \mathbf{A}_c^n)||_{0,4}||\theta_1^n||_{0,4}\big] \\
&\leq C_\varepsilon \Delta t\, h^{2(k+1)}||\mathbf{A}^n||_{k+1}^2 + \varepsilon\Delta t||\theta_1^n||_1^2.
\end{aligned}
$$

Rewrite (III) as

$$
\begin{aligned}
(III) &= \int_{I_n}\int_\Omega i(\phi_c\psi_1 - \phi_c^n\psi_1^n)\, \theta_1^{n*}\; d\Omega dt + \Delta t\int_\Omega i\phi_c^n(\psi_1^n - I_h\psi_1^n)\, \theta_1^{n*} d\Omega \\
&\qquad +\Delta t\int_\Omega i\phi_c^n(I_h\psi_1^n - \psi_{1h}^n)\, \theta_1^{n*} d\Omega,
\end{aligned}
$$

then we have

$$
\begin{aligned}
|\Re(III)| &\leq C\Delta t^{\frac{3}{2}}\left[\int_{I_n}||\frac{\partial(\phi_c\psi_1)}{\partial t}||_0^2\,dt\right]^{\frac{1}{2}}||\theta_1^n||_0 \\
&\quad +C\Delta t\,h^k||\phi_c^n||_{0,4}\,||\psi_1^n||_{k+1}\,||\theta_1^n||_0 \\
&\quad +\Delta t\,\Re\int_\Omega i\phi_c^n|\theta_1^n|^2d\Omega \\
&\leq C\Delta t^2\int_{I_n}\left[\,||\phi_c||_{0,4}^2\,||\frac{\partial\psi_1}{\partial t}||_{0,4}^2+||\psi_1||_{0,4}^2\,||\frac{\partial\phi_c}{\partial t}||_{0,4}^2\,\right]dt \\
&\quad +C\Delta t\,h^{2k}\,||\phi_c||_{0,4}^2\,||\psi_1^n||_{k+1}^2+2\Delta t||\theta_1^n||_0^2.
\end{aligned}
$$

By using the notation of function g_1 in lemma 4.4.2, we rewrite (IV) as

$$
\begin{aligned}
(IV) &= \int_{I_n}\int_\Omega(g_1(\psi_1)-g_1(\psi_1^n))\,\theta_1^{n*}d\Omega dt+\Delta t\int_\Omega(g_1(\psi_1^n)-g_1(I_h\psi_1^n))\,\theta_1^{n*}d\Omega \\
&\quad +\Delta t\int_\Omega(g_1(I_h\psi_1^n)-g_1(\psi_{1h}^n))\,\theta_1^{n*}d\Omega,
\end{aligned}
$$

then by the regularity assumption (4.12), $||\psi_1^n||_{0,\infty}\leq C$ and by lemma 4.4.1, $||I_h\psi_i^n||_{0,\infty}\leq C_{\psi_i}$. Thus by lemma 4.4.2, we get

$$
\begin{aligned}
|\Re(IV)| &\leq C\Delta t^{\frac{3}{2}}\left[\int_{I_n}||\frac{\partial g_1(\psi_1)}{\partial t}||_0^2\,dt\right]^{\frac{1}{2}}||\theta_1^n||_0 \\
&\quad +C_{\psi_1}\Delta t||\psi_1^n-I_n\psi_1^n||_0\,||\theta_1^n||_0+\Delta t|\mathcal{T}_1|\,||\theta_1^n||_0^2 \\
&\leq C\Delta t^{\frac{3}{2}}\left[\int_{I_n}||\frac{\partial g_1(\psi_1)}{\partial t}||_0^2\,dt\right]^{\frac{1}{2}}||\theta_1^n||_0 \\
&\quad +C_{\psi_1}\Delta t\,h^{k+1}||\psi_1^n||_{k+1}\,||\theta_1^n||_0+\Delta t|\mathcal{T}_1|\,||\theta_1^n||_0^2 \\
&\leq C\Delta t^2\int_{I_n}||\frac{\partial g_1(\psi_1)}{\partial t}||_0^2\,dt \\
&\quad +C_{\psi_1}\Delta t\,h^{2(k+1)}||\psi_1^n||_{k+1}^2+\Delta t(2+|\mathcal{T}_1|)\,||\theta_1^n||_0^2.
\end{aligned}
$$

Rewrite (V) as

$$
\begin{aligned}
(V) &= \int_{I_n}\int_\Omega\eta(\psi_2-\psi_2^n)\,\theta_1^{n*}d\Omega dt+\Delta t\int_\Omega\eta(\psi_2^n-I_h\psi_2^n)\,\theta_1^{n*}d\Omega, \\
&\quad +\Delta t\int_\Omega\eta(I_h\psi_2^n-\psi_{2h}^n)\,\theta_1^{n*}d\Omega,
\end{aligned}
$$

then we have

$$
\begin{aligned}
|\Re(V)| &\leq C\Delta t^{\frac{3}{2}}\left[\int_{I_n}||\frac{\partial\psi_2}{\partial t}||_0^2\,dt\right]^{\frac{1}{2}}||\theta_1^n||_0+C\Delta t\,h^{k+1}\,||\psi_2^n||_{k+1}\,||\theta_1^n||_0 \\
&\quad +\Delta t\,||\theta_2^n||_0\,||\theta_1^n||_0 \\
&\leq C\Delta t^2\int_{I_n}||\frac{\partial\psi_2}{\partial t}||_0^2\,dt+C\Delta t\,h^{2(k+1)}||\psi_2^n||_{k+1}^2+\frac{1}{4}\Delta t||\theta_2^n||_0^2+\Delta t||\theta_1^n||_0^2.
\end{aligned}
$$

Rewrite (VI) as

$$(VI) \;=\; \gamma_1 \frac{\xi_1^2}{x_0^2}\Big[\int_{I_n}\int_{\partial\Omega}(\psi_1-\psi_1^n)\,\theta_1^{n*}dSdt - \Delta t\int_{\partial\Omega}(\psi_1^n - I_h\psi_1^n)\,\theta_1^{n*}dS\Big],$$

then using an analogue result of the lemma 4.4.4 and the boundary interpolation error (4.4), we get

$$\begin{aligned}
|\Re(VI)| \;\le\; & C\gamma_1\frac{\xi_1^2}{x_0^2}\Big[\Delta t^{\frac{3}{2}}\Big[\int_{I_n}||\frac{\partial\psi_1}{\partial t}||^2_{0,\partial\Omega}\,dt\Big]^{\frac{1}{2}}||\theta_1^n||_{0,\partial\Omega} \\
&\quad +\Delta t\,h^{(k+1)-\frac{1}{2}}\,||\psi_1^n||_{(k+1)-\frac{1}{2},\partial\Omega}\,||\theta_1^n||_{0,\partial\Omega}\Big] \\
&\le C\Delta t^2\int_{I_n}||\frac{\partial\psi_1}{\partial t}||^2_{0,\partial\Omega}\,dt + C\Delta t\,h^{2(k+\frac{1}{2})}\,||\psi_1^n||^2_{k+1} + \Delta t\frac{\gamma_1}{2}\frac{\xi_1^2}{x_0^2}||\theta_1^n||^2_{0,\partial\Omega},
\end{aligned}$$

where in the second term of the last inequality, we have used the trace embedding theorem.

Combining all the above estimates, the inequality (4.64) becomes

$$\begin{aligned}
&\frac{1}{2}\big[\,||\theta_1^n||_0^2 - ||\theta_1^{n-1}||_0^2\,\big] + \Delta t\,\frac{\xi_1^2}{x_0^2}||\theta_1^n||_1^2 + \Delta t\gamma_1\frac{\xi_1^2}{x_0^2}||\theta_1^n||^2_{0,\partial\Omega} \\
&\quad\le (8+|\mathcal{T}_1|)\Delta t||\theta_1^n||_0^2 + \frac{1}{4}\Delta t||\theta_2^n||_0^2 + 7\varepsilon\Delta t||\theta_1^n||_1^2 + \Delta t\frac{\gamma_1}{2}\frac{\xi_1^2}{x_0^2}||\theta_1^n||^2_{0,\partial\Omega} \\
&\qquad + C_\varepsilon\Delta t||\Theta^n||_0^2 + \;\Delta t\left(\frac{\xi_1}{x_0}\right)^2||\theta_1^n||_0^2 + \Delta t\left(\frac{x_0}{\lambda_1}\right)^2||(\mathbf{A}_h^n+\mathbf{A}_c^n)\theta_1^n||_0^2 \\
&\qquad + Ch^{2k}\int_{I_n}||\frac{\partial\psi_1}{\partial t}||_s^2dt + C_\varepsilon\Delta t^2\int_{I_n}\big[\,||\frac{\partial\psi_1}{\partial t}||_1^2 + ||\frac{\partial}{\partial t}(\mathbf{A}+\mathbf{A}_c)||_0^2\,\big]dt \\
&\qquad + C_\varepsilon\Delta t\,h^{2k}||\psi_1^n||^2_{k+1} + C_\varepsilon\Delta t\,h^{2(k+1)}||\mathbf{A}^n||^2_{k+1} \\
&\qquad + C_\varepsilon\Delta t^2\int_{I_n}||\frac{\partial}{\partial t}(\mathbf{A}+\mathbf{A}_c)||_0^2\,dt + C\Delta t\,h^{2k}\,||\phi_c||^2_{0,4}\,||\psi_1^n||^2_{k+1} \\
&\qquad + C\Delta t^2\int_{I_n}\big[\,||\phi_c||^2_{0,4}\,||\frac{\partial\psi_1}{\partial t}||^2_{0,4} + ||\psi_1||^2_{0,4}\,||\frac{\partial\phi_c}{\partial t}||^2_{0,4}\,\big]dt \\
&\qquad + C\Delta t^2\int_{I_n}||\frac{\partial g_1(\psi_1)}{\partial t}||_0^2\,dt + C_{\psi_1}\Delta t\,h^{2(k+1)}||\psi_1^n||^2_{k+1} \\
&\qquad + C\Delta t^2\int_{I_n}||\frac{\partial\psi_2}{\partial t}||_0^2\,dt + C\Delta t\,h^{2(k+1)}||\psi_2^n||^2_{k+1} \\
&\qquad + C\Delta t^2\int_{I_n}||\frac{\partial\psi_1}{\partial t}||^2_{0,\partial\Omega}\,dt + C\Delta t\,h^{2(k+\frac{1}{2})}\,||\psi_1^n||^2_{k+1},
\end{aligned}\tag{4.65}$$

where $s = 2$ if $k = 1$ and $s = k$ if $k \ge 2$.

Now since

$$
\begin{aligned}
||(\mathbf{A}_h^n + \mathbf{A}_c^n)\theta_1^n||_0^2 &\le ||(\mathbf{A}_h^n + \mathbf{A}_c^n)||_{0,4}^2 \, ||\theta_1^n||_{0,4}^2 \\
&\le ||(\mathbf{A}_h^n + \mathbf{A}_c^n)||_{0,4}^2 \left(C||\theta_1^n||_0 \, ||\theta_1^n||_1 \right) \\
&\le C_\varepsilon \Delta t ||\theta_1^n||_0^2 + \varepsilon \Delta t ||\theta_1^n||_1^2,
\end{aligned}
$$

the inequality (4.65) becomes

$$
\begin{aligned}
&\frac{1}{2}\big[\, ||\theta_1^n||_0^2 - ||\theta_1^{n-1}||_0^2 \,\big] + \Delta t \frac{\xi_1^2}{x_0^2} ||\theta_1^n||_1^2 + \Delta t \gamma_1 \frac{\xi_1^2}{x_0^2} ||\theta_1^n||_{0,\partial\Omega}^2 \\
&\quad \le C_\varepsilon \Delta t ||\theta_1^n||_0^2 + \frac{1}{4}\Delta t ||\theta_2^n||_0^2 + 8\varepsilon \Delta t ||\theta_1^n||_1^2 + \Delta t \frac{\gamma_1}{2}\frac{\xi_1^2}{x_0^2} ||\theta_1^n||_{0,\partial\Omega}^2 + C_\varepsilon \Delta t ||\Theta^n||_0^2. \\
&\qquad + C h^{2k} \int_{I_n} ||\frac{\partial \psi_1}{\partial t}||_s^2 dt + C_\varepsilon \Delta t^2 \int_{I_n} \big[\, ||\frac{\partial \psi_1}{\partial t}||_1^2 + ||\frac{\partial}{\partial t}(\mathbf{A} + \mathbf{A}_c)||_0^2 \,\big] dt \\
&\qquad + C_\varepsilon \Delta t \, h^{2k} ||\psi_1^n||_{k+1}^2 + C_\varepsilon \Delta t \, h^{2(k+1)} ||\mathbf{A}^n||_{k+1}^2 \\
&\qquad + C_\varepsilon \Delta t^2 \int_{I_n} ||\frac{\partial}{\partial t}(\mathbf{A} + \mathbf{A}_c)||_0^2 \, dt + C \Delta t \, h^{2k} \, ||\phi_c||_{0,4}^2 \, ||\psi_1^n||_{k+1}^2 \\
&\qquad + C \Delta t^2 \int_{I_n} \big[\, ||\phi_c||_{0,4}^2 \, ||\frac{\partial \psi_1}{\partial t}||_{0,4}^2 + ||\psi_1||_{0,4}^2 \, ||\frac{\partial \phi_c}{\partial t}||_{0,4}^2 \,\big] dt \\
&\qquad + C \Delta t^2 \int_{I_n} ||\frac{\partial g_1(\psi_1)}{\partial t}||_0^2 \, dt + C_{\psi_1} \Delta t \, h^{2(k+1)} ||\psi_1^n||_{k+1}^2 \\
&\qquad + C \Delta t^2 \int_{I_n} ||\frac{\partial \psi_2}{\partial t}||_0^2 \, dt + C \Delta t \, h^{2(k+1)} ||\psi_2^n||_{k+1}^2 \\
&\qquad + C \Delta t^2 \int_{I_n} ||\frac{\partial \psi_1}{\partial t}||_{0,\partial\Omega}^2 \, dt + C \Delta t \, h^{2(k+\frac{1}{2})} \, ||\psi_1^n||_{k+1}^2. \qquad (4.66)
\end{aligned}
$$

Similarly, by applying the same technique to the equation (4.25) and to the weak form for ψ_2^ϵ of the problem $(\mathbf{WP}^\epsilon)$, we get

$$
\begin{aligned}
&\frac{1}{2}\big[\,||\theta_2^n||_0^2 - ||\theta_2^{n-1}||_0^2\,\big] + \Delta t\frac{\xi_2^2}{x_0^2}||\theta_2^n||_1^2 + \Delta t\gamma_2\frac{\xi_2^2}{x_0^2}||\theta_2^n||_{0,\partial\Omega}^2 \\
&\quad \le C_\varepsilon\Delta t||\theta_2^n||_0^2 + \frac{1}{4}\Delta t||\theta_1^n||_0^2 + 8\varepsilon\Delta t||\theta_2^n||_1^2 + \Delta t\frac{\gamma_2}{2}\frac{\xi_2^2}{x_0^2}||\theta_2^n||_{0,\partial\Omega}^2 + C_\varepsilon\Delta t||\Theta^n||_0^2. \\
&\qquad + Ch^{2k}\int_{I_n}||\frac{\partial\psi_2}{\partial t}||_s^2 dt + C_\varepsilon\Delta t^2\int_{I_n}\big[\,||\frac{\partial\psi_2}{\partial t}||_1^2 + ||\frac{\partial}{\partial t}(\mathbf{A}+\mathbf{A}_c)||_0^2\,\big]dt \\
&\qquad + C_\varepsilon\Delta t\, h^{2k}||\psi_2^n||_{k+1}^2 + C_\varepsilon\Delta t\, h^{2(k+1)}||\mathbf{A}^n||_{k+1}^2 \\
&\qquad + C_\varepsilon\Delta t^2\int_{I_n}||\frac{\partial}{\partial t}(\mathbf{A}+\mathbf{A}_c)||_0^2\, dt + C\Delta t\, h^{2k}\,||\phi_c||_{0,4}^2\,||\psi_2^n||_{k+1}^2 \\
&\qquad + C\Delta t^2\int_{I_n}\big[\,||\phi_c||_{0,4}^2\,||\frac{\partial\psi_2}{\partial t}||_{0,4}^2 + ||\psi_2||_{0,4}^2\,||\frac{\partial\phi_c}{\partial t}||_{0,4}^2\,\big]dt \\
&\qquad + C\Delta t^2\int_{I_n}||\frac{\partial g_2(\psi_2)}{\partial t}||_0^2\, dt + C_{\psi_2}\Delta t\, h^{2(k+1)}||\psi_2^n||_{k+1}^2 \\
&\qquad + C\Delta t^2\int_{I_n}||\frac{\partial\psi_1}{\partial t}||_0^2\, dt + C\Delta t\, h^{2(k+1)}||\psi_1^n||_{k+1}^2 \\
&\qquad + C\Delta t^2\int_{I_n}||\frac{\partial\psi_2}{\partial t}||_{0,\partial\Omega}^2\, dt + C\Delta t\, h^{2(k+\frac{1}{2})}\,||\psi_2^n||_{k+1}^2, \qquad (4.67)
\end{aligned}
$$

where $s = 2$ if $k = 1$ and $s = k$ if $k \ge 2$.

Now by choosing the test function $\tilde{\mathbf{A}}_h = \Delta t\Theta^n \in \Lambda_h$ in (4.26) and $\tilde{\mathbf{A}} = \Delta t\Theta^n$ in the weak form for $\mathbf{A}^\epsilon$ of the problem ($\mathbf{WP}^\epsilon$), and subtracting (4.26) by $1/\Delta t\int_{I_n}$ of the weak form for $\mathbf{A}^\epsilon$ of the problem ($\mathbf{WP}^\epsilon$), we get (after the superscript ϵ is dropped)

$$
\begin{aligned}
&\sigma\frac{x_0^2}{\lambda_1^2}(\delta_t\Theta^n,\,\Delta t\Theta^n)+\Delta t(\mathrm{curl}\Theta^n,\,\mathrm{curl}\Theta^n)+\epsilon\Delta t(\mathrm{div}\Theta^n,\,\mathrm{div}\Theta^n)\\
&\quad=(\delta_t(\mathbf{A}^n-I_h\mathbf{A}^n),\,\Delta t\Theta^n)\\
&\qquad+\Big[\int_{I_n}\big[(\mathrm{curl}\mathbf{A},\,\mathrm{curl}\Theta^n)+\epsilon(\mathrm{div}\mathbf{A},\,\mathrm{div}\Theta^n)\big]dt\\
&\qquad-\Delta t(\mathrm{curl}\mathbf{I}_h\mathbf{A}^n,\,\mathrm{curl}\Theta^n)-\epsilon\Delta t(\mathrm{div}\mathbf{I}_h\mathbf{A}^n,\,\mathrm{div}\Theta^n)\Big]\\
&\qquad-\Big[\int_{I_n}\Re\Big(\big(-i\frac{\xi_1}{x_0}\nabla\psi_1-\frac{x_0}{\lambda_1}(\mathbf{A}+\mathbf{A}_c)\psi_1\big),\,\frac{x_0}{\lambda_1}\psi_1\Theta^n\Big)dt\\
&\qquad+\Delta t\Re\Big(\big(-i\frac{\xi_1}{x_0}\nabla\psi_{1h}^n-\frac{x_0}{\lambda_1}(\mathbf{A}_h^n+\mathbf{A}_c^n)\psi_{1h}^n\big),\,\frac{x_0}{\lambda_1}\psi_{1h}^n\Theta^n\Big)\Big]\\
&\qquad-\Big[\int_{I_n}\Re\Big(\big(-i\frac{\xi_2}{x_0}\nabla\psi_2-\nu\frac{x_0}{\lambda_2}(\mathbf{A}+\mathbf{A}_c)\psi_2\big),\,\frac{1}{\nu}\frac{x_0}{\lambda_2}\psi_2\Theta^n\Big)dt\\
&\qquad+\Delta t\Re\Big(\big(-i\frac{\xi_2}{x_0}\nabla\psi_{2h}^n-\nu\frac{x_0}{\lambda_2}(\mathbf{A}_h^n+\mathbf{A}_c^n)\psi_{2h}^n\big),\,\frac{1}{\nu}\frac{x_0}{\lambda_2}\psi_{2h}^n\Theta^n\Big)\Big]\\
&\qquad+\Big[\Delta t(\mathbf{H}_e^n,\,\mathrm{curl}\Theta^n)-\int_{I_n}(\mathbf{H}_e,\,\mathrm{curl}\Theta^n)\Big]\\
&\quad=(VII)+(VIII)+(IX)+(X)+(XI).
\end{aligned}
\tag{4.68}
$$

So by using the identity $2(a-b,a)=|a|^2-|b|^2+|a-b|^2$, we get

$$
\begin{aligned}
&\sigma\frac{x_0^2}{2\lambda_1^2}\big[||\Theta^n||_0^2-||\Theta^{n-1}||_0^2\big]+\epsilon\Delta t||\mathrm{div}\,\Theta^n||_0^2+\Delta t||\mathrm{curl}\,\Theta^n||_0^2\\
&\qquad\le(VII)+(VIII)+(IX)+(X)+(XI).
\end{aligned}
\tag{4.69}
$$

We now estimate the terms (VII)-(XI). Similar to the term (I), we have

$$
|(VII)| \;\;\le\;\; C\,h^{2k}\int_{I_n}||\frac{\partial\mathbf{A}^n}{\partial t}||_s^2dt+\Delta t||\Theta^n||_0^2,
$$

where $s=2$ if $k=1$ and $s=k$ if $k\ge 2$.

Rewrite $(VIII)$ as

$$
\begin{aligned}
(VIII) \;\;=\;\; &\int_{I_n}\big[(\mathrm{curl}(\mathbf{A}-\mathbf{A}^n),\,\mathrm{curl}\Theta^n)+\epsilon(\mathrm{div}(\mathbf{A}-\mathbf{A}^n),\,\mathrm{div}\Theta^n)\big]dt\\
&+\Delta t(\mathrm{curl}(\mathbf{A}^n-\mathbf{I}_h\mathbf{A}^n),\,\mathrm{curl}\Theta^n)+\epsilon\Delta t(\mathrm{div}(\mathbf{A}^n-\mathbf{I}_h\mathbf{A}^n),\,\mathrm{div}\Theta^n),
\end{aligned}
$$

so we have

$$
\begin{aligned}
|(VIII)| &\leq C\Delta t^{\frac{3}{2}}\Big[\int_{I_n} ||\frac{\partial \mathrm{curl}\mathbf{A}}{\partial t}||_0^2 + \epsilon||\frac{\partial \mathrm{div}\mathbf{A}}{\partial t}||_0^2\,dt\Big]^{\frac{1}{2}}||\Theta^n||_1 \\
&\quad + \Delta t||(\mathbf{A}^n - \mathbf{I}_h\mathbf{A}^n)||_1\,||\Theta^n||_1 \\
&\leq C\Delta t^{\frac{3}{2}}\Big[\int_{I_n} ||\frac{\partial \mathbf{A}}{\partial t}||_1^2\,dt\Big]^{\frac{1}{2}}||\Theta^n||_1 + C\Delta t\,h^k||\mathbf{A}^n||_{k+1}\,||\Theta^n||_1 \\
&\leq C_{\varepsilon'}\Big[\Delta t^2\int_{I_n} ||\frac{\partial \mathbf{A}}{\partial t}||_1^2\,dt + C\Delta t\,h^{2k}||\mathbf{A}^n||_{k+1}^2\Big] + \varepsilon'\Delta t\,||\Theta^n||_1^2
\end{aligned}
$$

Rewrite (IX) as

$$
\begin{aligned}
&(IX) = \\
&\int_{I_n}\Re\int_\Omega \big[\big(-i\frac{\xi_1}{x_0}\nabla\psi_1^n - \frac{x_0}{\lambda_1}(\mathbf{A}^n + \mathbf{A}_c^n)\psi_1^n\big)\frac{x_0}{\lambda_1}\psi_1^{n*} \\
&\qquad -\big(-i\frac{\xi_1}{x_0}\nabla\psi_1 - \frac{x_0}{\lambda_1}(\mathbf{A} + \mathbf{A}_c)\psi_1\big)\frac{x_0}{\lambda_1}\psi_1^*\big]\cdot\Theta^n d\Omega\,dt \\
&\quad + \Delta t\,\Re\int_\Omega \big(-i\frac{\xi_1}{x_0}\nabla(\psi_{1h}^n - I_h\psi_1^n) - \frac{x_0}{\lambda_1}(\mathbf{A}_h^n + \mathbf{A}_c^n)(\psi_{1h}^n - I_h\psi_1^n)\big)\ \frac{x_0}{\lambda_1}\psi_{1h}^{n*}\cdot\Theta^n\,d\Omega \\
&\quad + \Delta t\,\Re\int_\Omega \big(-i\frac{\xi_1}{x_0}\nabla(I_h\psi_1^n - \psi_1^n) - \frac{x_0}{\lambda_1}(\mathbf{A}_h^n + \mathbf{A}_c^n)(I_h\psi_1^n - \psi_1^n)\big)\ \frac{x_0}{\lambda_1}\psi_{1h}^{n*}\cdot\Theta^n\,d\Omega \\
&\quad + \Delta t\,\Re\int_\Omega \big(-i\frac{\xi_1}{x_0}\nabla\psi_1^n - \frac{x_0}{\lambda_1}(\mathbf{A}^n + \mathbf{A}_c^n)\psi_1^n\big)\frac{x_0}{\lambda_1}(\psi_{1h}^{n*} - I_h\psi_1^{n*})\cdot\Theta^n d\Omega \\
&\quad + \Delta t\,\Re\int_\Omega \big(-i\frac{\xi_1}{x_0}\nabla\psi_1^n - \frac{x_0}{\lambda_1}(\mathbf{A}^n + \mathbf{A}_c^n)\psi_1^n\big)\frac{x_0}{\lambda_1}(I_h\psi_1^{n*} - \psi_1^{n*})\cdot\Theta^n d\Omega \\
&\quad + \Delta t\,\Re\int_\Omega \frac{x_0}{\lambda_1}\big((\mathbf{A}^n + \mathbf{A}_c^n) - (\mathbf{I}_h\mathbf{A}^n + \mathbf{A}_c^n)\big)\psi_1^n\frac{x_0}{\lambda_1}\psi_{1h}^{n*}\cdot\Theta^n d\Omega \\
&\quad + \Delta t\,\Re\int_\Omega \frac{x_0}{\lambda_1}\big((\mathbf{I}_h\mathbf{A}^n + \mathbf{A}_c^n) - (\mathbf{A}_h^n + \mathbf{A}_c^n)\big)\psi_1^n\frac{x_0}{\lambda_1}\psi_{1h}^{n*}\cdot\Theta^n d\Omega \\
&= (IX)_1 + (IX)_2 + (IX)_3 + (IX)_4 + (IX)_5 + (IX)_6 + (IX)_7.
\end{aligned}
$$

then we have

$$
\begin{aligned}
|(IX)_1| &\leq C\Delta t^{\frac{3}{2}}\Big[\int_{I_n} ||\frac{\partial}{\partial t}\big[\big(\nabla\psi_1 + (\mathbf{A} + \mathbf{A}_c)\psi_1\big)\psi_1^*\big]||_0^2 dt\Big]^{\frac{1}{2}}||\Theta^n||_0 \\
&\leq C\Delta t^{\frac{3}{2}}\Big[\int_{I_n}\big[\,||\psi_1||_{0,\infty}^2||\frac{\partial\nabla\psi_1}{\partial t}||_0^2 + ||\nabla\psi_1||_{0,4}^2||\frac{\partial\psi_1}{\partial t}||_{0,4}^2 \\
&\qquad + ||\psi_1||_{0,\infty}^2||\frac{\partial(\mathbf{A} + \mathbf{A}_c)}{\partial t}||_0^2 + ||\mathbf{A} + \mathbf{A}_c||_{0,4}^2||\psi_1||_{0,\infty}^2||\frac{\partial\psi_1}{\partial t}||_{0,4}^2\,\big]dt\Big]^{\frac{1}{2}}||\Theta^n||_0 \\
&\leq C\Delta t^2\int_{I_n}\big[\,||\frac{\partial\psi_1}{\partial t}||_1^2 + ||\frac{\partial}{\partial t}(\mathbf{A} + \mathbf{A}_c)||_0^2\,\big]dt + \Delta t||\Theta^n||_0^2.
\end{aligned}
$$

By the stability result that $||\psi_{1h}^n||_{0,4} \le C$ and $||\mathbf{A}_h^n + \mathbf{A}_c^n||_{0,4} \le C$, we get

$$\begin{aligned}
|(IX)_2| &\le C\Delta t\big[\, ||\nabla(\psi_{1h}^n - I_h\psi_1^n)||_0 + ||\mathbf{A}_h^n + \mathbf{A}_c^n||_{0,4}\, ||\psi_{1h}^n - I_h\psi_1^n||_{0,4} \,\big]\; ||\psi_{1h}^n||_{0,4}\, ||\Theta^n||_{0,4} \\
&\le \varepsilon\Delta t||\theta_1^n||_1^2 + C_\varepsilon \Delta t||\Theta^n||_{0,4}^2 \\
&\le \varepsilon\Delta t||\theta_1^n||_1^2 + C_\varepsilon \Delta t(||\Theta^n||_0\, ||\Theta^n||_1) \\
&\le \varepsilon\Delta t||\theta_1^n||_1^2 + C_{\varepsilon,\varepsilon'} \Delta t||\Theta^n||_0^2 + \varepsilon'\Delta t||\Theta^n||_1^2,
\end{aligned}$$

$$\begin{aligned}
|(IX)_3| &\le \Delta t\big[\, ||\nabla(I_h\psi_1^n - \psi_1^n)||_0 + ||\mathbf{A}_h^n + \mathbf{A}_c^n||_{0,4}\, ||I_h\psi_1^n - \psi_1^n||_{0,4} \,\big]\; ||\psi_{1h}^n||_{0,4}\, ||\Theta^n||_{0,4} \\
&\le C_{\varepsilon'}\Delta t\, h^{2k}||\psi_1^n||_{k+1}^2 + \varepsilon'\Delta t||\Theta^n||_1^2,
\end{aligned}$$

$$\begin{aligned}
|(IX)_4| &\le C\Delta t\big[\, ||\nabla\psi_1^n||_{0,4} + ||\mathbf{A}^n + \mathbf{A}_c^n||_{0,4}\, ||\psi_1^n||_{0,\infty}\big]\; ||\theta_1^n||_{0,4}\, ||\Theta^n||_0 \\
&\le C_\varepsilon\Delta t||\Theta^n||_0^2 + \varepsilon\Delta t||\theta_1^n||_1^2,
\end{aligned}$$

$$\begin{aligned}
|(IX)_5| &\le C\Delta t\big[\, ||\nabla\psi_1^n||_{0,4} + ||\mathbf{A}^n + \mathbf{A}_c^n||_{0,4}\, ||\psi_1^n||_{0,\infty}\big]\; ||I_h\psi_1^n - \psi_1^n||_{0,4}\, ||\Theta^n||_0 \\
&\le C\Delta t\, h^{2k}||\psi_1^n||_{k+1}^2 \; + \Delta t||\Theta^n||_0^2,
\end{aligned}$$

$$\begin{aligned}
|(IX)_6 &\le C\Delta t\, h^k||\mathbf{A}^n||_{k+1}\, ||\psi_1^n||_{0,\infty}\, ||\psi_{1h}^n||_{0,4}\, ||\Theta^n||_0 \\
&\le C\Delta t\, h^{2k}||\mathbf{A}^n||_{k+1}^2 + \Delta t||\Theta^n||_0^2,
\end{aligned}$$

$$\begin{aligned}
|(IX)_7| &\le C\Delta t||\Theta^n||_{0,4}\, ||\psi_1^n||_{0,\infty}\, ||\psi_{1h}^n||_{0,4}\, ||\Theta^n||_0 \\
&\le C_{\varepsilon'}\Delta t||\Theta^n||_0^2 + \varepsilon'\Delta t||\Theta^n||_1^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|(IX)| \le\;& 2\varepsilon\Delta t\, ||\theta_1^n||_1^2 + C_{\varepsilon,\varepsilon'}\Delta t\, ||\Theta^n||_0^2 + 3\varepsilon'\Delta t||\Theta^n||_1^2 \\
&+ C\Delta t^2 \int_{I_n} \big[\, ||\frac{\partial\psi_1}{\partial t}||_1^2 + ||\frac{\partial}{\partial t}(\mathbf{A} + \mathbf{A}_c)||_0^2 \,\big] dt \\
&+ C_{\varepsilon'}\Delta t\, h^{2k}||\psi_1^n||_{k+1}^2 + C\Delta t\, h^{2k}||\mathbf{A}^n||_{k+1}^2.
\end{aligned}$$

Similarly, we have for the term (X),

$$\begin{aligned}
|(X)| \le\;& 2\varepsilon\Delta t\, ||\theta_2^n||_1^2 + C_{\varepsilon,\varepsilon'}\Delta t\, ||\Theta^n||_0^2 + 3\varepsilon'\Delta t||\Theta^n||_1^2 \\
&+ C\Delta t^2 \int_{I_n} \big[\, ||\frac{\partial\psi_2}{\partial t}||_1^2 + ||\frac{\partial}{\partial t}(\mathbf{A} + \mathbf{A}_c)||_0^2 \,\big] dt \\
&+ C_{\varepsilon'}\Delta t\, h^{2k}||\psi_2^n||_{k+1}^2 + C\Delta t\, h^{2k}||\mathbf{A}^n||_{k+1}^2.
\end{aligned}$$

For the last term (XI), we have

$$|(XI)| \;\leq\; C_{\varepsilon'}\Delta t^2 \int_{I_n} ||\frac{\partial \mathbf{H}_e}{\partial t}||_0^2\, dt + \varepsilon' \Delta t ||\Theta^n||_1^2.$$

Combining all the estimates for Θ^n and by using the inequality (4.1), $D||\Theta^n||_1^2 \leq ||\mathrm{curl}\, \Theta^n||_0^2 + ||\mathrm{div}\, \Theta^n||_0^2$, the inequality (4.69) becomes

$$\begin{aligned}
&\sigma\frac{x_0^2}{2\lambda_1^2}\big[||\Theta^n||_0^2 - ||\Theta^{n-1}||_0^2\big] + \epsilon D\Delta t||\Theta^n||_1^2 \\
&\quad\leq C_{\varepsilon,\varepsilon'}\Delta t||\Theta^n||_0^2 + 8\varepsilon'\Delta t\,||\Theta^n||_1^2 + 2\varepsilon\Delta t||\theta_1^n||_1^2 + 2\varepsilon\Delta t||\theta_2^n||_1^2 \\
&\qquad + Ch^{2k}\int_{I_n}||\frac{\partial \mathbf{A}^n}{\partial t}||_s^2 dt + C_{\varepsilon'}\left[\Delta t^2\int_{I_n}||\frac{\partial \mathbf{A}}{\partial t}||_1^2\, dt + C\Delta t\, h^{2k}||\mathbf{A}^n||_{k+1}^2\right] \\
&\qquad + C\Delta t^2\int_{I_n}\big[\,||\frac{\partial \psi_1}{\partial t}||_1^2 + ||\frac{\partial \psi_2}{\partial t}||_1^2 + ||\frac{\partial}{\partial t}(\mathbf{A}+\mathbf{A}_c)||_0^2\,\big]dt \\
&\qquad + C_{\varepsilon'}\Delta t\, h^{2k}||\psi_1^n||_{k+1}^2 + C_{\varepsilon'}\Delta t\, h^{2k}||\psi_2^n||_{k+1}^2 + C\Delta t\, h^{2k}||\mathbf{A}^n||_{k+1}^2 \\
&\qquad + C_{\varepsilon'}\Delta t^2\int_{I_n}||\frac{\partial \mathbf{H}_e}{\partial t}||_0^2 dt,
\end{aligned} \tag{4.70}$$

where $s = 2$ if $k = 1$ and $s = k$ if $k \geq 2$.

Finally, by combining the inequalities (4.66), (4.67) and (4.70), we get

$$\begin{aligned}
&\frac{1}{2}\big[\,||\theta_1^n||_0^2 - ||\theta_1^{n-1}||_0^2\,\big] + \Delta t\frac{\xi_1^2}{x_0^2}||\theta_1^n||_1^2 + \Delta t\frac{\gamma_1}{2}\frac{\xi_1^2}{x_0^2}||\theta_1^n||_{0,\partial\Omega}^2 \\
&\quad + \frac{1}{2}\big[\,||\theta_2^n||_0^2 - ||\theta_2^{n-1}||_0^2\,\big] + \Delta t\frac{\xi_2^2}{x_0^2}||\theta_2^n||_1^2 + \Delta t\frac{\gamma_2}{2}\frac{\xi_2^2}{x_0^2}||\theta_2^n||_{0,\partial\Omega}^2 \\
&\quad + \sigma\frac{x_0^2}{2\lambda_1^2}\big[||\Theta^n||_0^2 - ||\Theta^{n-1}||_0^2\big] + \epsilon D\Delta t||\Theta^n||_1^2 \\
&\qquad \leq C_\varepsilon\Delta t||\theta_1^n||_0^2 + 10\varepsilon\Delta t||\theta_1^n||_1^2 \\
&\qquad\quad + C_\varepsilon\Delta t||\theta_2^n||_0^2 + 10\varepsilon\Delta t||\theta_2^n||_1^2 \\
&\qquad\quad + C_{\varepsilon,\varepsilon'}\Delta t||\Theta^n||_0^2 + 8\varepsilon'\Delta t\,||\Theta^n||_1^2 + (h^{2k} + \Delta t^2)\Psi^\epsilon,
\end{aligned} \tag{4.71}$$

where

$$
\begin{aligned}
\Psi^{\epsilon} \;\; = \;\; & C\int_{I_n}||\frac{\partial\psi_1}{\partial t}||_s^2 dt + C_\varepsilon\int_{I_n}\big[\,||\frac{\partial\psi_1}{\partial t}||_1^2 + ||\frac{\partial}{\partial t}(\mathbf{A}+\mathbf{A}_c)||_0^2\,\big]dt \\
& + C_\varepsilon\Delta t||\psi_1^n||_{k+1}^2 + C_\varepsilon\Delta t\, h^2||\mathbf{A}^n||_{k+1}^2 \\
& + C_\varepsilon\int_{I_n}||\frac{\partial}{\partial t}(\mathbf{A}+\mathbf{A}_c)||_0^2\,dt + C\Delta t\,||\phi_c||_{0,4}^2\,||\psi_1^n||_{k+1}^2 \\
& + C\int_{I_n}\big[\,||\phi_c||_{0,4}^2\,||\frac{\partial\psi_1}{\partial t}||_{0,4}^2 + ||\psi_1||_{0,4}^2\,||\frac{\partial\phi_c}{\partial t}||_{0,4}^2\,\big]dt \\
& + C\int_{I_n}||\frac{\partial g_1(\psi_1)}{\partial t}||_0^2\,dt + C_{\psi_1}\Delta t\, h^2||\psi_1^n||_{k+1}^2 \\
& + C\int_{I_n}||\frac{\partial\psi_2}{\partial t}||_0^2\,dt + C\Delta t\, h^2||\psi_2^n||_{k+1}^2 \\
& + C\int_{I_n}||\frac{\partial\psi_1}{\partial t}||_{0,\partial\Omega}^2\,dt + C\Delta t\, h\,||\psi_1^n||_{k+1}^2 \\
& + C\int_{I_n}||\frac{\partial\psi_2}{\partial t}||_s^2 dt + C_\varepsilon\int_{I_n}\big[\,||\frac{\partial\psi_2}{\partial t}||_k^2 + ||\frac{\partial}{\partial t}(\mathbf{A}+\mathbf{A}_c)||_0^2\,\big]dt \\
& + C_\varepsilon\Delta t||\psi_2^n||_{k+1}^2 + C_\varepsilon\Delta t\, h^2||\mathbf{A}^n||_{k+1}^2 \\
& + C_\varepsilon\int_{I_n}||\frac{\partial}{\partial t}(\mathbf{A}+\mathbf{A}_c)||_0^2\,dt + C\Delta t||\phi_c||_{0,4}^2\,||\psi_2^n||_{k+1}^2 \\
& + C\int_{I_n}\big[\,||\phi_c||_{0,4}^2\,||\frac{\partial\psi_2}{\partial t}||_{0,4}^2 + ||\psi_2||_{0,4}^2\,||\frac{\partial\phi_c}{\partial t}||_{0,4}^2\,\big]dt \\
& + C\int_{I_n}||\frac{\partial g_2(\psi_2)}{\partial t}||_0^2\,dt + C_{\psi_2}\Delta t\, h^2||\psi_2^n||_{k+1}^2 \\
& + C\int_{I_n}||\frac{\partial\psi_1}{\partial t}||_0^2\,dt + C\Delta t\, h^2||\psi_1^n||_{k+1}^2 \\
& + C\int_{I_n}||\frac{\partial\psi_2}{\partial t}||_{0,\partial\Omega}^2\,dt + C\Delta t\, h\,||\psi_2^n||_{k+1}^2 \\
& + C\int_{I_n}||\frac{\partial\mathbf{A}^n}{\partial t}||_s^2 dt + C_{\varepsilon'}\Big[\int_{I_n}||\frac{\partial\mathbf{A}}{\partial t}||_1^2\,dt + C\Delta t||\mathbf{A}^n||_{k+1}^2\Big] \\
& + C\int_{I_n}\big[\,||\frac{\partial\psi_1}{\partial t}||_1^2 + ||\frac{\partial\psi_2}{\partial t}||_1^2 + ||\frac{\partial}{\partial t}(\mathbf{A}+\mathbf{A}_c)||_0^2\,\big]dt \\
& + C_{\varepsilon'}\Delta t\,||\psi_1^n||_{k+1}^2 + C_{\varepsilon'}\Delta t\,||\psi_2^n||_{k+1}^2 + C\Delta t\,||\mathbf{A}^n||_{k+1}^2 \\
& + C_{\varepsilon'}\int_{I_n}||\frac{\partial\mathbf{H}_e}{\partial t}||_0^2 dt, \qquad (4.72)
\end{aligned}
$$

where $s = 2$ if $k = 1$ and $s = k$ if $k \geq 2$.

All the terms in Ψ^{ϵ} can be bounded by constants. For example, for $i = 1, 2$, by the

Sobolev embedding theorem,

$$\int_{I_n} \big[\, ||\phi_c||_{0,4}^2 \, ||\frac{\partial \psi_i}{\partial t}||_{0,4}^2 \leq ||\phi_c||^2_{L^\infty(0,T;L^4(\Omega))} \, ||\frac{\partial \psi_i}{\partial t}||^2_{\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega))},$$

$$\begin{aligned}\int_{I_n} ||\frac{\partial g_i(\psi_i)}{\partial t}||_0^2 dt \;&=\; \int_{I_n} ||\frac{\partial}{\partial t}(|\psi_i|^2 - \mathcal{T}_i)\psi_i||_0^2 dt \\ &\leq\; (C + ||\psi_i||^4_{\mathcal{L}^\infty(0,T;\mathcal{L}^\infty(\Omega))}) \, ||\frac{\partial \psi_i}{\partial t}||^2_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega)},\end{aligned}$$

and by the trace theorem,

$$\int_{I_n} ||\frac{\partial \psi_i}{\partial t}||^2_{0,\partial\Omega} \, dt \leq C||\frac{\partial \psi_i}{\partial t}||^2_{\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega))}.$$

By the regularity assumptions in (4.12)-(4.21), we get $\Psi^\epsilon \leq C^\epsilon$, where the constant C^ϵ is independent of Δt, h, and N but is dependent on ϵ due to the fact that we have used the stability estimates which depend on ϵ to get the generic constants appeared in the inequality (4.72), and $C_{\varepsilon'}$ depends on ϵ, see below for the choice of ε'.

Now choose ε small enough such that $\xi_1^2/x_0^2 - 10\varepsilon > 0$ and $\xi_2^2/x_0^2 - 10\varepsilon > 0$; also choose ε' small enough such that $D - 8\epsilon\varepsilon' > 0$. Next choose Δt small enough such that $1/2 - C_\varepsilon \Delta t > 0$ and $\sigma x_0^2/2\lambda_1^2 - C_{\varepsilon,\varepsilon'}\Delta t > 0$. Then inequality (4.71) becomes

$$\begin{aligned}&\big[\, c_1||\theta_1^n||_0^2 - ||\theta_1^{n-1}||_0^2 \,\big] + \Delta t c_2||\theta_1^n||_1^2 + \Delta t c_3||\theta_1^n||^2_{0,\partial\Omega} \\ &\quad + \big[\, c_4||\theta_2^n||_0^2 - ||\theta_2^{n-1}||_0^2 \,\big] + \Delta t c_5||\theta_2^n||_1^2 + \Delta t c_6||\theta_2^n||^2_{0,\partial\Omega} \\ &\quad + \big[c_7||\Theta^n||_0^2 - ||\Theta^{n-1}||_0^2\big] + \epsilon c_8 \Delta t||\Theta^n||_1^2 \\ &\quad \leq (h^{2k} + \Delta t^2)\Psi^\epsilon, \end{aligned} \tag{4.73}$$

where the constant $c_i, i = 1, \cdots, 8$ are positive and independent of ϵ, Δt, h and N.

Then by applying the discrete Gronwall inequality to the inequality (4.73), we obtain

$$\begin{aligned}&\max_{1\leq n\leq N} \big[\, ||\theta_1^n||_0^2 + ||\theta_2^n||_0^2 + ||\Theta^n||_0^2 \,\big] \\ &\quad + \sum_{n=1}^{N} \Delta t \, \big[\, ||\theta_1^n||_1^2 + ||\theta_2^n||_1^2 + +\epsilon||\Theta^n||_1^2 \,\big] + \sum_{n=1}^{N} \Delta t \big[\, ||\theta_1^n||^2_{0,\partial\Omega} + ||\theta_2^n||^2_{0,\partial\Omega} \,\big] \\ &\leq C_N^\epsilon (h^{2k} + \Delta t^2),\end{aligned}$$

where the constant C_N^ϵ is independent of Δt and h but is dependent of ϵ and N. Note that in the above estimate, we have used the fact that $||\theta_i^0||_0 = 0$ and $||\Theta^0||_0 = 0$, this is because we assumed in the discrete problem ($\mathbf{DP}^\epsilon$) that $\psi_{ih}^0 = I_h\psi_{i0} = I_h\psi_i^0$ and $\mathbf{A}_h^0 = \mathbf{I}_h\mathbf{A}_0 = \mathbf{I}_h\mathbf{A}^0$.

Now since by the finite element interpolation errors (4.2) to (4.4), we have for $i = 1, 2$ and $m = 0, 1$,

$$\begin{aligned} ||\rho_i^n||_m^2 &\leq Ch^{2(k+1-m)}||\psi_i^n||_{k+1}^2, \\ ||\Phi^n||_m^2 &\leq Ch^{2(k+1-m)}||\mathbf{A}^n||_{k+1}^2, \\ ||\rho_i^n||_{0,\partial\Omega}^2 &\leq Ch^{2(k+1-\frac{1}{2})}||\psi_i^n||_{k+1-\frac{1}{2},\partial\Omega}^2 \\ &\leq Ch^{2(k+\frac{1}{2})}||\psi_i^n||_{k+1}^2, \end{aligned}$$

where the last inequality is obtained from the trace embedding theorem. Finally, an application of the triangle inequality completes the proof. ∎

CHAPTER 5

Computational Results

In this chapter we present some computational results and investigate the properties and dynamics of the 2B-TDGL model in response to an applied magnetic field and/or an applied current under various Ginzburg-Landau (GL) parameter settings. In particular, we will focus our study on the following two-dimensional simulation topics:

1. Steady-state vortex lattices under the effect of a steady applied magnetic field. This includes cases involving samples consisting of Type-I/Type-II and Type-II/Type-II condensates, with two distinct critical temperatures.

2. Vortex dynamics under the effect of either a stationary or non-stationary applied current with or without an application of a steady applied magnetic field. A superconductor-normal metal (S-N) interface type boundary condition is used in simulations with current involved.

Our computational model is based on the 2B-TDGL equations gauged with the "current gauge" on a two-dimensional rectangular sample. Again, as in the analysis and finite element approximation, we assume that the gradient coupling effect is negligible, i.e., we set $\eta_1 = 0$. The external magnetic field is assumed to be applied in a direction perpendicular to the two-dimensional surface of the sample. In the simulation cases with an applied current, the applied current is injected in the y-direction at the two sides of the sample parallel to the x-axis. We ignore the existence of the physical current leads which should otherwise be in contact with the superconductor to feed the current. However, we include the S-N interface type boundary condition in our computational code. Our computational code is implemented by finite element methods. We use quadratic elements on a regular, uniform triangulations for approximation in space and the backward Euler discretization for approximation in time. To solve the system of nonlinear equations resulting from the discretization, we utilize Newton's linearization method on the fully discretized equations to obtain a system of linear equations which can then be solved by standard linear solvers.

5.1 Steady-State Vortices under Stationary Magnetic Field

Our main references in this section are some detailed studies done by E. Babaev in [34] and [35] (also see [36] and [39]) and by L.F. Chibotara *et al.* in [40]. Following the work in [34], two vortices generated by two distinct order parameters concentered to form one vortex core are together called a composite vortex and two vortex sublattices which consist of composite vortices are together called a composite lattice. Define a phase change quantity around a vortex core as

$$\Delta\theta := \oint_c \nabla\theta dl, \tag{5.1}$$

where c is a closed curve winding around the vortex core, and θ is the phase of the order parameter ψ. If the phase change $\Delta\theta$ around a vortex core is 2π, then the vortex carries one flux quantum. When an external magnetic field is applied to a two-band superconductor sample at low temperature, the system is in a state such that $\Delta(\theta_1 - \theta_2) = 0$ and $\Delta(\theta_1 + \theta_2) = 4\pi$, and both the sublattices generated by the two distinct order parameters form a composite lattice which is energetically favorable. The phase changes $\Delta\theta_1$ and $\Delta\theta_2$ of both order parameters winding around a composite vortex are equal to 2π, i.e., the composite vortex carries one magnetic flux quantum. This phenomenon holds in both cases of the zero and nonzero coupling parameter η, where in the former case, the only coupling between the two order parameters is through the vector potential $\mathbf{A}$. One interesting result stated in [34] is that this phenomenon also holds when one band is of Type-I condensate and the other is of Type-II condensate, even when the external magnetic field has exceeded the critical field of the Type-I condensate. However, Babaev's studies are based on the assumption that the sample is of infinite dimension. For a finite dimension sample in which non-negligible boundary effects must be taken into account, the existence of thermodynamically stable noncomposite vortices in small samples with size of the order of the coherent length ξ is studied by L.F. Chibotara *et al.* in [40].

We will present the results of many simulations and show that the composite and noncomposite lattice phenomena mentioned above appear only with special combinations of values of the coupling parameter η and the applied field H_e; different phenomena appear in other combinations of the values of η and H_e. In the dimensional two-band TDGL free energy functional (2.3), the dimensional interband coupling energy term $\mathcal{F}_{int} := \int_\Omega \epsilon(\psi_1\psi_2^* + \psi_1^*\psi_2)d\Omega$ can be rewritten as $\mathcal{F}_{int} = \int_\Omega \epsilon|\psi_1|^2\,|\psi_2|^2\cos(\theta_1 - \theta_2)d\Omega$. Observe that in the case of $\epsilon \neq 0$, in order to perform an evolutional simulation with a stable initial state, we need to set the initial condition of ψ_1 and ψ_2 such that the interband energy term $\mathcal{F}_{int}$ is minimized. Therefore, if $\epsilon > 0$ (or $\eta > 0$ for nondimensionalized equations), we set the phase difference $\theta_1 - \theta_2 = \pi$, and if $\epsilon < 0$ (or $\eta < 0$), we set $\theta_1 - \theta_2 = 0$. In our simulations, we set $\psi_{r1} = 0.8|\mathcal{T}_1|$, $\psi_{i1} = 0.6|\mathcal{T}_1|$ and $\psi_{r2} = -0.8\,\mathrm{sign}(\eta)|\mathcal{T}_2|$, $\psi_{i2} = -0.6\,\mathrm{sign}(\eta)|\mathcal{T}_2|$, where ψ_{rj} and ψ_{ij} are the real and the imaginary parts of the order parameter ψ_j, respectively. The values of the real and imaginary parts of the order parameters are chosen to make the simulations start with a near superconducting state. As we have just mentioned, when $\epsilon = 0$ (or $\eta = 0$), the only coupling between the two bands is through the same magnetic field induced by the vector potential $\mathbf{A}$.

We use the two upper critical fields of the two individual bands together as reference values to set the values of the applied field in our numerical simulations. We first derive an equation for the dimensionless upper critical field H_{c2}^j, where $j = 1, 2$, for band one, band two. From the conventional one-band GL theory we know that the dimensional thermodynamic critical field H_c^j at temperature $\mathcal{T}$ close to and below $\mathcal{T}_c$ (see section 1.2.4, also see, e.g., [62]), is given as

$$H_c^j(\mathcal{T}) = \frac{\Phi_0}{2\sqrt{2}\pi\lambda_j(\mathcal{T})\xi_j(\mathcal{T})}, \tag{5.2}$$

where $\Phi_0 = 2\pi\hbar c/|e^*|$. By using the approximation $\alpha_j(\mathcal{T}) = \alpha_j(0)[\mathcal{T}/\mathcal{T}_c - 1]$ and $\beta_j(\mathcal{T}) = \beta_j(0)$ which gives $\lambda_j(\mathcal{T}) = \lambda_j(0)/[1 - \mathcal{T}/\mathcal{T}_{cj}]^{1/2}$ and $\xi_j(\mathcal{T}) = \xi_j(0)/[1 - \mathcal{T}/\mathcal{T}_{cj}]^{1/2}$ (see (2.16)), we get

$$H_c^j(\mathcal{T}) = H_c^j(0)\left[1 - \frac{\mathcal{T}}{\mathcal{T}_{cj}}\right], \tag{5.3}$$

where

$$H_c^j(0) = \frac{\Phi_0}{2\sqrt{2}\pi\lambda_j(0)\xi_j(0)}. \tag{5.4}$$

Again from the one-band GL theory the dimensional critical field H_{c2}^j is given by

$$H_{c2}^j(\mathcal{T}) = \frac{\Phi_0}{2\pi\xi_j^2(\mathcal{T})} = \sqrt{2}\kappa_j H_c^j(\mathcal{T}) = \sqrt{2}\kappa_j H_c^j(0)\left[1 - \frac{\mathcal{T}}{\mathcal{T}_{cj}}\right]. \tag{5.5}$$

By using the nondimensionalization equation for the magnetic field defined in (2.16), namely, $H_e = \sqrt{2}H_c^1(0)H_e'$, where H_e' is the magnitude of the nondimensionalized field, we obtain the dimensionless field (with $'$ dropped)

$$H_{c2}^1(\mathcal{T}) = \kappa_1\left[1 - \frac{\mathcal{T}}{\mathcal{T}_{c1}}\right] = \kappa_1\mathcal{T}_1, \tag{5.6}$$

$$H_{c2}^2(\mathcal{T}) = \kappa_2\frac{H_c^2(0)}{H_c^1(0)}\left[1 - \frac{\mathcal{T}}{\mathcal{T}_{c2}}\right] = \frac{\lambda_2(0)\xi_2(0)}{\lambda_1(0)\xi_1(0)}\kappa_2\mathcal{T}_2 = \nu\kappa_2\mathcal{T}_2. \tag{5.7}$$

Remark All of the above results are based on a nondimensionalization of the 2B-TDGL equations by using $\alpha_j(0)$ and $\beta_j(0)$, $j = 1, 2$, as the base parameters. When doing so, all the dimensionless paramters appear in the nondimensionalized 2B-TDGL equations are referred at $\mathcal{T} = 0.0$; we obtain the temperature dependent terms $(|\psi_j|^2 - \mathcal{T}_j)\psi_j$ in the nondimensionalized 2B-TDGL equations; and the upper critical fields are expressed as in (5.6) and (5.7). In view of this nondimensionalization, $(|\psi_j|^2 - 1.0)\psi_j$ will mean that the operating temperature is equal to $\mathcal{T} = 0.0$. On the other hand, if we use $\alpha_j(\mathcal{T})$ and $\beta_j(\mathcal{T})$ as the base parameters to nondimensionalize the 2B-TDGL equations, all the parameters appear in the nondimensionalized 2B-TDGL equations are now referred at $\mathcal{T}$ equals to the operating temperature. The temperature dependent terms in the nondimensionalized 2B-TDGL equations now become $(|\psi_j|^2 - 1.0)\psi_j$ and the upper critical fields (5.6) and (5.7) become

$$H_{c2}^1(\mathcal{T}) = \kappa_1, \tag{5.8}$$

$$H_{c2}^2(\mathcal{T}) = \kappa_2\frac{H_c^2(\mathcal{T})}{H_c^1(\mathcal{T})} = \nu\kappa_2, \tag{5.9}$$

where now $\nu = (\lambda_2(\mathcal{T})\xi_2(\mathcal{T}))/(\lambda_1(\mathcal{T})\xi_1(\mathcal{T}))$. Note that we don't have to distinguish $\kappa_j(0)$ from $\kappa_j(\mathcal{T})$, since they are the same by our approximation $\beta(0) = \beta(\mathcal{T})$. So when we set $\mathcal{T}_j = 1.0$, we could mean $\mathcal{T} = 0.0$ or $\mathcal{T}$ equals to whatever the operating temperature is, depending on what nondimensionalization we intend to use.

Now we are going to investigate in details our simulation results of the 2B-TDGL model. From an observation on a large amount of simulation results that we obtained, we are able to speculate that there exists a vortex phase diagram, as shown in Figure 1 below, which governs how the vortex phase of each band behaves under various combinations of the values of the coupling parameter η and the applied field H_e with a fixed operating temperature $\mathcal{T}$. After a discussion of the meaning of this phase diagram, we will present some representative numerical examples to illustrate how the vortex phases change in the regions on a $\eta - H_e$ plane.

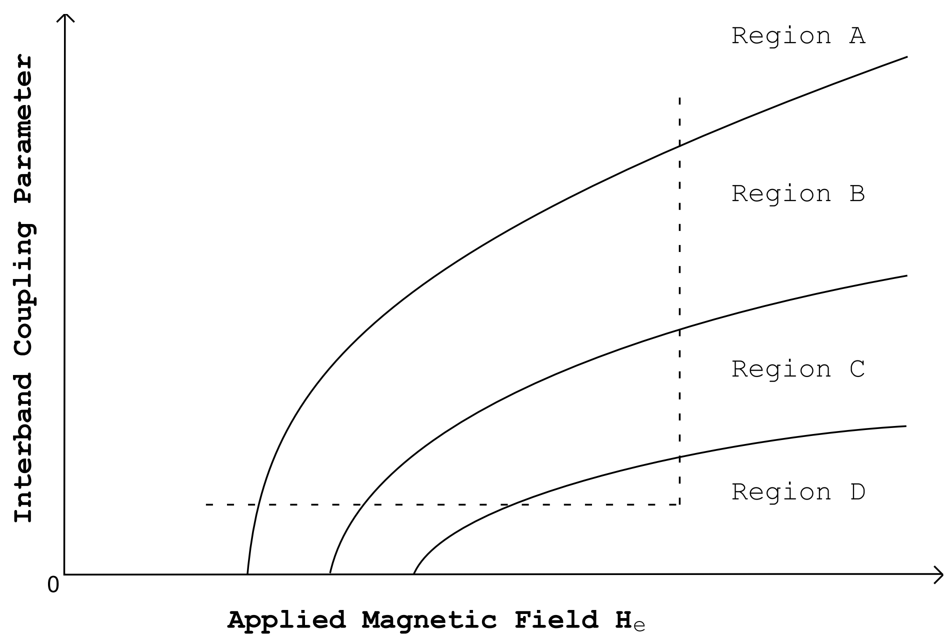

Figure 5.1: Vortex Phase Diagram (Coupling Parameter η vs. Applied Field H_e) for the 2B-TDGL model under fixed temperature $\mathcal{T}$ and GL parameters.

In Figure 5.1, the Region A, B, C and D are regions on the η-H_e plane. Region A is a region in which the values of η and H_e together give no vortex nucleation inside the two-band superconductor. Region B is a region in which the values of η and H_e together cause the superconductor to produce composite vortices which can be comprised of a set of strong vortices corresponding to one band (called the strong band) and a set of strong or weak vortices corresponding to the other band (called the weak band in case of weak vortices). Region C is a zone in which η and H_e together cause the superconductor to produce noncomposite vortices which can be comprised of a set of strong vortices corresponding to the strong band and a set of strong or weak vortices corresponding to the other band. The two noncomposite sublattices, each corresponding to one particular band, may have the same topological structure despite the fact that the vortices are spatially not concentric, or they may have two totally different topological structures in which the two set of vortices from both bands are distributed evenly in some sense over the spatial domain of the sample. When two noncomposite sublattices with different topological structure are formed in steady state, we discovered that the vortices of the weak band (with weak vortices) always go through a series of evolutional transitions- as time elapsing, they first form composite vortices but then shrink or merge to form blocks of large normal region (called giant vortices) and finally they split to form a noncomposite sublattice which has a totally different topological structure from that of the stronger band at steady state. Lastly, Region D is a zone in which the values of η and H_e together cause at least one band to reside in the normal state over the whole spatial domain or form some large blocks of normal region (giant vortices) in the domain. We want to stress that our numerical data show that Region C only happens with relatively small or null coupling effects, i.e., noncomposite vortices only happen when η is relatively small or equal to zero. The parabolic curves that define the boundaries of the regions are used to convey the conceptual idea of the existence of the regions, their actual shapes are not known without additional numerical results.

Suppose we perform a series of simulations on a sample of fixed size at fixed temperature by varying the values of η and H_e along the vertical dashed-line shown in Figure 5.1, i.e., by keeping the applied field H_e fixed while changing the value of η. Starting with the values in Region D, we will obtain a vortex phase in which one band, say band one, with its domain in whole or in part resides in the normal state or with giant vortices, while band two is either in the same phase as band one or in a phase with or without vortices. When we increase the value of η to Region C, we will obtain a result with noncomposite vortices. When we increase the value of η to Region B, we will obtain a result with composite vortices. However, when we further increase the value of η, we will obtain a vortex phase with no vortices in both bands in the bulk, i.e., a strong enough coupling suppresses vortex nucleation. However, this last result is not too exciting, as we will see later that a sample with the same characteristic, i.e., with the same GL parameters and the same η, but of larger size will behave in a vortex phase in a region other than Region A under the same applied field H_e. We will obtain the same results if now we vary the values of η and H_e along the horizontal dashed-line as shown in Figure 5.1, i.e., by changing the value of H_e while keeping the value of η fixed. We want to stress that there is no clean cutoff boundaries between the regions- there are results that seem to fit into two adjacent regions. Also, the area of a region, particular that of Region B, Region C or Region D, can be diminished or even disappear with some combinations of η and H_e for some superconductors.

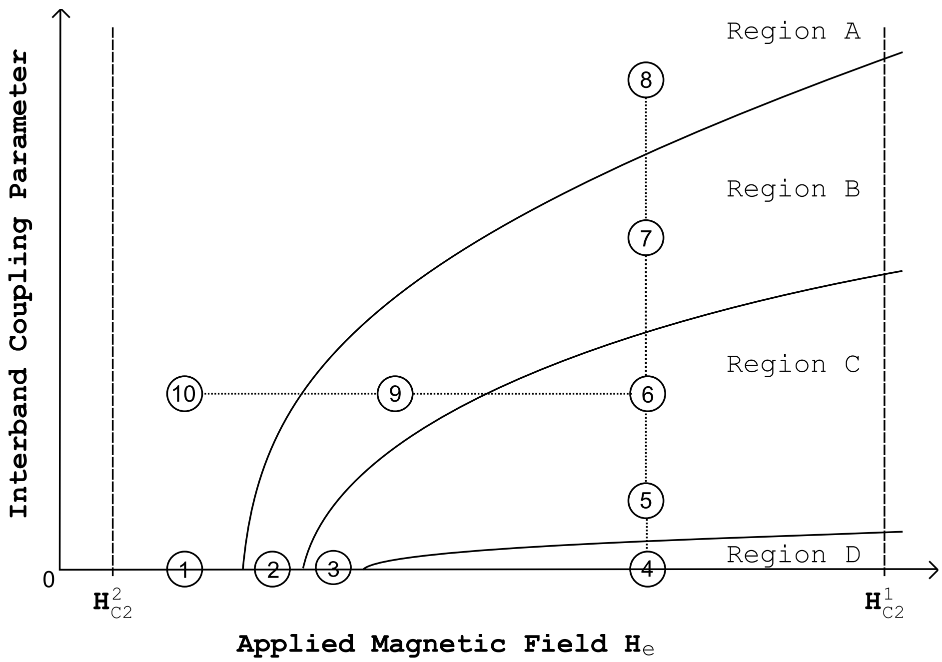

Figure 5.2: Example-set 1, Vortex Phase Diagram with $H_{c2}^2 << H_e < H_{c2}^1$.

Example-Set 1 (Figure 5.2)

Our first set of numerical examples use the following parameters:

$$\lambda_1(0) = 0.6,\ \lambda_2(0) = 0.2,\ \xi_1(0) = 0.05,\ \xi_2(0) = 0.1,\ \gamma_1 = \gamma_2 = 0.0,\ \mathcal{T}_1 = 0.7,\ \mathcal{T}_2 = 0.2$$

, and size of sample $= 20\,\xi_1(0) \times 20\,\xi_1(0)$, mesh $= 1.95$ points per $\xi_1(0)$. This gives

$$\nu = 0.67,\ \kappa_1 = 12.0,\ \kappa_2 = 2.0,\ H_{c2}^1 = 8.4,\ H_{c2}^2 = 0.27,$$

where the upper critical fields are calculated according to equations (5.6)-(5.7). Note that we are using a Type-II/Type-II superconductor here and we do not use S-N interface type boundary conditions in the simulations of this example set, i.e., we set $\gamma_1 = \gamma_2 = 0.0$ (see

(2.88)-(2.89) as boundary conditions and (4.24)-(4.25) in weak form). The applied field we are going to use varies in values that are much greater that the upper critical field of the second band, i.e., H_{c2}^2 but always smaller that H_{c2}^1, see Figure 5.2 above.

The purpose of the arrangement of this set of examples, namely with $H_{c2}^2 << H_e < H_{c2}^1$, is to demonstrate how the vortices of a strong band in a sample with strong coupling can induce a formation of vortices, either composite or noncomposite, in a weak band which supposedly to be in the normal state in a one band setting due to $H_{c2}^2 << H_e$. Each circled number in Figure 5.2 represents the example named with that number.

Example 1.1. (see Figure 5.3 and Figure 5.4) With $\eta = 0.0$ and $H_e = 0.5$. No vortex nucleation in the bulk, so the vortex phase is in Region A.

Example 1.2. We have not found any example in the Region B with $\eta = 0$. This would mean that this region is very narrow, or even may not exist at all on the $\eta = 0$ line.

Example 1.3. We have not found any example in the Region C with $\eta = 0$. This would mean that this region is very narrow, or even may not exist at all on the $\eta = 0$ line.

Example 1.4. (see Figure 5.5 and Figure 5.6) With $\eta = 0.0$ and $H_e = 1.6$. The vortex phase is in Region D.

Example 1.5. (see Figure 5.7 and Figure 5.8) With $\eta = 0.000005$ and $H_e = 1.6$. These two set of vortices are not concentric and band 2 has very weak vortices with $|\psi_2| \leq 0.00003$, i.e., the vortex phase is weakly noncomposite in Region C. Notice that η is very small, this shows that the Region D would be a very thin phase region.

Example 1.6. (see Figure 5.9 and Figure 5.10) With $\eta = 0.1$ and $H_e = 1.6$. These two set of vortices are not concentric, i.e., the vortex phase is noncomposite in Region C.

Example 1.7. (see Figure 5.11 and Figure 5.12) With $\eta = 0.8$ and $H_e = 1.6$. These two set of vortices are concentric, i.e., the vortex phase is composite in Region B.

Example 1.8. (see Figure 5.13 and Figure 5.14) With $\eta = 1.0$ and $H_e = 1.6$. The vortex phase is in Region A.

Example 1.9. We have not found any example in the Region B with $\eta = 0.1$. This would mean that this region is very narrow, or even may not exist at all on the $\eta = 0.1$ line.

Example 1.10. (see Figure 5.15 and Figure 5.16) With $\eta = 0.1$ and $H_e = 0.6$. The vortex phase is in Region A.

All the vortex phases depicted in the above examples occur only in two-band superconductors. It would be interesting to see what vortex phase would be generated by a corresponding one-band superconductor with the same material characteristic (i.e., with the same λ, ξ and $\mathcal{T}_c$) as that of one particular band of a two-band superconductor, under the same operating temperature and applied field. The following two numerical examples use the same material parameters of one of the band as in Example 1.1 to Example 1.10 but now they are simulated with a one-band TDGL code.

Example 1.11. (see Figure 5.17) This example use the following parameters

$$\lambda(0) = 0.6,\ \xi(0) = 0.05,\ \mathcal{T} = 0.7$$

This gives $\kappa = 12.0,\ H_{c2} = 8.4$. Size of sample $= 20\,\xi(0) \times 20\,\xi(0)$, mesh $= 1.95$ points per $\xi(0)$. Applied field $H_e = 0.6$. This one band parameter setting is exactly the same as the parameter setting for the band one in the two-band superconductors we used in Example 1.1 to Example 1.10.

Example 1.12. (see Figure 5.18) This example use the following parameters

$$\lambda(0) = 0.2,\ \xi(0) = 0.1,\ \mathcal{T} = 0.2$$

This gives $\kappa = 2.0,\ H_{c2} = 0.27$. Size of sample and applied field are the same as that in Example 1.11. This one band parameter setting is exactly the same as the parameter setting for the band two in the two-band superconductors we used in Example 1.1 to Example 1.10.

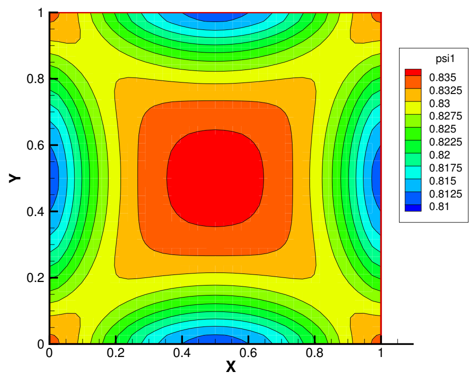

Figure 5.3: Example 1.1: ψ_1, $\eta = 0.0$, $H_e = 0.5$

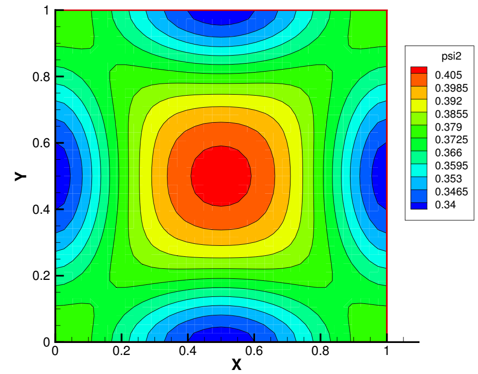

Figure 5.4: Example 1.1: ψ_2, $\eta = 0.0$, $H_e = 0.5$

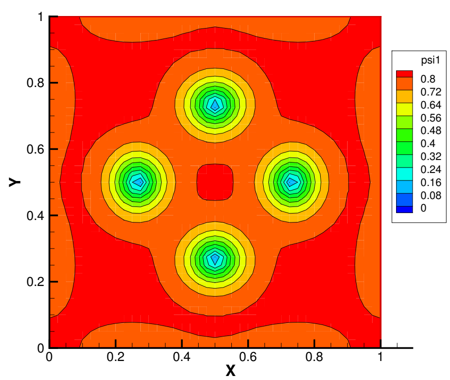

Figure 5.5: Example 1.4: ψ_1, $\eta = 0.0$, $H_e = 1.6$

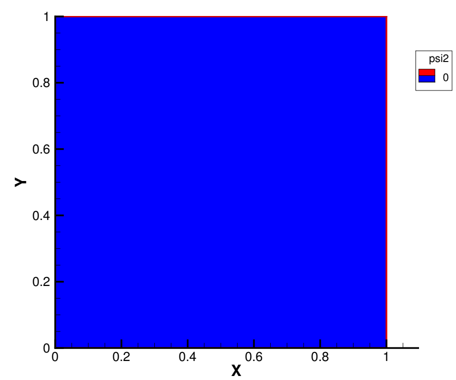

Figure 5.6: Example 1.4: ψ_2, $\eta = 0.0$, $H_e = 1.6$

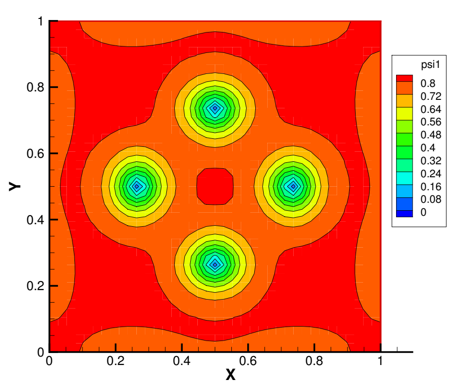

Figure 5.7: Example 1.5: ψ_1, $\eta = 0.1$, $H_e = 1.6$

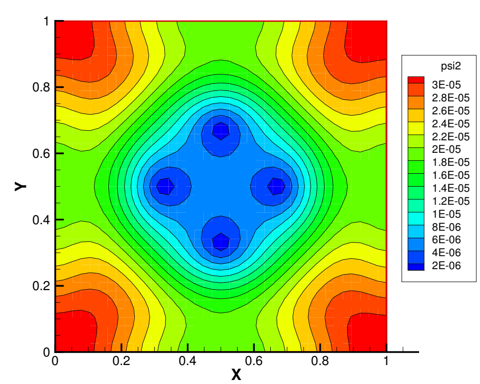

Figure 5.8: Example 1.5: ψ_2, $\eta = 0.1$, $H_e = 1.6$

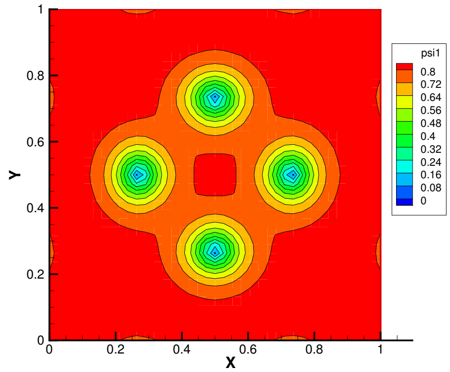

Figure 5.9: Example 1.6: ψ_1, $\eta = 0.1$, $H_e = 1.6$

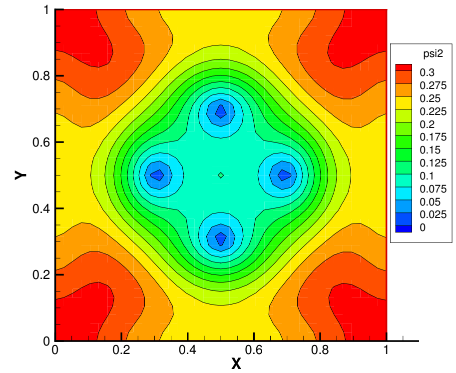

Figure 5.10: Example 1.6: ψ_2, $\eta = 0.1$, $H_e = 1.6$

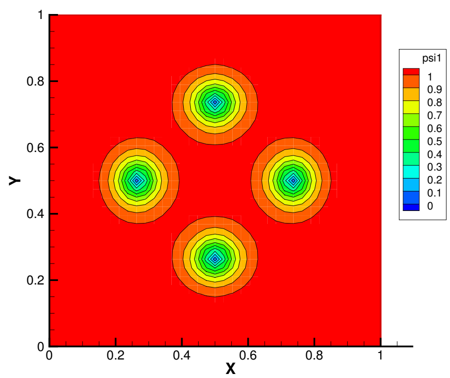

Figure 5.11: Example 1.7: ψ_1, $\eta = 0.8$, $H_e = 1.6$

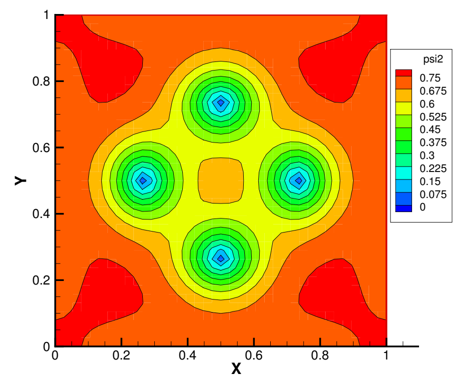

Figure 5.12: Example 1.7: ψ_2, $\eta = 0.8$, $H_e = 1.6$

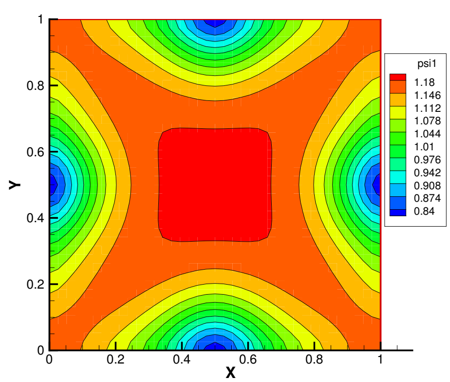

Figure 5.13: Example 1.8: ψ_1, $\eta = 1.0$, $H_e = 1.6$

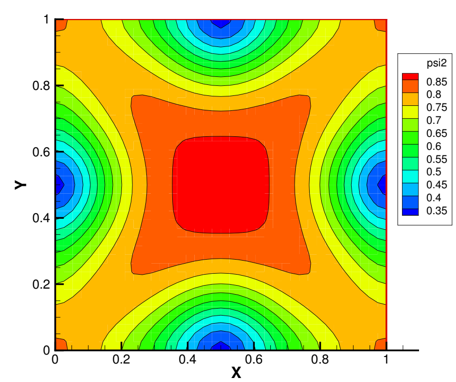

Figure 5.14: Example 1.8: ψ_2, $\eta = 1.0$, $H_e = 1.6$

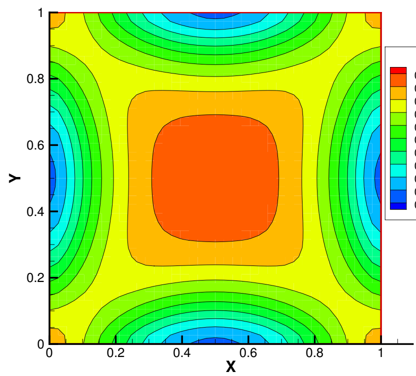

Figure 5.15: Example 1.10: ψ_1, $\eta = 0.1$, $H_e = 0.6$

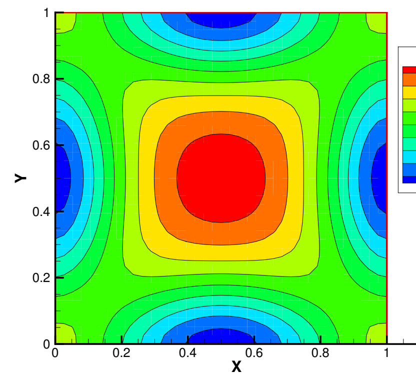

Figure 5.16: Example 1.10: ψ_2, $\eta = 0.1$, $H_e = 0.6$

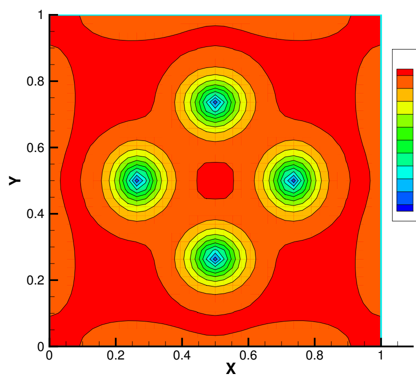

Figure 5.17: Example 1.11: ψ, $\lambda = 0.6$, $\xi = 0.05$, $\mathcal{T} = 0.7$, $H_e = 1.6$.

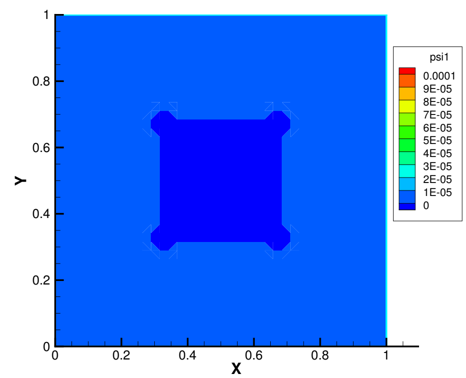

Figure 5.18: Example 1.12: ψ, $\lambda = 0.2$, $\xi = 0.1$, $\mathcal{T} = 0.2$, $H_e = 1.6$.

Note that due to $H_{c2} << H_e$, the one-band superconductor is in the normal state.

In the above Example-set 1, we observe that with no coupling, i.e., $\eta = 0$, the coupling through the vector potential $\mathbf{A}$ alone does not seem to cause the strong band one which has already generated vortices to induce vortices, either composite or noncomposite, in the weak band two. In other words, there is no Region B and Region C on the $\eta = 0$ line (the x-axis) in our Example-set 1. We want to ask if this is generally true. In Example-set 3 below, we will show that there are vortex phases occurring in Region B and Region C on the $\eta = 0$ line under different parameter settings. On the other hand, we also observe that with strong enough coupling η, the vortices of the strong band one can induce the band two which supposedly to be in the normal state in a one band setting due to $H_{c2}^2 << H_e$ as shown in Example 1.12, to produce vortices, either composite as shown in Example 1.7 or noncomposite as shown in Example 1.5 and Example 1.6. We suspect whether this same situation will happen in an analogous case where the band two is a type-I band with $H_c^2 << H_e$. The following Example-set 2 will show that indeed this is true.

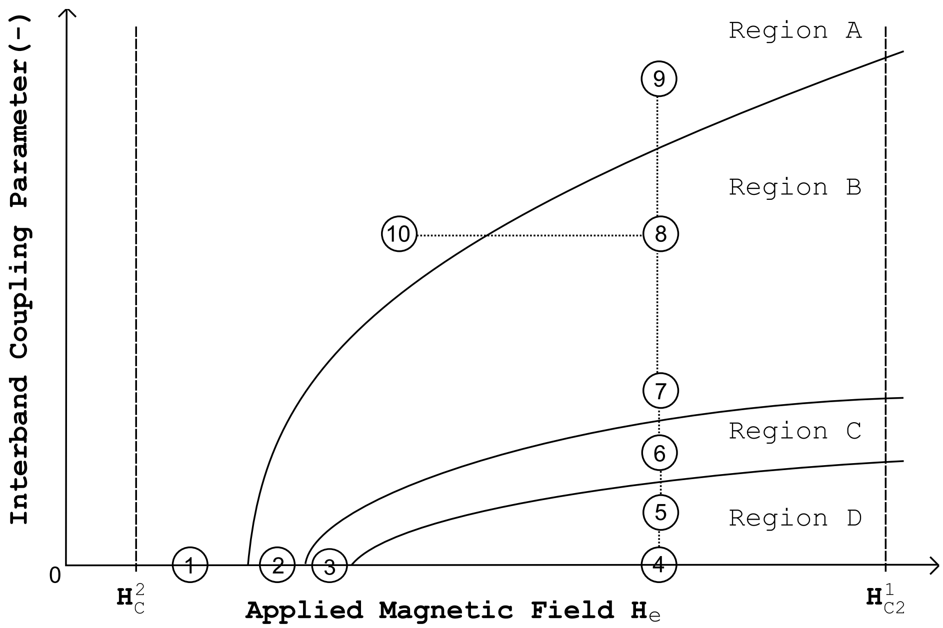

Figure 5.19: Example-set 2, Vortex Phase Diagram with $H_c^2 << H_e < H_{c2}^1$. Note that the η axis is in negative values.

Example-Set 2 (Figure 5.19)

Our second set of numerical examples use the following parameters:

$$\lambda_1(0) = 0.3,\ \lambda_2(0) = 0.06,\ \xi_1(0) = 0.05,\ \xi_2(0) = 0.1,\ \gamma_1 = \gamma_2 = 0.0,\ \mathcal{T}_1 = 0.7,\ \mathcal{T}_2 = 0.2,$$

and size of sample $= 20\,\xi_1(0) \times 20\,\xi_1(0)$, mesh $= 1.95$ points per $\xi_1(0)$. This gives

$$\nu = 0.4,\ \kappa_1 = 6.0,\ \kappa_2 = 0.6,\ H_{c2}^1 = 4.2,\ H_{c2}^2 = 0.048.$$

Note that we are working on a Type-II/Type-I superconductor here and we do not use S-N interface type boundary conditions in the simulations of this example set. Here $H_{c2}^2 = 0.048$

is used as a reference for H_c^2 which is greater that H_{c2}^2 in a Type-I superconductor setting. The applied field we are going to use varies in values that are much greater that the upper critical field of the Type-I band two, i.e., H_{c2}^2, but always smaller that H_{c2}^1, see Figure 5.19 above.

Example 2.1. (see Figure 5.21 and Figure 5.22) With $\eta = 0.0$ and $H_e = 0.5$. The vortex phase is in Region A.

Example 2.2. We have not found any example in the Region B with $\eta = 0$. This would mean that this region is very narrow, or even may not exist at all on the $\eta = 0.0$ line.

Example 2.3. We have not found any example in the Region C with $\eta = 0$. This would mean that this region is very narrow, or even may not exist at all on the $\eta = 0.0$ line.

Example 2.4. (see Figure 5.23 and Figure 5.24) With $\eta = 0.0$ and $H_e = 1.6$. The vortex phase is in Region D.

Example 2.5. (see Figure 5.25 and Figure 5.26) With $\eta = -0.00001$ and $H_e = 1.6$. The vortex phase is in Region D.

Example 2.6. We have not found any example in the Region C with $H_e = 1.6$. This would mean that this region is very narrow, or even may not exist at all on the $H_e = 1.6$ line. Note that in Example 2.5, with $H_e = 1.6$, $\eta = -0.00001$ is in Region D; and in Example 2.7 below, $\eta = -0.0005$ is in Region B.

Example 2.7. (see Figure 5.27 and Figure 5.28) With $\eta = -0.0005$ and $H_e = 1.6$. Note that band 2 has very weak vortices, with $|\psi_2| \leq 0.0002$. The two set of vortices are weakly composite in Region B.

Example 2.8. (see Figure 5.29 and Figure 5.30) With $\eta = -0.8$ and $H_e = 1.6$. The two set of vortices are composite in Region B.

Example 2.9. (see Figure 5.31 and Figure 5.32) With $\eta = -1.2$ and $H_e = 1.6$. The vortex phase is in Region A.

Example 2.10. (see Figure 5.33 and Figure 5.34) With $\eta = -0.8$ and $H_e = 1.35$. The vortex phase is in Region A.

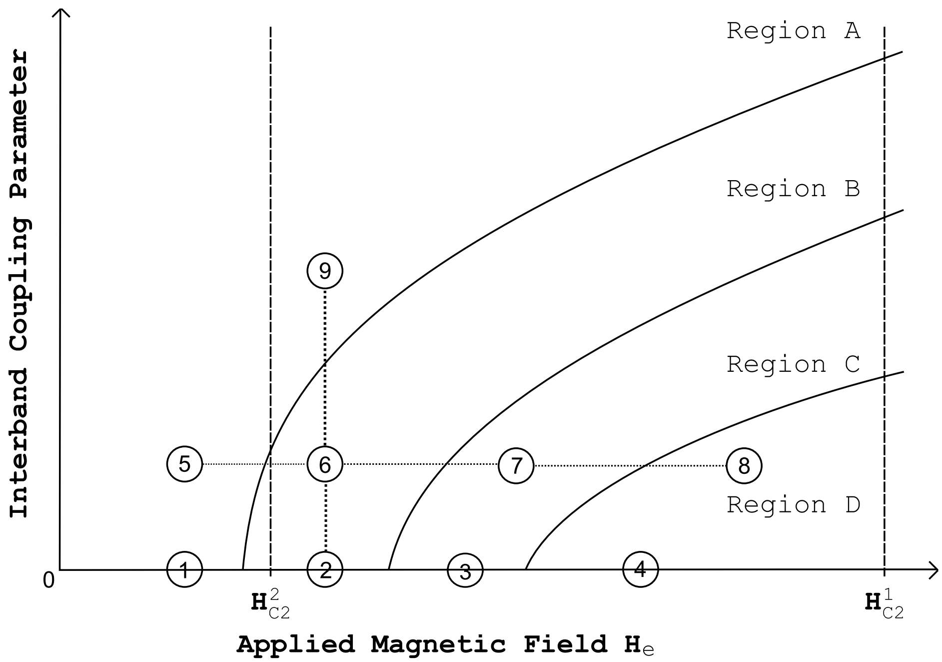

Figure 5.20: Example-set 3, Vortex Phase Diagram with $H^2_{c2} \approx H_e < H^1_{c2}$.

Example-Set 3 (Figure 5.20)

Our third set of numerical examples use the following parameters:

$$\lambda_1(\mathcal{T}) = 0.6,\ \lambda_2(\mathcal{T}) = 0.2,\ \xi_1(\mathcal{T}) = 0.05,\ \xi_2(\mathcal{T}) = 0.1,\ \gamma_1 = \gamma_2 = 0.0,\ \mathcal{T}_1 = 1.0,\ \mathcal{T}_2 = 1.0,$$

and size of sample = $20\,\xi_1(\mathcal{T}) \times 20\,\xi_1(\mathcal{T})$, mesh = 1.95 points per $\xi_1(\mathcal{T})$. This gives

$$\nu = 0.67,\ \kappa_1 = 12.0,\ \kappa_2 = 2.0,\ H^1_{c2} = 12.0,\ H^2_{c2} = 1.33.$$

Note that we are working on a Type-II/Type-II sample here and as before, we do not use S-N interface boundary conditions in this example set. As we mentioned before, we can interpret $\mathcal{T}_j = 1.0$ in either way- with $\mathcal{T} = 0.0$ or with nondimensionalization using $\alpha_j(\mathcal{T})$

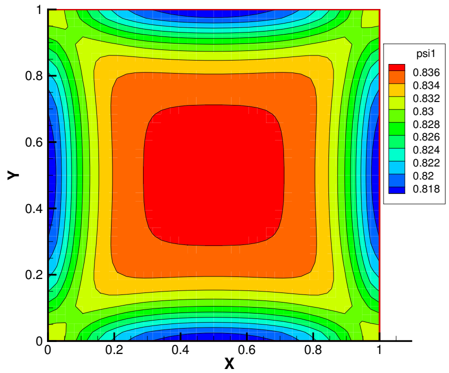

Figure 5.21: Example 2.1: ψ_1, $\eta = 0.0$, $H_e = 0.5$

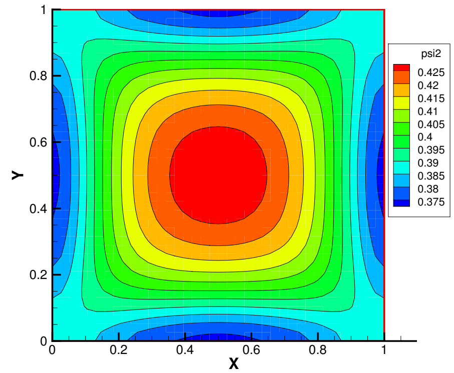

Figure 5.22: Example 2.1: ψ_2, $\eta = 0.0$, $H_e = 0.5$

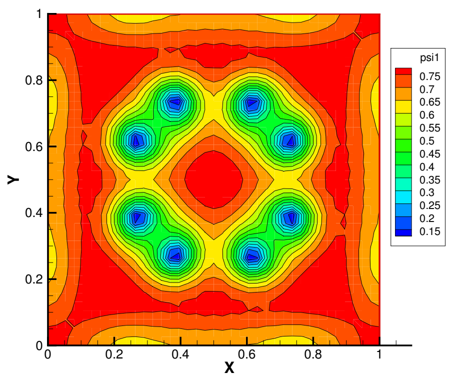

Figure 5.23: Example 2.4: ψ_1, $\eta = 0.0$, $H_e = 1.6$

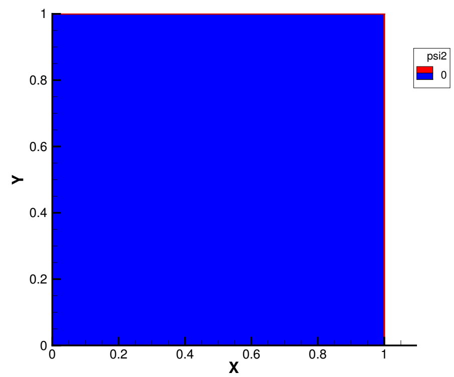

Figure 5.24: Example 2.4: ψ_2, $\eta = 0.0$, $H_e = 1.6$

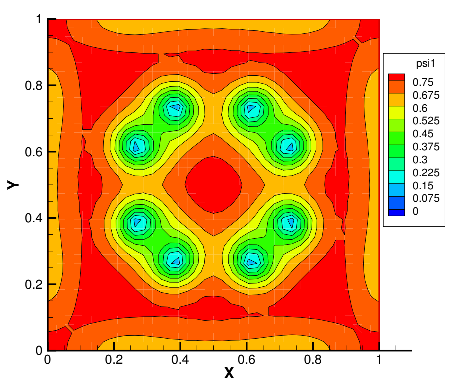

Figure 5.25: Example 2.5: ψ_1, $\eta = -0.00001$, $H_e = 1.6$

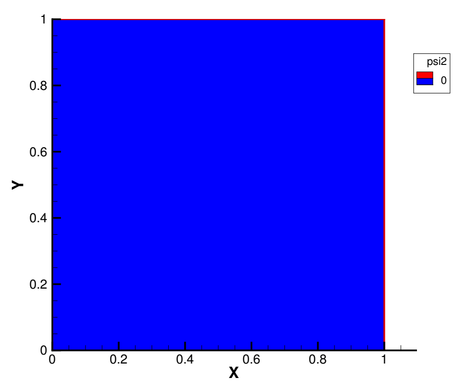

Figure 5.26: Example 2.5: ψ_2, $\eta = -0.00001$, $H_e = 1.6$

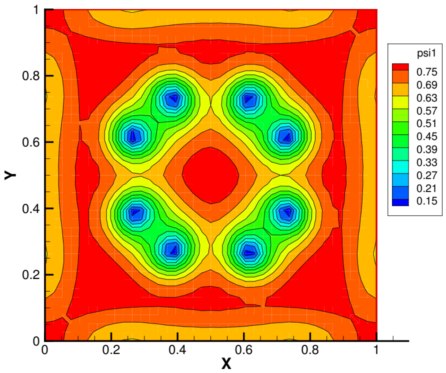

Figure 5.27: Example 2.7: ψ_1, $\eta = -0.0005$, $H_e = 1.6$

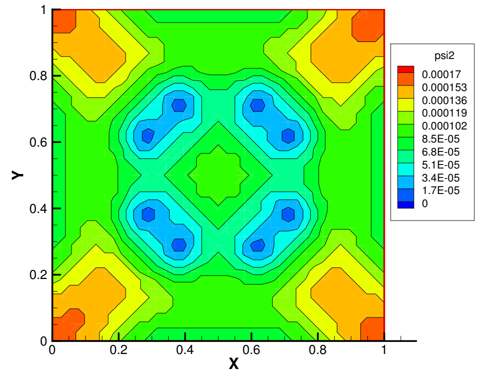

Figure 5.28: Example 2.7: ψ_2, $\eta = -0.0005$, $H_e = 1.6$

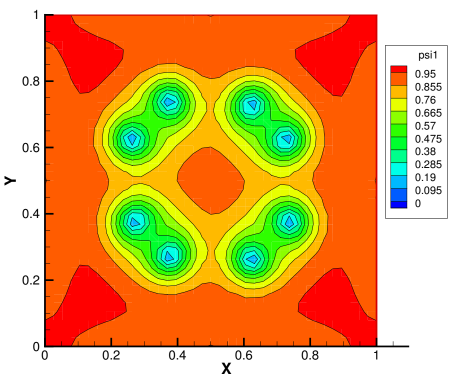

Figure 5.29: Example 2.8: ψ_1, $\eta = -0.8$, $H_e = 1.6$

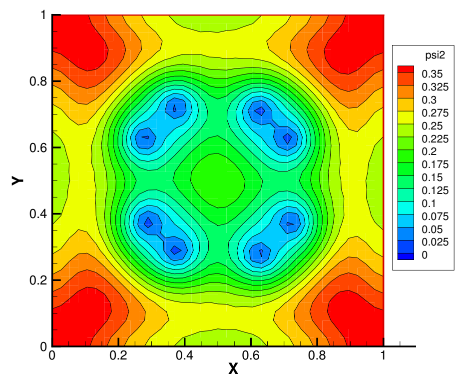

Figure 5.30: Example 2.8: ψ_2, $\eta = -0.8$, $H_e = 1.6$

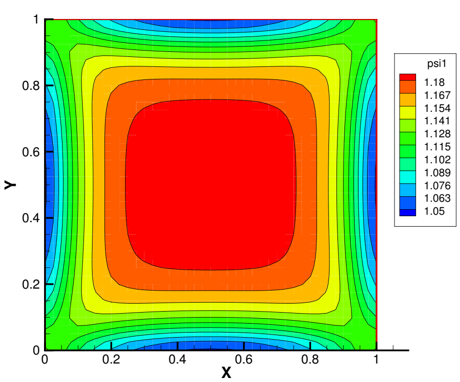

Figure 5.31: Example 2.9: ψ_1, $\eta = -1.2$, $H_e = 1.6$

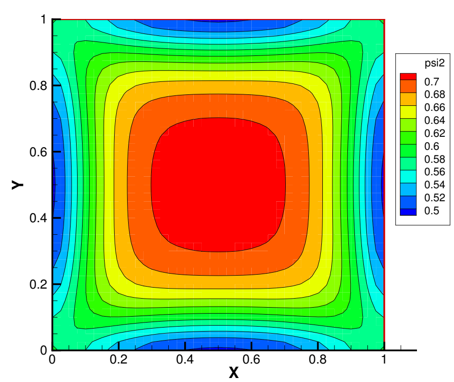

Figure 5.32: Example 2.9: ψ_2, $\eta = -1.2$, $H_e = 1.6$

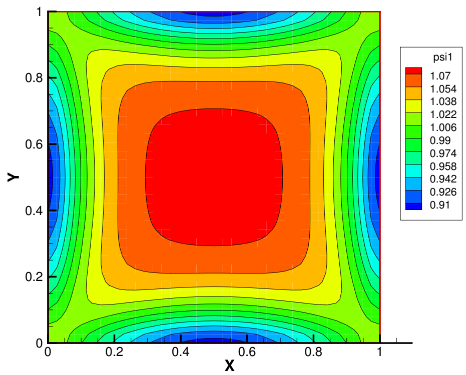

Figure 5.33: Example 2.10: ψ_1, $\eta = -0.8$, $H_e = 1.35$

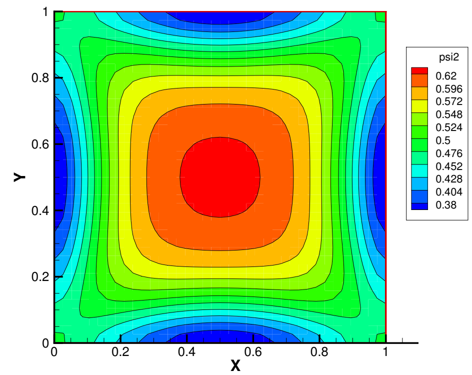

Figure 5.34: Example 2.10: ψ_2, $\eta = -0.8$, $H_e = 1.35$

and $\beta_j(\mathcal{T})$ as base parameters, in this case $\mathcal{T}$ is equals to whatever the operating temperature is. In the latter case, the second critical fields are calculated according to equations (5.8)-(5.9). The applied field we are going to use varies in values around H_{c2}^2 but always smaller that H_{c2}^1, see Figure 5.20 above.

Example 3.1. (see Figure 5.35 and Figure 5.36) With $\eta = 0.0$ and $H_e = 1.0$. The vortex phase is in Region A.

Example 3.2. (see Figure 5.37 and Figure 5.38) With $\eta = 0.0$ and $H_e = 1.6$. The two set of vortices are composite in Region B.

Example 3.3. (see Figure 5.39 and Figure 5.40) With $\eta = 0.0$ and $H_e = 3.0$. These two vortex sublattices have completely different topologies, so the vortex phase is noncomposite in Region C.

Example 3.4. (see Figure 5.41 and Figure 5.42) With $\eta = 0.0$ and $H_e = 5.0$. The vortex phase is in Region D.

Example 3.5. (see Figure 5.43 and Figure 5.44) With $\eta = 0.05$ and $H_e = 1.0$. The vortex phase is in Region A.

Example 3.6. (see Figure 5.45 and Figure 5.46) With $\eta = 0.05$ and $H_e = 1.6$. The two set of vortices are composite in Region B.

Example 3.7. (see Figure 5.47 and Figure 5.48) With $\eta = 0.05$ and $H_e = 2.5$. These two set of vortices are not concentric, i.e., the vortex phase is noncomposite in Region C.

Example 3.8. (see Figure 5.49 and Figure 5.50) With $\eta = 0.05$ and $H_e = 4.0$. The vortex phase is in Region D.

Example 3.9. (see Figure 5.51 and Figure 5.52) With $\eta = 0.08$ and $H_e = 1.6$. The vortex phase is in Region A.

We observe that in Example 3.9 (also in Example 1.8 and Example 2.9) that a strong coupling η inhibits the generation of vortices in both bands in a $20\,\xi_1(\mathcal{T}) \times 20\,\xi_1(\mathcal{T})$ sample. However, if we increase the size of the sample, keeping every other parameters fixed, the new sample will response in a vortex phase other than the no-vortices phase Region A. The following example demonstrates this phenomenon.

Example 4. (see Figure 5.53 and Figure 5.54) All parameters are the same as those in Example 3.9, except now the sample's size is increased to $30\,\xi_1(\mathcal{T}) \times 30\,\xi_1(\mathcal{T})$. In contrast to example 3.9, now a larger sample produces composite vortices in Region B.

Let us make a remark to this section. We found a rough phase diagram that relates the combinations of the values of η and H_e to the vortex phases of a two-band superconductor modeled by the 2B-TDGL equations. However, we have not tried to add another dimension to the phase diagram to include the effect of the operating temperature $\mathcal{T}$ and size of the sample. According to Chibotara *et al.*'s paper [40], the vortex phase diagram depends on the sample's size R, the operating temperature $\mathcal{T}$ and the applied field strength H. All of our comparable numerical results are in consistent with the results mentioned in Babaev's paper [34] and [35] about composite vortices in Type-I/Type-II and Type-II/Type-II superconductors (in the Region B); and also in Chibotara *et al.*'s paper [40] about phase changes versus applied

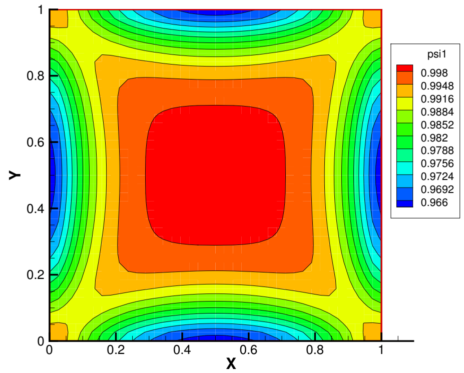

Figure 5.35: Example 3.1: ψ_1, $\eta = 0.0$, $H_e = 1.0$

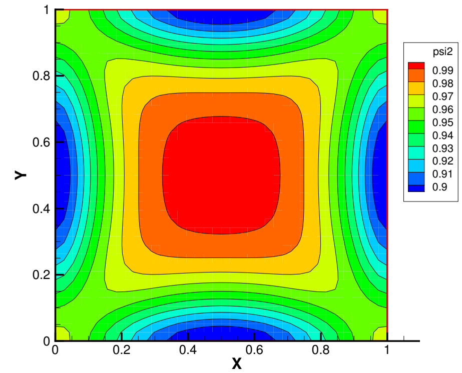

Figure 5.36: Example 3.1: ψ_2, $\eta = 0.0$, $H_e = 1.0$

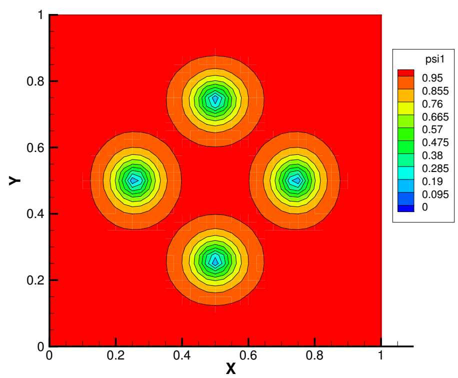

Figure 5.37: Example 3.2: ψ_1, $\eta = 0.0$, $H_e = 1.6$

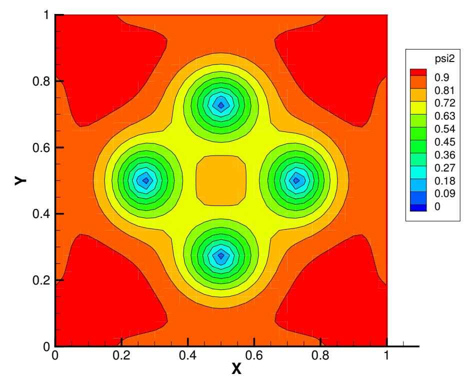

Figure 5.38: Example 3.2: ψ_2, $\eta = 0.0$, $H_e = 1.6$

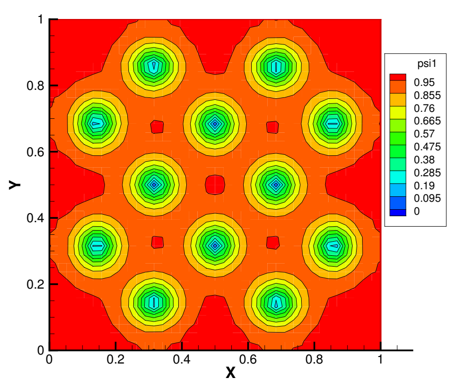

Figure 5.39: Example 3.3: ψ_1, $\eta = 0.0$, $H_e = 3.0$

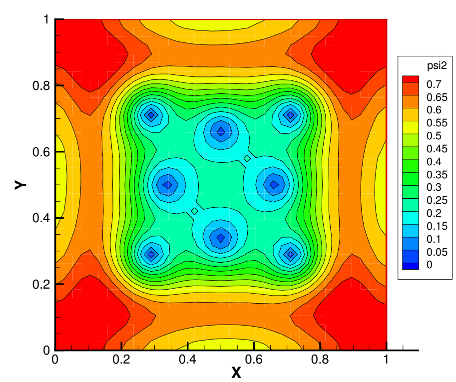

Figure 5.40: Example 3.3: ψ_2, $\eta = 0.0$, $H_e = 3.0$

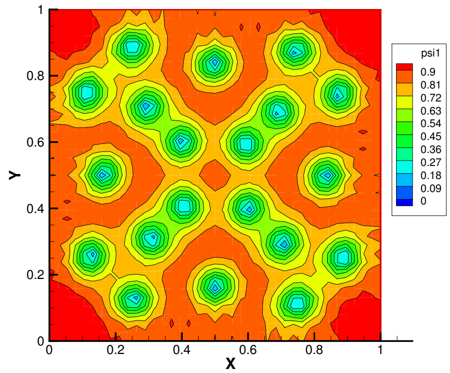

Figure 5.41: Example 3.4: ψ_1, $\eta = 0.0$, $H_e = 5.0$

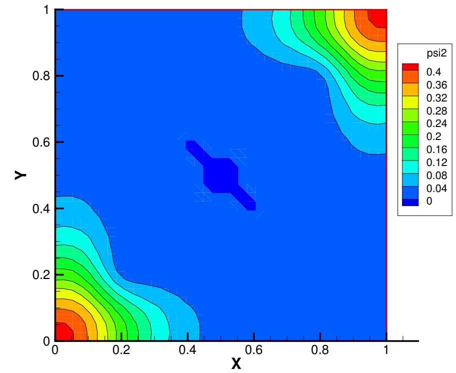

Figure 5.42: Example 3.4: ψ_2, $\eta = 0.0$, $H_e = 5.0$

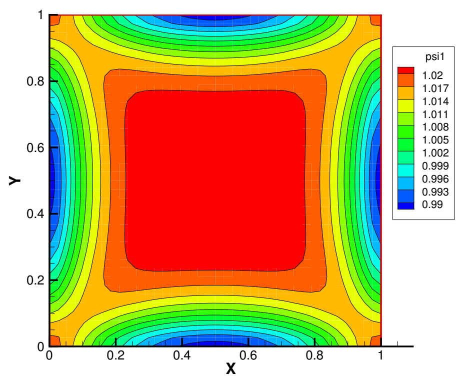

Figure 5.43: Example 3.5: ψ_1, $\eta = 0.05$, $H_e = 1.0$

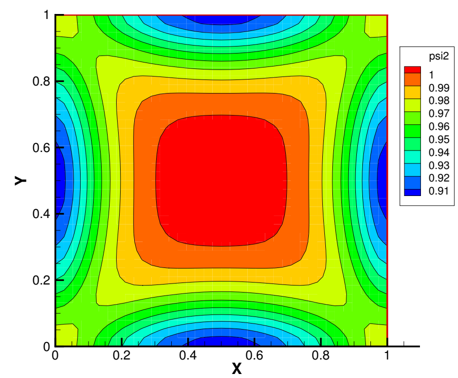

Figure 5.44: Example 3.5: ψ_2, $\eta = 0.05$, $H_e = 1.0$

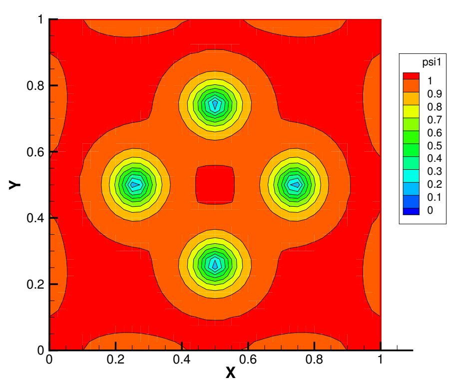

Figure 5.45: Example 3.6: ψ_1, $\eta = 0.05$, $H_e = 1.6$

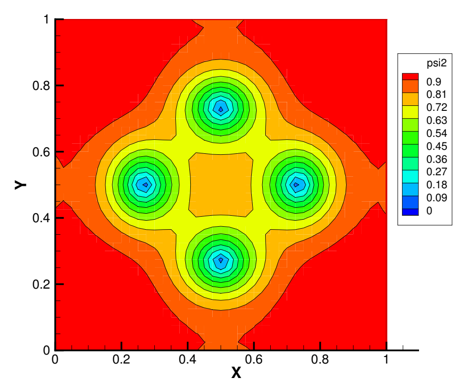

Figure 5.46: Example 3.6: ψ_2, $\eta = 0.05$, $H_e = 1.6$

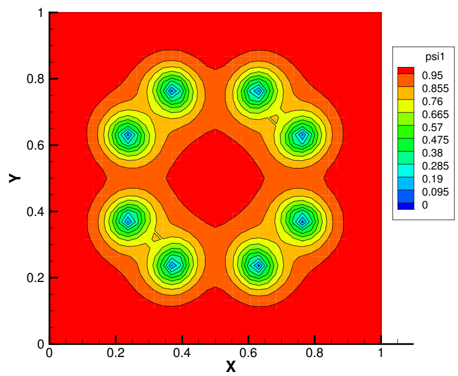

Figure 5.47: Example 3.7: ψ_1, $\eta = 0.05$, $H_e = 2.5$

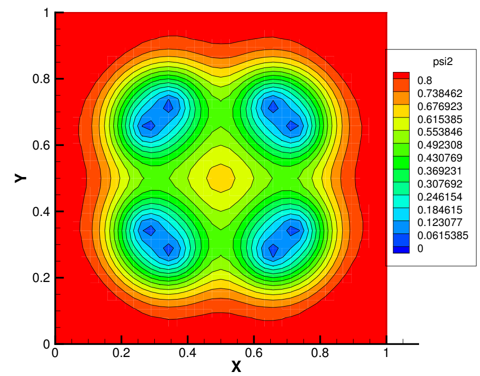

Figure 5.48: Example 3.7: ψ_2, $\eta = 0.05$, $H_e = 2.5$

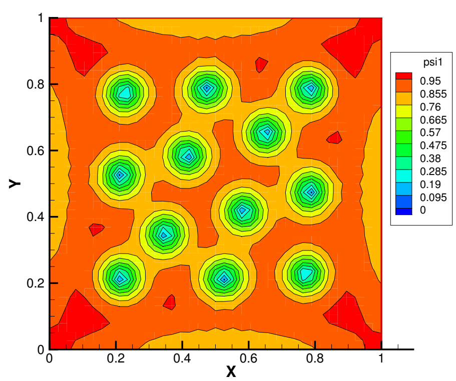

Figure 5.49: Example 3.8: ψ_1, $\eta = 0.05$, $H_e = 4.0$

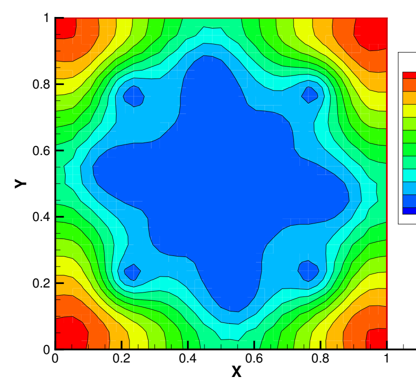

Figure 5.50: Example 3.8: ψ_2, $\eta = 0.05$, $H_e = 4.0$

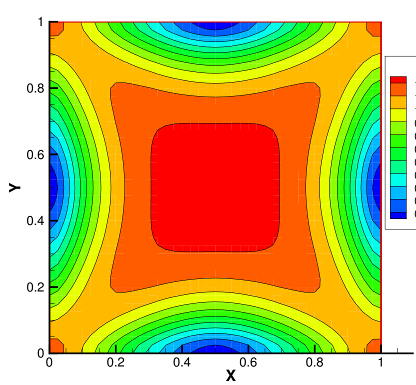

Figure 5.51: Example 3.9: ψ_1, $\eta = 0.08$, $H_e = 1.6$; size $=20\,\xi_1(\mathcal{T}) \times 20\,\xi_1(\mathcal{T})$.

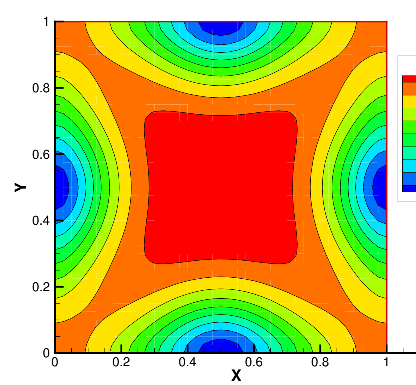

Figure 5.52: Example 3.9: ψ_2, $\eta = 0.08$, $H_e = 1.6$; size $=20\,\xi_1(\mathcal{T}) \times 20\,\xi_1(\mathcal{T})$.

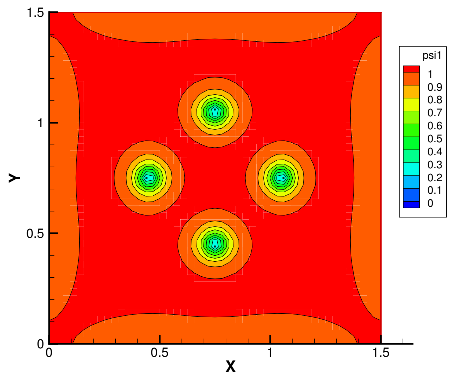

Figure 5.53: Example 4: ψ_1, $\eta = 0.08$, $H_e = 1.6$; size $=30\,\xi_1(\mathcal{T}) \times 30\,\xi_1(\mathcal{T})$.

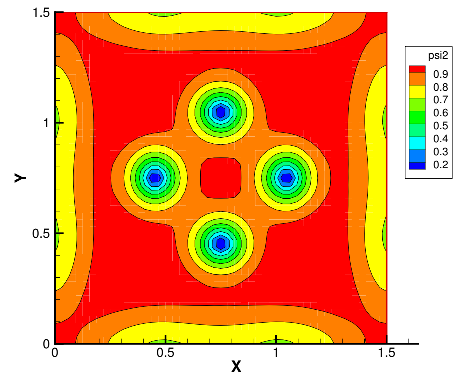

Figure 5.54: Example 4: ψ_2, $\eta = 0.08$, $H_e = 1.6$; size $=30\,\xi_1(\mathcal{T}) \times 30\,\xi_1(\mathcal{T})$.

field (at fixed $\mathcal{T}$ and size). We have seen that the Region C which exhibits noncomposite lattices only exists for zero or small η. According to Babaev's paper in [34], in the presence of an external magnetic field noncomposite vortices is energetically prohibited in an infinite dimensional sample. However, in [40], noncomposite vortices are observed in small sample of finite dimension. Our numerical results show that besides composite and noncomposite vortices, at least in small samples with sizes in an order of ten of the smallest coherent length of the two individual bands, there are also phases belong to Region A and Region D. Region A is in particularly more interesting when we are considering a sample of the size last mentioned in the presence of an applied magnetic field or current- we can explore the possibilities of modifying material properties of a two-band superconductor to enhance its interband coupling strong enough to inhibit the generation of vortices while hoping other critical material properties such as the critical temperature is not weakened too much if not strengthened.

5.2 Vortex Dynamics under Non-Stationary Applied Current and Stationary Magnetic Field

In this section, we are going to investigate the influence of an applied current on the vortex dynamics. The currents we added to the finite element simulations are always constant in space but varying in time as a sin function. In some cases in addition to the current, we also added an external magnetic field to the model to induce vortices. Denote the applied current appearing on the boundary of the sample as $\mathbf{j}_a|_{\partial\Omega}$. Suppose we set

$$\mathbf{j}_a|_{\partial\Omega} = \begin{cases} (0,\, j_a \sin(\omega t)) & \text{on} \quad \partial\Omega_a, \\ (0,\, 0) & \text{on} \quad \partial\Omega/\partial\Omega_a, \end{cases} \tag{5.10}$$

where $(\cdot,\, \cdot)$ denotes a vector on an X-Y plane and ω is the harmonic frequency of the current. The boundary $\partial\Omega_a \subset \partial\Omega$ is the subset of the sample's boundary on which we injected the current into the sample and we set it to be the two opposite sides of the sample parallel to the x-axis. The direction of the current is in the y-axis direction. Therefore, we have $\mathbf{j}_a \cdot \mathbf{n} = j_a \sin(\omega t)$ on $\partial\Omega_a$ and $\mathbf{j}_a \cdot \mathbf{n} = 0$ on $\partial\Omega/\partial\Omega_a$.

Recall that we denoted the normal current inside of the superconductor corresponding to the applied current $\mathbf{j}_a|_{\partial\Omega}$ as $\mathbf{j}_c$. For any time $t \geq 0$, the current $\mathbf{j}_c(\mathbf{x},\, t)$ inside the superconductor sample in the normal state as a good conductor can be found by solving (2.39)-(2.40). It is easy to see that for any fixed $t \geq 0$, $\mathbf{j}_c(\mathbf{x},\, t) = (0,\, j_a \sin(\omega t))$ is a solution to this problem with boundary condition (5.10). So we have $\mathbf{j}_c(\mathbf{x},\, t) = (0,\, j_a \sin(\omega t))$ in Ω. Since for any fixed $t \geq 0$, $\mathbf{j}_c(\mathbf{x},\, t) = \text{constant in } \Omega$, so we have $\operatorname{curl} \mathbf{j}_c(t) = 0$ in Ω and thus $\mathbf{j}_c(t) = \nabla\phi(t)$ for some $\phi \in H^1(\Omega)$. Therefore from (2.48), also (2.50)-(2.51) and the comments after it, we see that we can find a unique ϕ_c such that for any $t \geq 0$, $\mathbf{j}_c(t) = -\frac{\sigma}{\kappa_1}\nabla\phi_c(t)$. Solving this we get $\phi_c = -\frac{\kappa_1}{\sigma} j_a \sin(\omega t) y$. On the other hand, we have $\mathbf{j}_c(t) = \operatorname{curl} H_c(t)$, solving this we get $H_c = -j_a \sin(\omega t) x$, here x and y are the 2-dimensional domain spatial coordinate variable. When replacing $\nabla\phi_c$ by H_c, we can roughly view the applied current $\mathbf{j}_c$ as a Type-B current (not exactly is, since we have a term involving ϕ_c in the 2B-TDGL equations for $\psi_{1,2}$). Taking into account the external magnetic field H_e, we can roughly view the system as being presented in an external field with magnitude equals to $H_e - j_a \sin(\omega t) x$.

Example 5. (see Figure 5.55 and Figure 5.56 to Figure 5.61 and Figure 5.62)) This numerical example uses the following parameters:

$$\lambda_1(\mathcal{T}) = 0.6,\ \lambda_2(\mathcal{T}) = 0.2,\ \xi_1(\mathcal{T}) = 0.05,\ \xi_2(\mathcal{T}) = 0.1,\ \gamma_1 = \gamma_2 = 0.1,$$
$$\eta = 1.5,\ j_a = 6.0,\ H_e = 0.0,\ \mathcal{T}_1 = 1.0,\ \mathcal{T}_2 = 1.0,$$

and size of sample $= 20\,\xi_1(\mathcal{T}) \times 20\,\xi_1(\mathcal{T})$, mesh $= 1.95$ points per $\xi_1(\mathcal{T})$. It shows that a strong enough direct current (dc) will induce vortices and antivortices which annihilate at the center of the sample. Note that the moving vortices and antivortices are not composite. The nucleation and annihilation processes shown in Figure 5.55 and 5.56 to 5.61 and 5.62 are repeated indefinitely as long as the dc is applied to the sample.

Example 6. (see Figure 5.63 and Figure 5.64) This numerical example uses the same parameters as in Example 5, except that now we increase the coupling to $\eta = 2.0$. It shows that a strong enough coupling will inhibit the generation of vortices and antivortices which would otherwise annihilate at the center of the sample. Note that the magnitudes of the order parameters are large, e.g., it is well over 1.6 for ψ_1. This is a result of the theorem 3.2.16 which says that for $j = 1, 2$, $|\psi_j| \leq \sqrt{4\max\{|\eta|, |\eta|\nu^2\} + \max\{|\mathcal{T}_1|, |\mathcal{T}_2|\}}$.

Observe that in Example 5, without the help of an external magnetic field, we need to use a very large current and a large coupling η to induce vortices and antivortices in the bulk superconductor (numerical results showed that no vortices could be generated with a smaller η or $\mathbf{j}_a$), this is due in part to the gradient of the induced field $H_c = j_a x$ which has null value at the center vertical line of the sample. To investigate the influence of a current, stationary or nonstationary, with parameters that can be projected to the vortex phase diagram 5.2, we add an applied magnetic field to the sample, with values of H_e in the range that was used the in Example-set 1. By doing so, vortices are first provoked by a reasonably small applied field with a small η, and then a much smaller applied current can be used to affect the dynamics of the vortices.

Example 7. (see Figure 5.65 and Figure 5.66 to Figure 5.73 and Figure 5.74)) This numerical

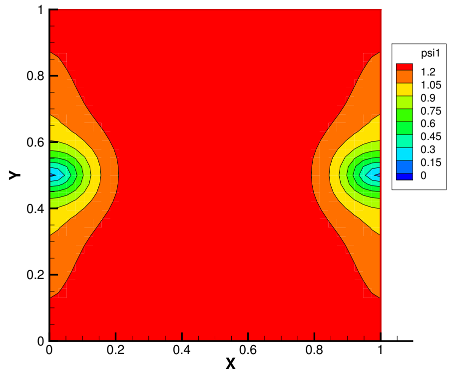

Figure 5.55: Example 5: ψ_1, $\eta = 1.5$, $H_e = 0.0$, $j_a = 6.0$, $\gamma_1 = \gamma_2 = 0.1$, $t = 10$.

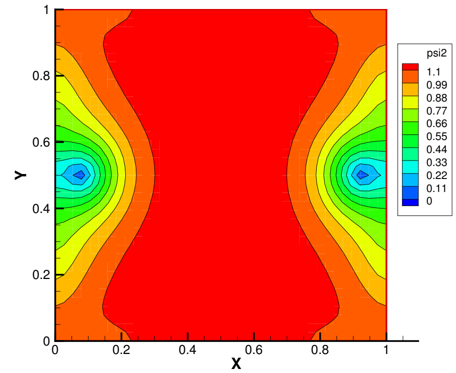

Figure 5.56: Example 5: ψ_2, $\eta = 1.5$, $H_e = 0.0$, $j_a = 6.0$, $\gamma_1 = \gamma_2 = 0.1$, $t = 10$.

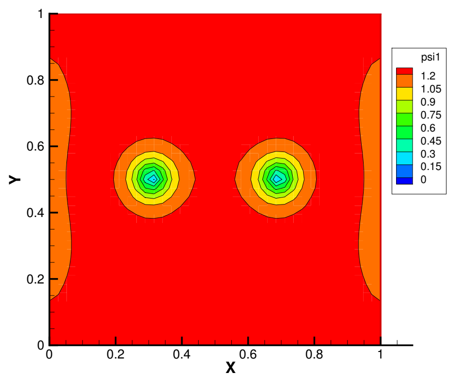

Figure 5.57: Example 5: ψ_1, $\eta = 1.5$, $H_e = 0.0$, $j_a = 6.0$, $\gamma_1 = \gamma_2 = 0.1$, $t = 18$.

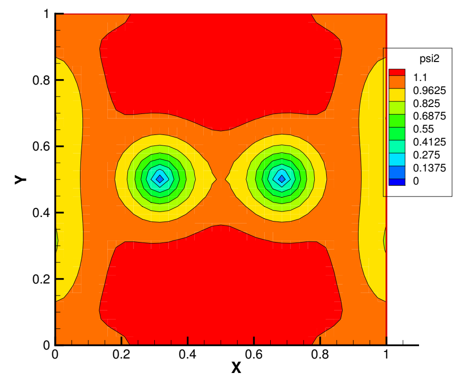

Figure 5.58: Example 5: ψ_2, $\eta = 1.5$, $H_e = 0.0$, $j_a = 6.0$, $\gamma_1 = \gamma_2 = 0.1$, $t = 18$.

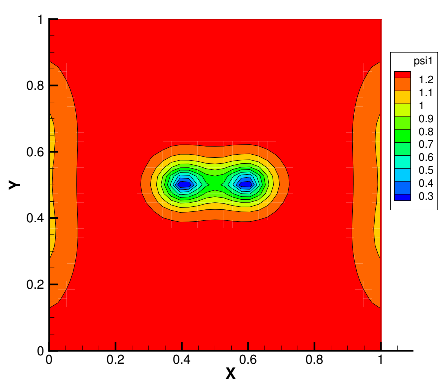

Figure 5.59: Example 5: ψ_1, $\eta = 1.5$, $H_e = 0.0$, $j_a = 6.0$, $\gamma_1 = \gamma_2 = 0.1$, $t = 21$.

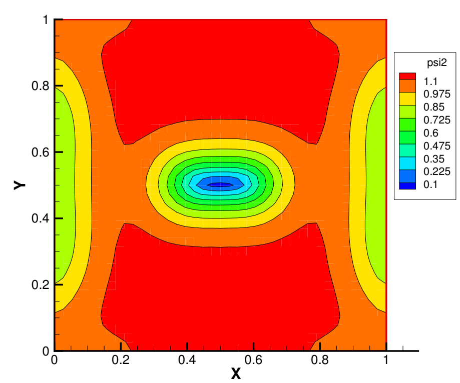

Figure 5.60: Example 5: ψ_2, $\eta = 1.5$, $H_e = 0.0$, $j_a = 6.0$, $\gamma_1 = \gamma_2 = 0.1$, $t = 21$.

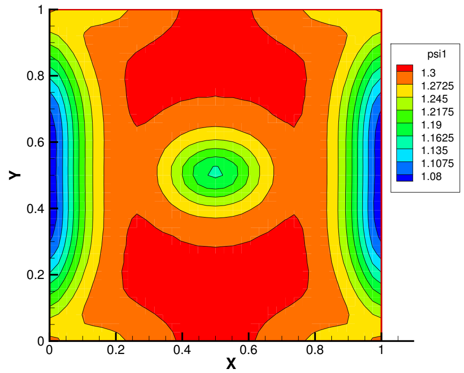

Figure 5.61: Example 5: ψ_1, $\eta = 1.5$, $H_e = 0.0$, $j_a = 6.0$, $\gamma_1 = \gamma_2 = 0.1$, $t = 22$.

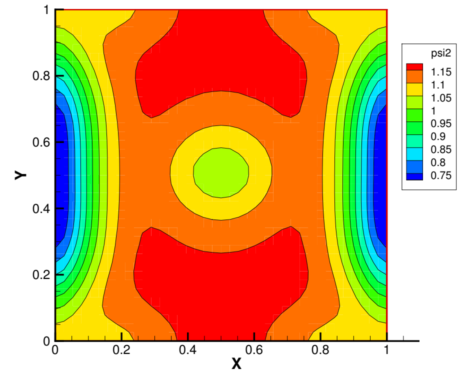

Figure 5.62: Example 5: ψ_2, $\eta = 1.5$, $H_e = 0.0$, $j_a = 6.0$, $\gamma_1 = \gamma_2 = 0.1$, $t = 22$.

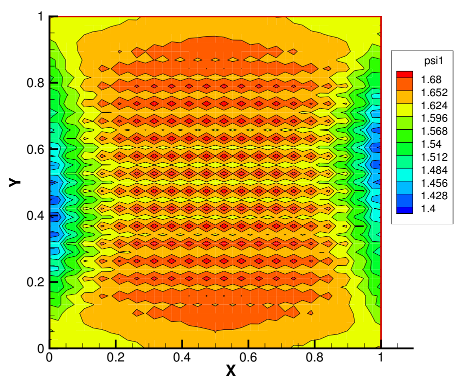

Figure 5.63: Example 6: ψ_1, $\eta = 2.0$, $H_e = 0.0$, $j_a = 6.0$, $\gamma_1 = \gamma_2 = 0.1$, $t > 8$.

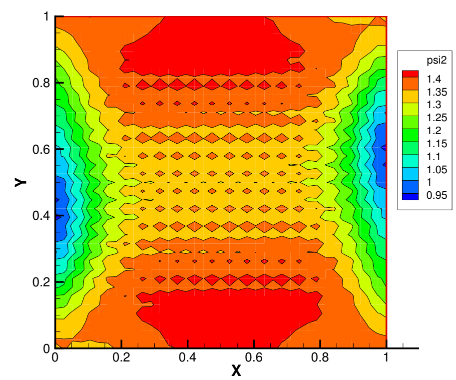

Figure 5.64: Example 6: ψ_2, $\eta = 2.0$, $H_e = 0.0$, $j_a = 6.0$, $\gamma_1 = \gamma_2 = 0.1$, $t > 8$.

example uses the following parameters:

$$\lambda_1(0) = 0.6,\ \lambda_2(0) = 0.2,\ \xi_1(0) = 0.05,\ \xi_2(0) = 0.1,\ \gamma_1 = \gamma_2 = 0.1,$$
$$\eta = 0.01,\ H_e = 1.6,\ \mathcal{T}_1 = 0.7,\ \mathcal{T}_2 = 0.2,$$

and size of sample $= 20\,\xi_1(0) \times 20\,\xi_1(0)$, mesh $= 1.95$ points per $\xi_1(\mathcal{T})$. Now we delay the application of an applied alternative current (ac) until steady state noncomposite vortices are generated in the bulk at $t = 60$. After the vortex phase has reached a steady state in

Region C, see Figure 5.65 and Figure 5.66, the following ac is then added to the sample.

$$j_a = \begin{cases} 0.0 & \text{for} \quad t < 60, \\ 2.0\sin(\omega t) \text{ with } \omega = 0.025 & \text{for} \quad t \geq 60. \end{cases} \tag{5.11}$$

From Figure 5.65 and Figure 5.66 to Figure 5.73 and Figure 5.74 we can see that the two sets of vortices, each corresponding to a separated distinct band, behave differently in dynamics and in topological patterns. When the ac flows in the y-axis direction, the vortices are forced by a non-symmetric magnetic field of magnitude equals to $H_e - j_a \sin(\omega t)x$ to shift to the right side of the sample. Eventually one vortex from each band escapes from the right and two vortices from each band enter at the left, see Figure 5.67 and Figure 5.68. When the ac changes its direction, the vortices are now forced by another non-symmetric magnetic field of magnitude equals to $H_e + j_a \sin(\omega t)x$ to shift to the left side of the sample. Eventually two vortices from band one escape from the left and two vortices from band two first merge and then split again and only one escape from the left, see Figure 5.71 and Figure 5.72. Then one vortex from each band enters at the right, see Figure 5.73 and Figure 5.74. The vortex phases corresponding to the second band are actually in a mixture of Region C which consists of noncomposite vortices, and Region D which consists of a large block of normal region or giant vortex, see for example, Figure 5.72 and Figure 5.74. Also note that the vortices corresponding to the second band are all weak vortices. The same dynamics start over repeatedly from Figure 5.65 and Figure 5.66 to Figure 5.73 and Figure 5.74 as long as the same magnetic field and current are applied to the sample.

Example 8. (see Figure 5.75 and Figure 5.76 to Figure 5.83 and Figure 5.84)) This numerical example uses the same material parameters and same applied field and current as in Example 7, but now we increase the coupling parameter to $\eta = 0.8$. In the absence of an applied current, this setting is exactly the same as that in Example 1.7 which showed that when in steady state, the sample generates composite vortices in Region B in the presence of an applied field of magnitude $H_e = 1.6$, see Figure 5.11 and Figure 5.12. Observe that the two vortex sublattices shown in Figure 5.11 and Figure 5.12 look the same as those shown in Figure 5.75 and Figure 5.76 which are the steady vortex lattices of the sample in this example, despite now a S-N interface (with $\gamma_1 = \gamma_2 = 0.1$) exists on the boundary of this sample. The same ac as in Example 7 is applied to the sample after it has reached the steady vortex phase Region B at $t = 60$. This simulation shows that a strong enough coupling "binds" the two set of vortices together, even when they move accordingly to the vortex dynamics induced by a strong current. The same dynamics start over repeatedly from Figure 5.75 and Figure 5.76 to Figure 5.81 and Figure 5.82 as long as the same magnetic field and current are applied to the sample. However, other simulations not shown in this work showed that a combination of stronger applied field and current may create more complex dynamics which includes transverse movements of vortices as we have seen here, as well as twists of vortex lattices which would make the topology of the vortex lattice highly non-symmetry. As we mentioned before, a S-N interface with $\gamma_1 = \gamma_2 = 0.1$ on the boundary of the sample does not make any obvious changes to the generation of steady state vortex sublattices. The next example shows that a S-N interface effect with larger $\gamma_1,\ \gamma_2$ does affect the generation of the vortices of both bands.

Example 9. (see Figure 5.85 and Figure 5.86 to Figure 5.93 and Figure 5.94) This numerical example uses the same material parameters and same applied field and current as in Example 8, but now we increase the S-N interface parameters to $\gamma_1 = \gamma_2 = 1.0$. In the case $\gamma_1 = \gamma_2 = 0.1$, the sample in Example 8 produces four steady vortices at $t = 60$ in the presence of an applied field, see Figure 5.75 and Figure 5.76 to Figure 5.77 and Figure 5.78. However, when the interface parameters are increased to $\gamma_1 = \gamma_2 = 1.0$, the sample in this example produces only two steady vortices at $t = 60$, see Figure 5.85 and Figure 5.86 to Figure 5.87 and Figure 5.88. As a result, the samples produce different vortex dynamics. However, for $t > 166$, the vortex dynamics of the sample in this example repeat in a sequence which resembles the dynamics shown in Figure 5.77 and Figure 5.78 to Figure 5.83 and Figure 5.84 of Example 8.

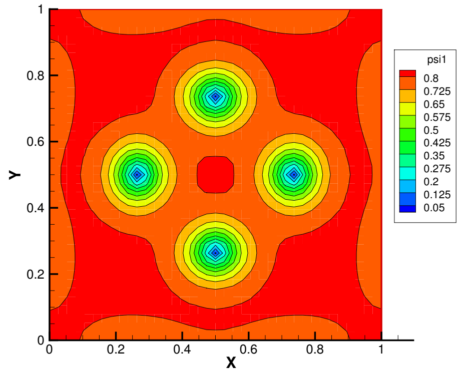

Figure 5.65: Example 7: ψ_1, $\eta = 0.01$, $H_e = 1.6$, $j_a = 0$, $\gamma_1 = \gamma_2 = 0.1$, $t = 60$. Steady state- noncomposite.

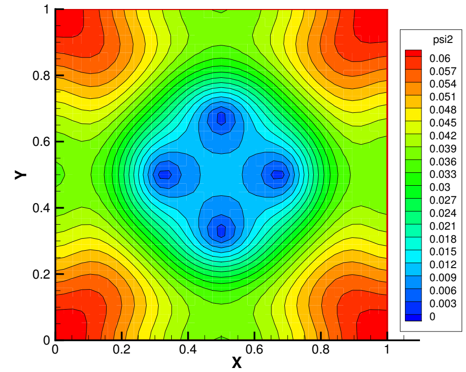

Figure 5.66: Example 7: ψ_2, $\eta = 0.01$, $H_e = 1.6$, $j_a = 0$, $\gamma_1 = \gamma_2 = 0.1$, $t = 60$. Steady state- noncomposite.

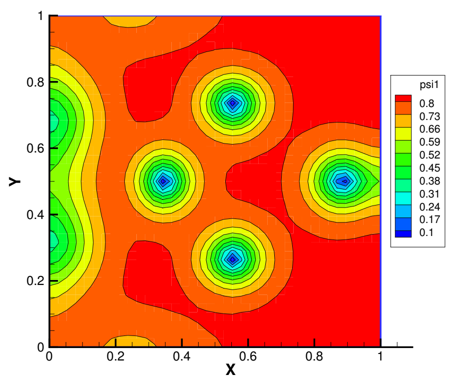

Figure 5.67: Example 7: ψ_1, $\eta = 0.01$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 0.1$, $t = 87$. One Vortex exits from the right. Two enter at the left.

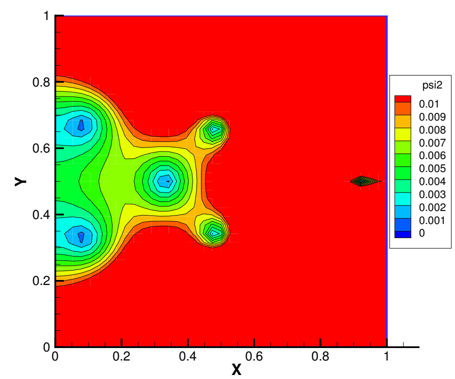

Figure 5.68: Example 7: ψ_2, $\eta = 0.01$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 0.1$, $t = 87$. One Vortex exits from the right. Two enter at the left.

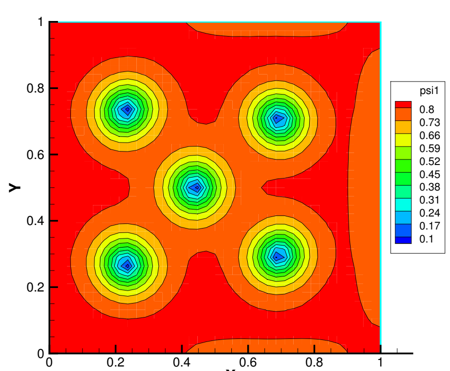

Figure 5.69: Example 7: ψ_1, $\eta = 0.01$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 0.1$, $t = 118$.

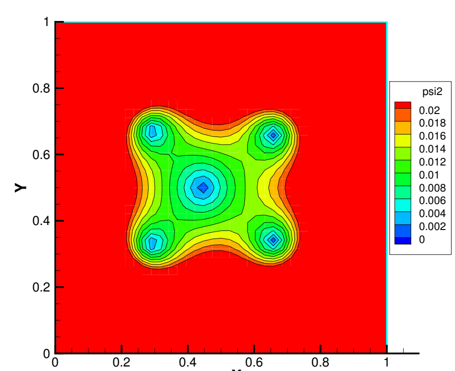

Figure 5.70: Example 7: ψ_2, $\eta = 0.01$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 0.1$, $t = 118$.

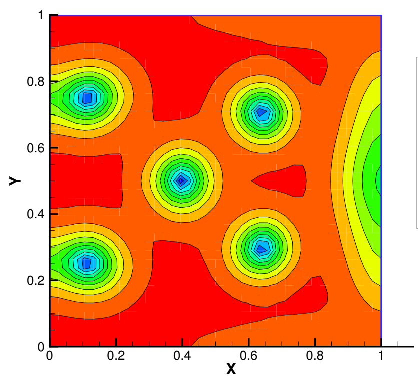

Figure 5.71: Example 7: ψ_1, $\eta = 0.01$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 0.1$, $t = 135$. Two vortices exit from the left.

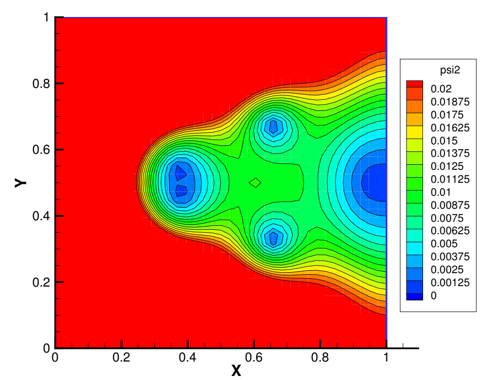

Figure 5.72: Example 7: ψ_2, $\eta = 0.01$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 0.1$, $t = 135$. Two vortices merging into one.

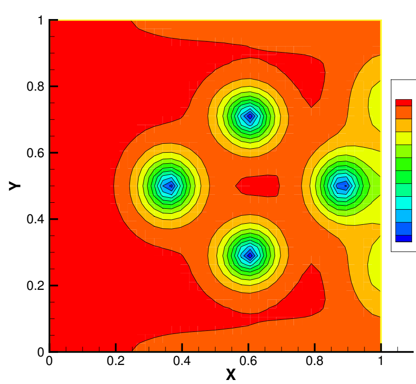

Figure 5.73: Example 7: ψ_1, $\eta = 0.01$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 0.1$, $t = 143$. One vortex enter at the right.

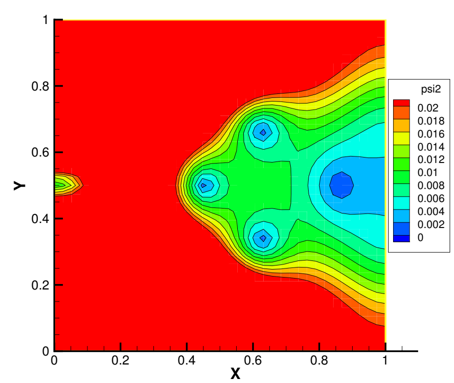

Figure 5.74: Example 7: ψ_2, $\eta = 0.01$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 0.1$, $t = 143$. One vortex exits from the left, one enters at the right.

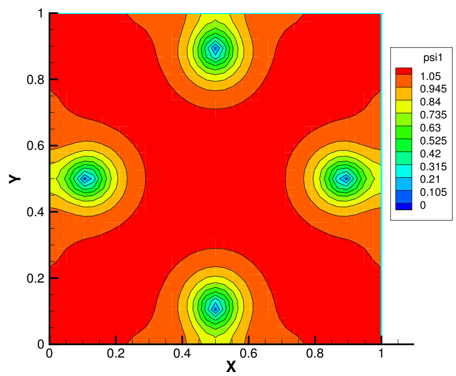

Figure 5.75: Example 8: ψ_1, $\eta = 0.8$, $H_e = 1.6$, $j_a = 0$, $\gamma_1 = \gamma_2 = 0.1$, $t = 28$.

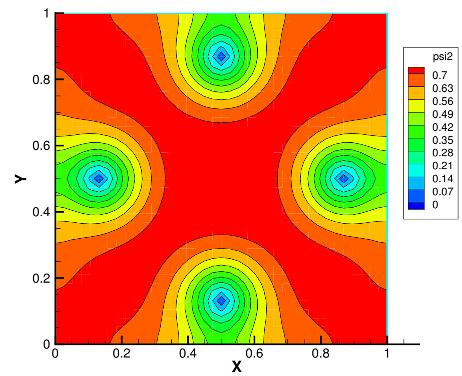

Figure 5.76: Example 8: ψ_2, $\eta = 0.8$, $H_e = 1.6$, $j_a = 0$, $\gamma_1 = \gamma_2 = 0.1$, $t = 28$.

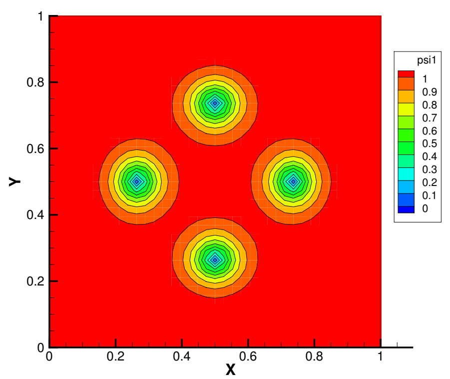

Figure 5.77: Example 8: ψ_1, $\eta = 0.8$, $H_e = 1.6$, $j_a = 0$, $\gamma_1 = \gamma_2 = 0.1$, $t = 60$. Steady state- composite.

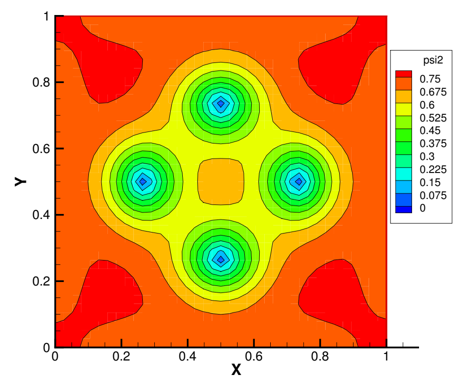

Figure 5.78: Example 8: ψ_2, $\eta = 0.8$, $H_e = 1.6$, $j_a = 0$, $\gamma_1 = \gamma_2 = 0.1$, $t = 60$. Steady state- composite

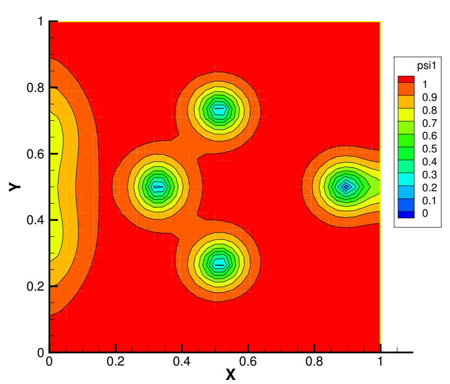

Figure 5.79: Example 8: ψ_1, $\eta = 0.8$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 0.1$, $t = 89$. One Vortex exits from the right.

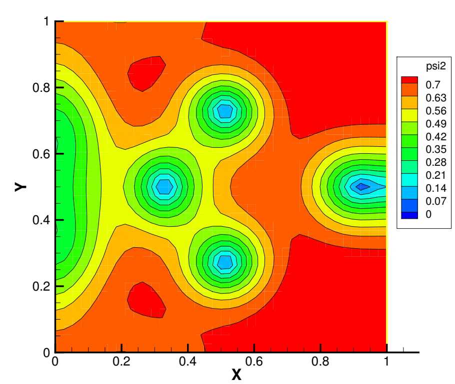

Figure 5.80: Example 8: ψ_2, $\eta = 0.8$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 0.1$, $t = 89$. One Vortex exits from the right.

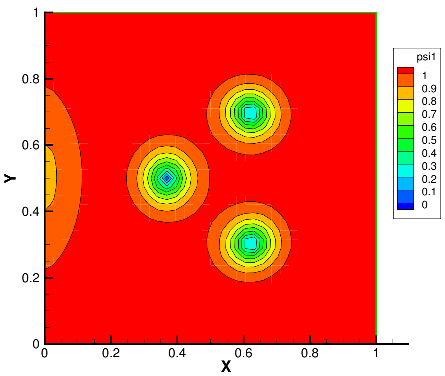

Figure 5.81: Example 8: ψ_1, $\eta = 0.8$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 0.1$, $t = 110$.

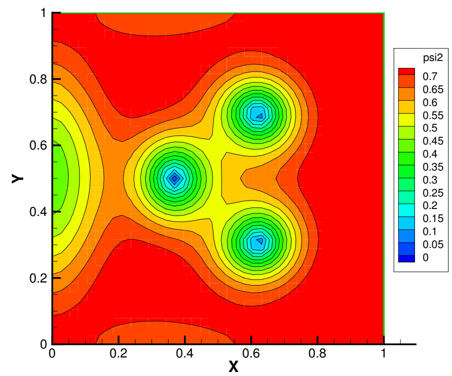

Figure 5.82: Example 8: ψ_2, $\eta = 0.8$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 0.1$, $t = 110$.

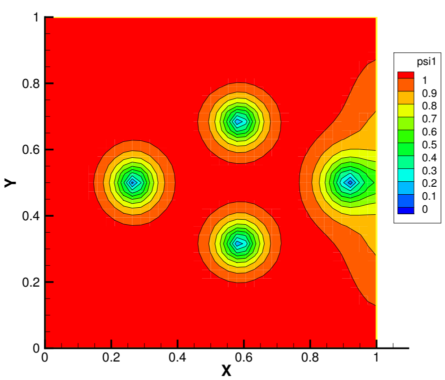

Figure 5.83: Example 8: ψ_1, $\eta = 0.8$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 0.1$, $t = 137$. One vortices enters at the right.

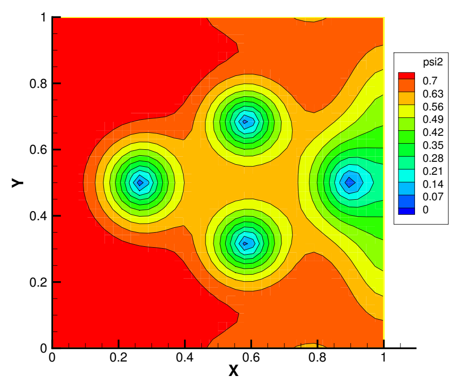

Figure 5.84: Example 8: ψ_2, $\eta = 0.8$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 0.1$, $t = 137$. One vortices enters at the right.

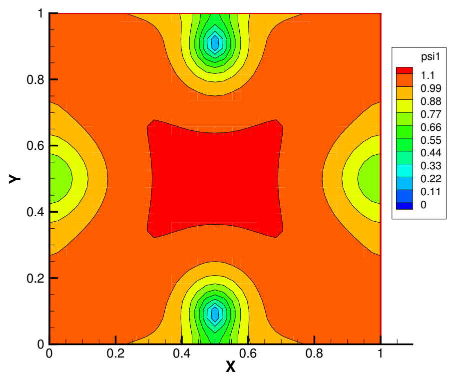

Figure 5.85: Example 9: ψ_1, $\eta = 0.8$, $H_e = 1.6$, $j_a = 0$, $\gamma_1 = \gamma_2 = 1.0$, $t = 25$.

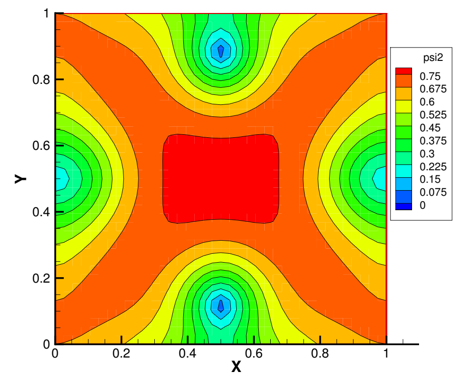

Figure 5.86: Example 9: ψ_2, $\eta = 0.8$, $H_e = 1.6$, $j_a = 0$, $\gamma_1 = \gamma_2 = 1.0$, $t = 25$.

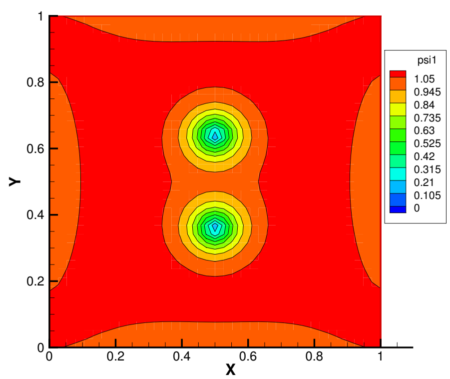

Figure 5.87: Example 9: ψ_1, $\eta = 0.8$, $H_e = 1.6$, $j_a = 0$, $\gamma_1 = \gamma_2 = 1.0$, $t = 60$. Steady state- composite.

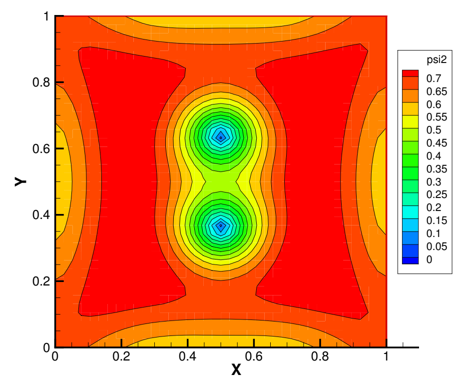

Figure 5.88: Example 9: ψ_2, $\eta = 0.8$, $H_e = 1.6$, $j_a = 0$, $\gamma_1 = \gamma_2 = 1.0$, $t = 60$. Steady state- composite

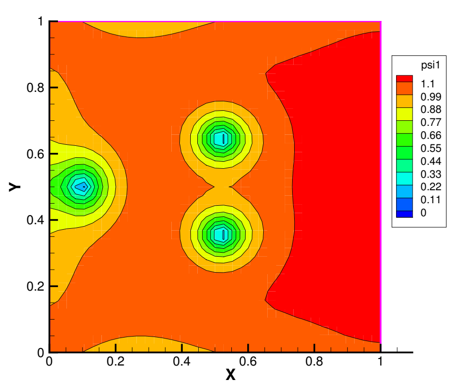

Figure 5.89: Example 9: ψ_1, $\eta = 0.8$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 1.0$, $t = 78$. One Vortex enters at the left.

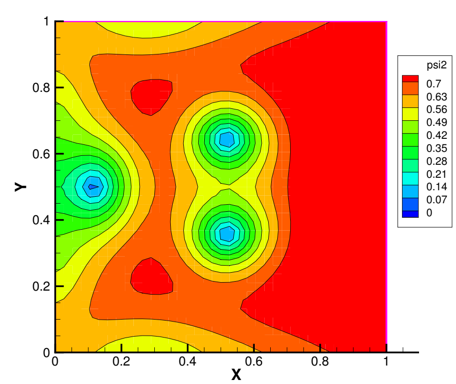

Figure 5.90: Example 9: ψ_2, $\eta = 0.8$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 1.0$, $t = 78$. One Vortex enters at the left.

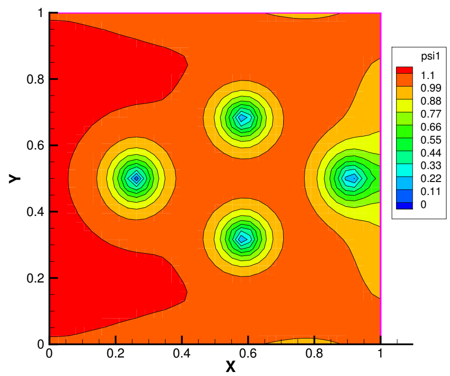

Figure 5.91: Example 9: ψ_1, $\eta = 0.8$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 1.0$, $t = 138$. One Vortex enters at the right.

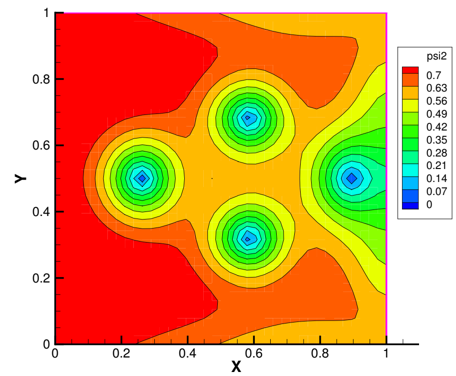

Figure 5.92: Example 9: ψ_2, $\eta = 0.8$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 1.0$, $t = 138$. One Vortex enters at the right.

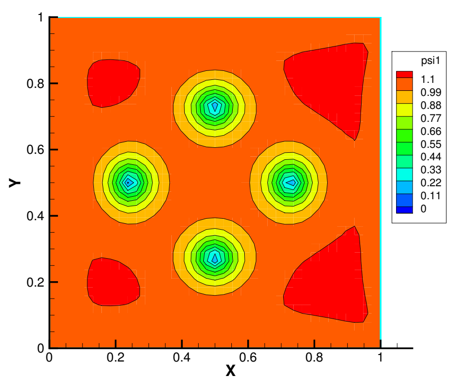

Figure 5.93: Example 9: ψ_1, $\eta = 0.8$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 1.0$, $t = 166$.

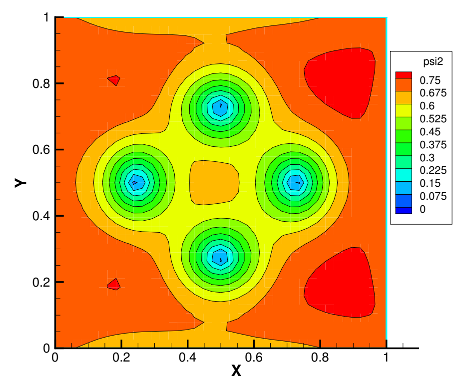

Figure 5.94: Example 9: ψ_2, $\eta = 0.8$, $H_e = 1.6$, $j_a = 2.0\sin(0.025t)$, $\gamma_1 = \gamma_2 = 1.0$, $t = 166$.

CHAPTER 6

Conclusions and Future Research

6.1 Conclusions

In this work we studied some analytical and approximation issues of a coupled isothermal, isotropic two-band time-dependent Ginzburg-Landau (2B-TDGL) equations in a two-dimensional bounded domain Ω with inclusions of time-dependent applied magnetic field and time-dependent applied current. We introduced a "current gauge" to take time-dependent applied current into account. Based on variational formulations of the 2B-TDGL equations gauged by this "current gauge" and modified to include a regularization term, we developed some theorems concerning the global existence, uniqueness and continuous initial data dependency of the solutions. We also proved that the solution satisfies a maximum principle. In the course of developing our results, we derived the necessary regularities for the data, namely, the time-dependent applied field in the domain Ω, the time-dependent current on the boundary $\partial\Omega$, and the initial condition at $t = 0$.

In the finite element analysis, a backward Euler finite element approximations of the 2B-TDGL equations under the "current gauge" were studied. The existence and uniqueness of the approximated solutions were proved. Stability and error estimates of the numerical scheme were developed. However, we have not sought higher regularities of the solutions and specified the smoothness of the domain which are needed to meet the regularities required by the finite-element analysis. Neither have we found the error estimate with respect to the regularization coefficient ϵ.

Through simulation results, we discovered that in a sample with size in the order of ten of the coherent length, there exists four vortex phases which change according to the strengths of the interband coupling and applied magnetic field. For example, in the presence of a fixed magnetic field, by merely changing the coupling strength of a sample, the sample would response in different vortex phases- a weak coupling would produce non-concentric vortices; a strong enough coupling would induce concentric vortices, however, a too strong coupling would inhibit the nucleation of vortices. The vortex dynamics induced by an applied current in the 2B-TDGL model with an inclusion of a superconductor-normal metal interface were also investigated. Numerical results showed that under an applied current, the motions of each set of vortices corresponding to each individual band may not be synchronized or concentric to the other set when the coupling strength is not strong enough. For the vortex phase phenomenon, we need to find experimental results to judge whether the mentioned numerical phenomenon produced by an isotropic 2B-TDGL model is physical. For the phenomenon under an applied current, it would be interesting to investigate the differences in pinning strength required to pin a pair of non-concentric vortices versus a pair of concentric vortices; or to see how the pinning strength affected by the coupling strength.

6.2 Future Research

To complete the current two-dimensional analytical work and for similar future work, we need to prove higher regularities of the solutions in order to meet the regularities required by the finite element analysis. Also we need to estimate the approximation error with respect to the regularization coefficient ϵ. In this present work, we have simplified the 2B-TDGL model by ignoring the gradient coupling term, as most physicists and scientists used only the non-gradient coupling term in their researches that involve a variant of the 2B-TDGL model. For a more complete work in analysis, we need to consider the inclusion of the gradient coupling term in the same analytical framework of our current work, and in computation, we need to investigate the influence of this additional term in various simulation cases. The study of the dynamical properties of the solutions of the 2B-TDGL model such as its long-time asymptotic behavior and the existence of global attractor is another interesting topic to explore.

The intermetallic compound superconductor MgB_2 is known to be anisotropic in physical parameters. From a practical point of view, it is necessary to study an anisotropic variant of the 2B-TDGL model, both analytically and numerically. A more realistic 2B-TDGL model could be a valuable numerical experiment tool for physicists and scientists to explore, understand and even modify the characteristics of multi-band superconductors under various settings of material parameters, operating temperature and external excitations. The development of analytical results and computational codes for a three-dimensional 2B-TDGL model which must now include the coupling of Maxwell equations at the exterior of the superconductor domain, is another important step to bring more practical analytical results and computational tools of the 2B-TDGL model to the real world. Computation of a two-dimensional finite element code for a sample of the size in the order of ten of the coherent length is already a challenging job, for three-dimensional finite element computations, speed-up of the code by using parallelizations such as domain decomposition methods is an indispensable job. Inhomogeneities and material defects of a superconductor sample play a crucial role in the study of vortex pinning and dynamics. It has been a common and easy practice to add spatial inhomogeneities such as normal inclusions to a TDGL model to act as vortex pinning sites. An equally important but more challenging issue in the study of vortex pinning and dynamics is to include random noise into the model. In real world where noises such as thermal fluctuation or random drift in applied field can not be ignored, a deterministic 2B-TDGL model may not be able to produce physically meaningful computational results. All of the above considerations are subjects of our future research.

REFERENCES

[1] Q. Du, M.D. Gunzburger and J. Peterson, *Analysis and Approximation of the Ginzburg-Landau Model of Superconductivity*, SIAM Review 34, 54-81, 1992.

[2] Q. Du, *Global Existence and Uniqueness of Solutions of the Time-Dependent Ginzburg-Landau Model for Superconductivity*, Applicable Analysis 53, 1-17, 1994.

[3] Q. Du, *Finite Elements Methods for the Time-Dependent Ginzburg-Landau Model of Superconductivity*, Applicable Analysis 53, 1-17, 1994.

[4] Q. Du, *Numerical Approximations of the Ginzburg-Landau Model of Superconductivity*, J. Math. Phy., 46, 095109, 2005.

[5] J. Chapman, Q. Du, M.D. Gunzburger, *A Ginzburg-Landau type Model of Superconducting/Normal Junctions including Josephson Junctions*, Euro. J. Appl. Math., 6, no. 2, 97-114, 1995.

[6] Q. Du, *Studies of a Ginzburg-Landau Model for d-wave Superconductors*, SIAM J. Appl. Math., 59, no. 4, 1225-1250, 1999.

[7] Q. Tang and S. Wang, *Time Dependent Ginzburg-Landau Equations of Superconductivity*, Physica D, 88, 139-166, 1995.

[8] Z.Chen, K.H. Hoffmann and J. Liang, *On a Non-Stationary Ginzburg-Landau Superconductivity Model*, Math. Meth. Appl. Sci., 16, 855-875, 1993.

[9] J.F. Pelle, H.G. Kaper, P. Takac, *Dynamics of the Ginzburg-Landau Equations of Superconductivity*, Nonlin. Anal. Theor. Meth. Appl., 32, no. 5, 647-665, 1998.

[10] F. Zaouch, *Global Existence and Boundedness of Solutions of the Time-dependent Ginzburg-Landau Equations with a Time-Dependent Magnetic Field*, Rostock. Mth. Kolloq., 57, 53-70, 2003.

[11] V. Georgescu, *Some Boundary Value Problems for Differential Forms on Compact Riemannian Manifolds*, Ann. Mat. Pura Appl., 4, 122, 159-198, 1979.

[12] D.N. Arnold, L.R. Scott, M. Vogelius, *Regular Inversion of the Divergence Operator with Dirichlet Boundary Conditions on a Polygon*, , IMA, Univ. Minnesota, 302, 1987.

[13] J. Fleckinger-Pellé, H.G. Kaper, *Gauges for the Ginzburg-Landau Equations of Superconductivity*, Proc. ICIAM'95, Z. Angew. Math. Mech., 76, 345-348, 1996.

[14] D. Phillips, E. Shin, *On the Analysis of a Non-isothermal Model for Superconductivity*, Euro. Jnl of Applied Mathematics, 15, 147-179, 2004.

[15] Z. Chen, K.H. Hoffmann, *Global Classical Solutions to a Non-isothermal Dynamical Ginzburg-Landau Model in Superconductivity*, Numer. Funct. Anal. Optimiz., 18, 901-920, 1997.

[16] Z. Chen, K.H. Hoffmann, *An error estimate for a finite-element scheme for a phase field model*, IMA Journal of Numerical Analysis 14, 234-255, 1994.

[17] Z. Chen, *Numerical Studies of a Non-Stationary Ginzburg-Landau model for Superconductivity*, Numer. Math. 76, 323-353, 1997.

[18] Z. Chen, *Mixed Finite Element Methods for a Dynamical Ginzburg-Landau Model of Superconductivity*, Numer. Math. 76, 323-353, 1997.

[19] G.D. Akrivis, V.A. Dougalis, O.A. Karakashian, *On Fully Discrete Galerkin methods of Second-Order Temporal Accuracy for the Nonlinear Schrödinger Equation*, Numer. Math. 59, 31-53, 1991.

[20] R. Karamikhova *Finite Element Analysis and Approximations of a Ginzburg-Landau Model of Superconductivity*, Nonlinear Analysis, 30, No. 3, 1893-1904, 1997.

[21] H. Suhl, B.T. Matthias and L.R. Walker, *Bardeen-Cooper-Schrieffer Theory of Superconductivity in the Case of Overlapping Bands*, Phys. Rev. Lett., 3, 552, 1959.

[22] L.P. Gor'kov, G.M. Éliashberg, *Generalization of the Ginzburg-Landau Equations for Non-Stationary Problems in the Case of Alloys with Paramagnetic Impurities*, Soviet Physics JETP, 27, 328-334, 1968.

[23] J. Nagamatsu, N. Nakagawa, T. Muranaka, Y. Zenitani and J. Akimitsu, *Superconductivity at 39 K in Magnesium Diboride*, Nature, 410, 63-64, 2001.

[24] H.J. Choi, D. Roundy, H. Sun, M.L. Cohen and S.G. Louie, *The Origin of the Anomalous Superconducting Properties of* MgB_2, Nature, 418, 2002.

[25] S.Souma, Y. Machida, T. Sato, *et al.*, *The Origin of Multiple Superconducting Gaps in* MgB_2, Nature, 423, 2003.

[26] E.J. Nicol and J.P. Carbotte, *Properties of the Superconducting State in a Two-Band Model*, arXiv:cond-mat/0409335 v2, 2005.

[27] A. Gurevich and V.M. Vinokur, *Interband Phase Modes and Nonequilibrium Soliton Structures in Two-Gap Superconductors*, Phys. Rev. Lett., 90, no.4, 047004, 2003.

[28] I.N. Askerzade and A. Gencer, *London Penetration Depth* $\lambda(T)$ *in Two-Band Ginzburg-Landau Theory: Application to* MgB_2, Solid State Comm., 123, 63-67, 2002.

[29] I.N. Askerzade and A. Gencer, *Thermodynamic Magnetic Field and Specific Heat Jump of a Bulk Superconductor MgB_2 Using Two-Band Ginzburg-Landau Theory*, J. Physical Society of Japan, 71, no. 7, 1637-1639, 2002.

[30] I.N. Askerzade, *Anisotropy of the Upper Critical Field in MgB_2: The Two-Band Ginzburg-Landau Theory*, JETP Lett., 81, no.11, 583-586, 2005.

[31] D.R. Tilley, *The Ginzburg-Landau Equations for Anisotropic Alloy*, Proc. Phys. Soc., vol. 86, 1965.

[32] A.E. Koshelev and A.A. Golubov, *Why Magnesium Diboride is Not Described by Anisotropic Ginzburg-Landau Theory*, Phys. Rev. Lett., 92, no.10, 107008, 2004.

[33] M. Angst, R. Puzniak, *Two Band Superconductivity in MgB_2: Basic Anisotropic Properties and Phase Diagram*, arXiv:cond-mat/0305048 v2, 2003.

[34] E. Babaev, *Vortices with Fractional Flux in Two-Gap Superconductors and in Extended Faddeev Model*, Phys. Rev. Lett., 89, no. 6, 067001, 2002.

[35] E. Babaev, *Phase Diagram of Planar $U(1) \times U(1)$ Superconductor: Condensation of Vortices with Fractional Flux and a Superfluid State*, arXiv:cond-mat/0201547 v7, 2004.

[36] E. Babaev, *Neither a Type-I nor a Type-II Superconductivity in a Two-Gap System*, arXiv:cond-mat/0302218 v3, 2004.

[37] E. Babaev, *Thermodynamically Stable Non-Local Vortices, Vortex Molecules and Semi-Meissner State in Neither a Type-I nor a Type-II Multicomponent Superconductors*, arXiv:cond-mat/0302218 v3, 2004.

[38] E. Babaev, *Semi-Meissner State and Neither Type-I nor Type-II Superconductivity in Multicomponent Systems*, arXiv:cond-mat/0411681 v2, 2005.

[39] E. Smorgrav, J. Smiseth, E. Babaev and A.Sudbo, *Vortex Sublattice Melting in a Two-Component Superconductor*, Phys. Rev. Lett., 94, no. 9, 096401, 2005.

[40] L.F. Chibotaru and V.H. Dao, *Thermodynamically Stable Noncomposite Vortices in Mesoscopic Two-Gap Superconductors*, arXiv:cond-mat/0704.3325 v1, 2007.

[41] M.E. Zhitomirsky, V.H. Dao, A. Ceulemans *Ginzburg-Landau Theory of Vortices in a Multi-Gap Superconductor*, arXiv:cond-mat/0309372 v1, 2003.

[42] R. Cubitt, M.R. Eskildsen, ect. *Effects of Two-Band Superconductivity on the Flux-Line Lattice in Magnesium Diboride*, Phys. Rev. Lett., 91, no. 4, 047002, 2003.

[43] R.A. Adams, J.J. Fournier, *Sobolev Spaces*, 2nd. ed., Academic Press, 2003.

[44] V. Girault, P.A. Raviart, *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*, Springer Verlag, 1986.

[45] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, AMS 2001.

[46] J.L. Lions, *Quelques méthodes de résolution des problémes aux limites non linéaires*, Dunod, Paris, 2002.

[47] O.A. Ladyzhenskaia, *Linear and Quasilinear Equations of Parabolic Type*, AMS, 1968.

[48] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer Verlag, 1992.

[49] J.L. Lions, E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications I, II*, Springer Verlag, 1972.

[50] J. Wloka, *Partial Differential Equations*, Cambridge Univ. Press, 1987.

[51] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, 1985.

[52] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer Verlag, 1977.

[53] E. Zeidler, *Nonlinear Functional Analysis and its Applications II/A: Linear Monotone Operators*, Springer Verlag, 1990.

[54] G.P. Galdi, *Introduction to the Mathematical Theory of the Navier-Stokes Equations: Linearized Steady Problems*, Springer Verlag, 1994.

[55] G.P. Galdi, *Introduction to the Mathematical Theory of the Navier-Stokes Equations: Nonlinear Steady Problems* , Springer Verlag, 1994.

[56] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, SIAM, 2002.

[57] P. Monk, *Finite Element Methods for Maxwell's Equations*, Oxford Univ Press, 2003.

[58] W.T. Reid, *Ordinary Differential Equations*, Wiley, 1971.

[59] K.H. Hoffmann, Q. Tang, *Ginzburg-Landau Phase Transition Theory and Superconductivity*, Birkhauser Verlag, 2001.

[60] N.B. Kopnin, *Theory of Nonequilibrium Superconductivity*, Oxford Univ. Press, 2001.

[61] P.G. De Gennes, *Superconductivity of Metal and Alloys*, Westview Press, 1999.

[62] M. Tinkham, *Introduction to Superconductivity*, Dover, 2nd. ed., 2004.

[63] C.P. Poole Jr., H.A. Farach, R.J. Creswick, *Superconductivity*, Academic Press, 1995.

[64] A. Barone, *Physics and Applications of the Josephson effect*, Wiley, 1982.

[65] E.J. Rothwell, M.J. Cloud, *Electromagnetics*, CRC Press, 2001.

BIOGRAPHICAL SKETCH

W.K. Chan

W.K. Chan was born in a small town in Sabah, Malaysia on Borneo island. His father, together with his brother and sister, built their own long house, their own woodwork factory, and their own furniture. The house and the factory were built in a forest nearby a small airport which was located near a beach. So the factory, the forest, the airport and the beach had become W.K.'s daily playgrounds at his school ages. And because of this, he loved to design and make things like wood cages for snakes, birds, bats, and some animals he couldn't name. He was so happy in his own world he once told his parents that he hated school and never wanted to study anymore.

In senior high school, he surprised everyone, including himself, that he scored very well in a college entrance exam. So he was admitted into the National Chiao Tung University, Taiwan, majoring control engineering. It was in the college that he became interested in mathematics and hated control engineering. After graduation in 1987, owing to his design-for-fun of a very complicated IBM compatible computer system in college, he was chosen by Acer Computer Inc. to become a computer digital hardware design engineer. And that started his eleven-year of happy career at Acer. He met his wife at Acer, they married and had two children and lived happily... but he had never forgotten his passion for mathematics. When his finance was strong enough, with full support of his wife, he made his biggest decision ever- he quit his promising career and pursued his dreams in the US. He first earned a master degree in mathematics with honor at California State University, Long Beach. Then, he was accepted by Dr. Gunzburger into Iowa State University to pursue his doctoral study in applied mathematics. A year later, he followed Dr. Gunzburger to Florida State University and earned his Ph.D. degree in applied and computational mathematics in Fall 2007. He enjoys his life living with his wife, his two daughters and a cat. He hope someday with the help of mathematical tools, he can invent something that would help the world and earn enough money to help some needy people and animals in some countries.