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Z-Sum Approach to Loop Integrals

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Z-SUM APPROACH TO LOOP INTEGRALS

By

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To my family

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ABSTRACT

We study the applicability of the Z-Sum approach to multi-loop calculations with massive particles in perturbative quantum field theory. We systematically analyze the case of one-loop scalar integrals, which represent the building blocks of any higher-loop calculation. We focus in particular on triangle one-loop integrals and identify strengths and limitations of the Z-Sum approach, extending our results to the case of one-loop box integrals when appropriate. We conclude with the calculation of a specific physical example: the calculation of heavy flavor corrections to the renormalized scattering amplitude for deep inelastic scattering.

CHAPTER 1

INTRODUCTION

Particle physics is the study of the most fundamental particles and their interactions. While humanity has for millennia wondered and developed theories about the constituents of matter, it has been only in the last few centuries that a real scientific approach has produced concrete results in the field. This progress ultimately led to the development in the last century of what is called Quantum Field Theory (QFT), the combination of quantum mechanics and special relativity. Within that framework, the Standard Model (SM) has been established as the most successful description of particle physics at the energy scales so far explored [3, 4, 5, 6, 7]. It is the result of a phenomenal effort, combining over the years an enormous amount of experimental and theoretical work. Its results have been shown to be extremely accurate, having been confirmed by experiments to an impressive degree of precision. It allowed for the prediction of several particles, among others the Z^0, W^+, W^- bosons, and top quark before their experimental discovery. One final prediction of the SM is the existence of a particle called the Higgs Boson which has not yet been observed.

While the SM has been well established at energy scales of approximately a few hundred GeV, few physicists believe it to be the ultimate theory. More likely it is an effective theory good up to some energy value, after which a larger theory takes over. The reasons for that are several. The most obvious may be the fact that gravity is simply not accounted for. Another comes from cosmology, where recent discoveries lead to the belief that known baryonic and radiation matter account only for about four percent of the energy in the universe, the rest being made up of dark energy and dark matter (DM), with the SM lacking a DM candidate. The SM also cannot fully account for the matter-antimatter asymmetry, the fact that we detect a lot more matter than anti-matter in the universe. Moreover, the hierarchy problem, the unexplained wide range of fundamental energies in the theory and the fine tuning required for large cancellations leading to observable quantities, is also seen as an indication of the limits of the theory.

The potential experimental discovery of the Higgs boson, and the expectation of new physics beyond the SM at the TeV scale are the main drive for the operation of current particle colliders: the Tevatron, a proton/anti-proton collider with 1.96 TeV center of mass energy located at Fermilab, IL, and the Large Hadron Collider (LHC), a proton/proton collider with 7 to 14 TeV center of mass energy located at CERN, Switzerland [8]. Experiments at these institutions are focused on the discovery of new particles and detection of inconsistencies between SM predictions and experimental values for observable quantities, requiring an unprecedented level of precision for theoretical predictions.

In this context the study of new methods that may allow to reach higher level of theoretical accuracy is extremely important, and will be the focus of this dissertation.

1.1 Standard Model: Particle Content and Interactions

The SM describes elementary particles and their interactions. Each particle has its own characteristics, including quantum numbers and mass, on which its interactions depend. One of the properties particles have is called spin. If a particle has half integer spin (in units of $\hbar$) it is called a fermion, and follows the Fermi-Dirac statistics, while if it has integer spin it is called a boson, and follows Bose-Einstein statistics. Matter particles are fermions and are divided into leptons and quarks. They come in three generations, with the first containing stable particles, and the second and third generations containing more massive unstable copies of the particles in the first generation. Within the experimental precision, quarks and leptons are considered indivisible, or elementary. All fermion particles also have a twin called anti-particle which has the same mass but opposite charges or quantum numbers (indicated by symbols with a bar e.g. $\bar{q}$). In the following we will cover a very qualitative description of their properties.

Table 1.1: Fermions in the Standard Model. Mass values are approximate only [2].

Leptons			Quarks		
Flavor	Mass (GeV/c^2)	Electric Charge	Flavor	Mass (GeV/c^2)	Electric Charge
ν_e	0	0	u	0.003	2/3
e	0.000511	-1	d	0.005	$-1/3$
ν_μ	0	0	c	1.3	2/3
μ	0.106	-1	s	0.1	$-1/3$
ν_τ	0	0	t	172	2/3
τ	1.777	-1	b	4.3	$-1/3$

Leptons. The first lepton generation is made of the electron (e) and the electron neutrino (ν_e). The next two generations are made of the muon (μ), muon neutrino (ν_μ), tau (τ) and tau neutrino (ν_τ). Muons and taus are not stable and that is why common matter only involves electrons, which are stable. Neutrinos were until recently believed to be massless (or more precisely stated, experimental mass measurements only yielded an upper bound), but the recent discovery of neutrino oscillations implies that at least two of them are not massless.

Quarks. The first quark generation is made of the up and down quark. They bind together into compound particles called protons (uud) and neutrons (udd) which, in addition to the electron, form all elements of the periodic table. The second and third generations include the charm, strange, top and bottom quarks. Besides protons and neutrons, quarks can combine into several more exotic and unstable particles.

Interactions. All interactions in the Standard Model occur through the exchange of force carrier particles, which are bosons. The Standard Model describes three interactions, Strong, Weak and Electromagnetic, with the last two combined into what is called Electroweak interactions. The Strong force is what binds nuclei together, and the Weak force is for instance responsible for some slow nuclear decays. The gravitational force is not included for two reasons. Although very obvious in our macroscopic world, it is much weaker than the other forces on smaller scales. The second reason is technical and involves difficulties in combining general relativity with quantum mechanics. It is believed, however, that any fundamental theory of particle physics will have to include gravitation.

Electromagnetism. The force carrier particle for the electromagnetic force is called photon. Because photons are massless, the range of interactions for the electromagnetic force is not limited, meaning we can measure it as a macroscopic force. A photon can interact with any particle that carries electromagnetic charge. In the lepton sector, electron, muon and tau particles carry an integer electromagnetic charge, while neutrinos are neutral. Quarks have fractional electromagnetic charge, with the up, charm and top quarks carrying +2/3 and down, strange and bottom carrying -1/3. Since it does not carry charge itself, a photon cannot directly interact with another photon. In the QFT framework, the theory that describes the electromagnetic force is called Quantum Electrodynamics (QED).

Table 1.2: Interactions in the Standard Model plus gravity.

		Electroweak		Strong	
	Gravity	Weak	Electromagnetic	Fundamental	Residual
Acts on:	Mass, energy	Flavor	Electric charge	Color charge	Residual color charge
Particles Experiencing	All	Quarks, Leptons	Electrically charged	Quarks, Gluons	Hadrons
Particles Mediating	Graviton	W^+, W^-, Z^0	γ (Photon)	Gluons	Mesons

Strong. The strong interaction is mediated thorough the exchange of particles called gluons. Gluons interact with particles that carry a quantum number called "color charge". This charge can have three different colors and their anti-colors. Quarks carry a color or anti-color, and gluons themselves carry a color and an anti-color, which means a gluon can directly interact with other gluons. Because the strong interaction becomes more intense as particles are pulled apart, colorful states cannot be observed, a feature called "color confinement". Only bound states with zero net color can be detected, meaning we cannot directly observe a quark or gluon, which makes the strong interaction short ranged even though the gluon is massless. In the QFT framework, the theory that describes the strong force is called Quantum Chromodynamics (QCD).

WEAK. Finally, the last interaction described by the SM is the weak interaction. The weak interaction has three force carriers, Z^0, W^+ and W^-. The last two carry an electric charge so they interact electromagnetically, while the Z^0 boson is neutral. All fermions predicted by the SM interact weakly, and the Z^0, W^+ and W^- interact with each other. The QFT description of the weak force unifies it with the electromagnetic force in what is known as the Electroweak theory (EW). One of the untested predictions of the EW theory is the existence of a particle called Higgs boson.

The $W^\pm$ and Z^0 carriers are very massive and this makes the weak interaction short ranged, as well as intrinsically different from electromagnetic and strong ones. In QFT, the properties of different interactions are directly related to the symmetries of a physical system. While electromagnetic and strong interactions correspond to exact symmetries, weak interactions arise from a broken symmetry. The existence of the Higgs particle directly relates to the breaking of the electroweak symmetry.

1.2 From Lagrangians to Feynman Rules

While the last section covers some of the basic qualitative aspects of the Standard Model, it does not discuss its formal structure nor how calculations are performed. We will try to clarify this in the current section, although only briefly.

The Standard Model is a quantum field theory based on a Lorentz invariant Lagrangian, where the basic dynamical degrees of freedom are fields ($\Phi_i(x_j)$, where x_j represent position in space-time). Particles arise as components of these fields in momentum space which satisfy the fundamental dispersion relation $p^2 = m^2$ (in units of $c = \hbar = 1$). A Lagrangian may display global and local symmetries, the latter involving transformations dependent on x_j that leave the action invariant. Furthermore, a symmetry may be apparent above a given energy but not below. In that case, it is called a broken symmetry and one must find a mechanism to introduce such behavior into the Lagrangian. Interactions are not included in an ad hoc manner, but arise dynamically from the imposition of symmetries in the Lagrangian. In particular, local symmetries and their realization (exact or approximate) establish the nature of the interactions.

The local symmetry of the SM is formed by the group of transformations $SU(3)_C \times SU(2)_L \times U(1)_Y$. The first is the symmetry group of QCD, with the C subscript referring to the color charge, and is left unbroken. The last two symmetry groups represent the Electroweak force, where the L subscript refers to the fact that gauge fields couple only to left handed particles, while the Y denotes to the hypercharge, a conserved quantum number related to electric charge and the third component of the weak isospin ($Q = T_3 + \frac{Y}{2}$). At energies of about 200 GeV, the Electroweak symmetry is broken and gives rise to the weak and electromagnetic interactions.

The EW breaking is realized in the SM Lagrangian through the Higgs mechanism, chosen because of its simplicity and minimal requirements of extra degrees of freedom. In it, a new complex scalar field is introduced in the Lagrangian with a non-trivial potential causing it to acquire a non-zero expectation value. Once the Lagrangian is expanded about the non-zero minimum of the potential, the initial generators of $SU(2)_L \times U(1)_Y$ are combined into three massive bosons, Z^0, W^+, W^-, and one massless one, the photon, while fermions acquire mass through arbitrary Yukawa-type interactions. Only one new physical degree of freedom is left in the Lagrangian as the Higgs boson, yet to be discovered.

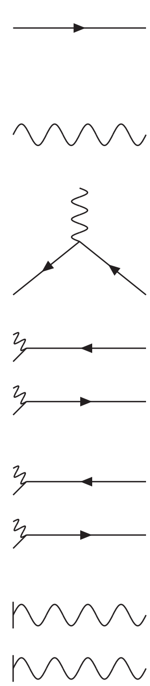
Experimental Observables. Experiments in high energy physics are performed by colliding particles and detecting the products of such reactions, thus obtaining a normalized distribution of probabilities for every possible final state. By comparing these results with theoretical predictions it is possible to test whether a specific model is a good description of nature.

Theoretical Predictions. Theoretical predictions for various physical observables are obtained from the Lagrangian using fundamental QFT techniques, allowing for the calculation of cross-section which are related to the probabilities for every state. Whenever the interaction coupling is small, calculations in QFT are performed perturbatively, that is, by expanding expressions in powers of the coupling, with more accurate results being obtained as more terms are included in the calculation. The common practice is to refer to the contributing powers as leading-order (LO), next-to-leading (NLO), next-to-next-to-leading order (NNLO or N^2LO) and so on. In the SM, the perturbative approach is applicable for QED, weak interactions, and QCD at high energies, which is particularly important since QCD has the most dominant effect at hadron colliders. To make a clear distinction between the low energy regime of QCD, applicable to when quarks and gluons are bound into hadrons, and the high energy regime, where particles behave almost as if free, we refer to the latter as perturbative QCD (pQCD).

To understand the kind of technical difficulties encountered in the higher-order calculations, let us illustrate the typical calculation of a scattering amplitude, that is, the quantum amplitude associated to the transitions from a given initial state to a given final state. The square of this amplitude, integrated over phase space, will give the cross section. A very powerful technique to calculate scattering amplitudes is to represent them in terms of Feynman diagrams, which are directly derived from the Lagrangian. There are three types of building blocks used in Feynman diagrams: propagators, vertices and external legs. An example is given in fig. 1.1 for the case of QED. Each of these correspond to a mathematical expression with a specific power of the expansion parameter. These are built together to create a diagram with the desired initial and final states, and by considering all possible diagrams up to a given order one can simply read the mathematical expression corresponding to the amplitude by using the Feynman rules.

For a given process with given initial and final states the lowest order scattering amplitude usually corresponds to a "tree diagram" (i.e. a diagram with no internal loops) with the given external legs. The next order corrections to the scattering amplitude corresponds to diagrams with one internal loop of particles, and so on. Each loop corresponds analytically to an integration over four-momentum of the particles in the loop. The true difficulty arises from the computation of these integrals, with complexity increasing as more external legs, massive internal particles, and number of loops are involved.

$$\text{Quantum Electrodynamics: } \mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \tfrac{1}{4}(F_{\mu\nu})^2 - e\bar{\psi}\gamma^\mu\psi A_\mu$$



Dirac propagator:	$= \frac{i(\not{p}+m)}{p^2-m^2+i\lambda}$
Photon propagator:	$= \frac{-ig_{\mu\nu}}{p^2+i\lambda}$
QED vertex:	$= ie\gamma^\mu$
External fermions:	$= u^s(p)$ (initial)
	$= \bar{u}^s(p)$ (final)
External antifermions:	$= v^s(p)$ (final)
	$= \bar{v}^s(p)$ (initial)
External photons:	$= \epsilon_\mu(p)$ (initial)
	$= \epsilon^*_\mu(p)$ (final)

Figure 1.1: Feynman Rules for QED using the Feynman gauge.

Finally it is important to mention that although the Feynman diagram approach is by far the most common it is not the only option. When calculating an amplitude, several different Feynman diagrams may lead to large cancellations, and so other methods have been developed in an attempt to avoid numerical instabilities and make the calculation more efficient. Whether using these new methods (generalized unitarity methods and the like) or the more traditional Feynman diagram approach, the final result will always depend on solving scalar loop integrals, and while some of these can be solved numerically, a systematic analytical approach to solving these integrals is desirable.

1.3 Z-Sums

The method discussed in this dissertation will introduce functions called Z-Sums. Z-Sums are composed of concatenated sums with a particular structure and are useful because they generalize several other important functions including multiple polylogarithms. Furthermore, they form a Hopf algebra including several operations like a product, convolution, and conjugation, which are instrumental in simplifying complicated summations.

Loop integrals can be expressed in terms of concatenated sums, and there are different method to achieving that result. In some situations, it is a known fact that these summations match the structure of the Z-Sum algebra, and so the Z-Sum approach is interesting because it may be used as a systematic procedure to express the initial integrals in terms of multiple polylogarithms.

The Z-Sum method has been studied in detail [9, 10] and has been applied to calculations involving up to two-loop diagrams, both massless [11, 12, 13, 14] and massive [15, 16, 17, 18, 19]. Until now, however, a full study of the applicability to any loop integration had never been performed, and that is one of the contributions of this work. In addition, we will for the first time present results for massive two-loop integrations using Taylor series as the expansion method.

Hopf algebras also seem to naturally emerge in the process of renormalization of a QFT (see [20] and references therein). While it is possible that there is a connection between the two, since loop diagrams are involved in the renormalization process, the relation is not well understood.

1.4 Layout

The layout of this work is as follow. In the first chapter we covered a basic qualitative description of High Energy Physics and the Standard Model with its successes and limitations, with the intent to motivate and justify the need to perform loop calculations.

Chapter 2 covers the standard steps involved in loop calculations, including regularization, and momentum integration of scalar integrals, which once performed lead to specific parametric integrations. We proceed with the discussion on reducing loops with more than four legs to bubbles, triangles and boxes, and the procedure to express tensor integrals in terms of scalar ones. The chapter ends with a discussion of different approaches to dealing with multi-loop diagrams, including the traditional method and Mellin-Barnes splitting method, emphasizing that single loops are used as building blocks for multiloop calculations. The main point of the chapter is that no matter what specific QFT model is being used or which diagram is being performed, we will always need to perform non trivial parametric integrations.

Chapter 3 covers a general parametric integration, with focus on developing a systematic procedure. We start with a study on different options in expanding the integrand using Taylor series, followed by the parameter integration, and finally we obtain a general expression. We briefly compare the Taylor expansion with an expansion performed using the inverse Mellin-Barnes transformation, but choose to focus our attention on the former since it is a simpler procedure. After the general expression has been obtained, we proceed to expand it in powers of the infinitesimal variable ε, introduced by the regularization procedure. This expansion will involve functions called Z and S-Sums, which are then studied in detail including definitions and properties. The purpose of this is to find a systematic method to reduce the expressions obtained from the Taylor series into pure Z-Sums and eventually into multiple polylogarithms, which are a generalization of the standard polylogarithm functions. With that in mind, we end the chapter listing all known algorithms available for such a reduction.

In chapter 4 we use the algorithms listed in the previous chapter and, in reverse order, obtain all possible loop-like integrations that would lead to a seamless reduction all the way from integrations to pure Z-Sums. We then cross check these integrals with the ones obtained from all single-loop triangle integrals that occur in the Standard Model and more generally in any Quantum Field Theory in order to establish how often the method using Taylor expansion and the available Z-Sum machinery is applicable. As we will find out, not every calculation will be feasible with such a procedure, and so we finish the chapter with a discussion on what new algorithms in the Z-Sum machinery would need to be developed in order for it to be always applicable.

We conclude in chapter 5 by applying the method to specific physically motivated calculations involving two loop massive integrals, necessary for instance for the calculation of heavy flavor coefficient functions in deep inelastic scattering.

CHAPTER 2

FEYNMAN LOOP DIAGRAMS

In the last chapter we motivated why accounting for higher order corrections in the theoretical predictions of physical observables is crucial to fully exploit the discovery potential of both the Tevatron and the LHC. In this chapter we start introducing the formal aspects of loop integrations which appear at higher order in perturbative QFT (pQFT). In particular, we will focus on the steps required to reduce the initial expression obtained from a Feynman diagram with internal loops to a scalar integration [21]. Most of these initial steps are standard and used regardless of the method applied to perform the scalar integrations.

As an example we will use the one-loop Feynman diagram shown in fig. 2.1. This diagram represents a correction to the cross section for $e^+e^- \to 3$ jets via the sub-channel $e^+e^- \to qg\bar{q}$, where $q(\bar{q})$ is a light quark (anti-quark) and g is a gluon.

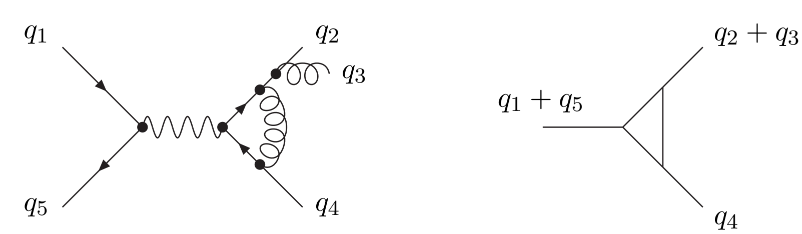


Figure 2.1: A Feynman diagram contributing to the process $e^+e^- \to qg\bar{q}$ and the associated topology.

Using the appropriate QCD and QED Feynman rules [3, 4] we can write the scattering amplitude corresponding to this diagram as:

$$\bar{v}(q_5)\gamma^\mu u(q_1)\frac{1}{q_{234}^2}\int\frac{d^4k_1}{(2\pi)^4}\frac{1}{k_2^2}\bar{u}(q_2)\not\epsilon(q_3)\frac{\not q_{23}}{q_{23}^2}\gamma_\nu\frac{\not k_1}{k_1^2}\gamma_\mu\frac{\not k_3}{k_3^2}\gamma^\nu v(q_4), \tag{2.1}$$

where we have neglected an overall factor of $iQe^2g^3t^at^bt^c$ containing both couplings and color structure. In Eq. (2.1) v, $\bar{v}$, u, $\bar{u}$ are Dirac spinors associated with the incoming and outgoing particles, γ^μ are the Dirac γ matrices and $\varepsilon^\lambda(q_3)$ is the polarization vector of the final state gluon. $\not{q}_i$ denotes the contraction of the four momentum q_i^μ with γ_μ, i.e. $\not{q}_i = q_i^\mu\gamma_\mu$. Moreover, $q_{23} = q_2 + q_3$, $q_{234} = q_2 + q_3 + q_4$, $k_2 = k_1 - q_{23}$, $k_3 = k_2 - q_4$, k_1 is the loop momentum, and all external momenta are on-shell, i.e. $q_i^2 = m_i^2 = 0$ for $i = 1, .., 5$ (light quarks are assumed to be massless).

The integral to be evaluated in Eq. (2.1),

$$\int \frac{d^4k_1}{(2\pi)^4}\frac{k_1^\rho k_3^\sigma}{k_1^2k_2^2k_3^2} \quad , \tag{2.2}$$

is however an ill defined mathematical expression as it diverges in the limit $k_1^2 \to \infty$.

The appearance of singularities in the momentum integration of a scattering amplitude is a well known phenomenon in QFT. Divergences appearing in the ultraviolet (UV) region of momentum, $k_1^2 \to \infty$, like the one encountered in eq. 2.2, are absorbed into the definition of physical couplings, masses and fields (possible in renormalizable theories like the SM); while divergences appearing in the infrared (IR) region of momenta ($k_1^2 \to 0$) cancel when virtual (or loop) corrections and real (or radiative) corrections are combined order by order perturbatively. To proceed with the integration, we need therefore to introduce a regularization procedure.

2.1 Regularization

In order to deal with UV or IR divergencies and calculate the integral one introduces a regulator, for which several different methods exist. Most modern QFT calculations use "dimensional regularization" [3, 4, 21] where the four-dimensional integration is replaced by a D-dimensional integration with $D = 4 - 2\varepsilon$ where ε is an infinitesimal, such that the original theory is regained in the limit $D \to 4$. All vectors in the integration are also assumed to be D dimensional with

$$k_D^\mu = k_4^\mu + k_{D-4}^\mu \tag{2.3}$$

and

$$k_{D-4} \cdot k_4 = 0 \ . \tag{2.4}$$

Dimensional regularization works not only around dimension four and D can be taken to be:

$$D = 2m - 2\varepsilon \ , \tag{2.5}$$

where m is an integer, ε is infinitesimal, and $2m$ is the original dimensionality of the integration.

The infinities in the initial integration will appear as poles in $1/\varepsilon$ when the result is expanded in powers of ε, and so can be handled analytically. The limit $\varepsilon \to 0$ is taken only when all poles have been canceled out and a finite and physical result is obtained. Equation 2.2 now becomes:

$$\mu^{2\varepsilon} \int \frac{d^{4-2\varepsilon}k_1}{(2\pi)^{4-2\varepsilon}} \frac{k_1^\rho k_3^\sigma}{k_1^2 k_2^2 k_3^2} \ , \tag{2.6}$$

which is a well defined object, where μ is a mass scale and the prefactor $\mu^{2\varepsilon}$ has been added to keep the dimensionality unaltered. From here on, we will not be including this factor for simplicity, but its presence is implied.

While all non-scalar objects within the integration are changed to D dimensions, there are different "schemes" for dealing with outside tensors, and they differ on how they treat momenta and polarization vectors of observed and unobserved particles, with the option of keeping all in four dimensions (with the exception of internal loop momenta), changing all to D dimensions or even keeping some in four dimensions and changing some to D dimensions. The choice of scheme affects the intermediate results, but in the limit $D \to 4$ all physical results are independent of the adopted regularization scheme. In our case, since we are simply focusing on the integration itself, the choice is not important as all schemes treat internal loop momenta in D dimensions.

In the same spirit, we will work with individual building blocks (integrals), not the full calculation, and so our results will still depend on the infinitesimal, usually being presented ordered by powers of ε.

2.2 One-Loop Scalar Integrals

Integrals such as the one in eq. 2.6 are called a "tensor integrals" because they contain factors of loop momentum with free Lorentz indices. When no such factors are present, we refer to them as "scalar integrals".

NOTATION. In this work we will be using the following notation for tensor and scalar loop integrations:

$$\begin{aligned} A_0 &= \int \frac{d^D k}{(2\pi)^D} \frac{1}{P_1} \ , \\ B_{0,\mu,\mu\nu,\ldots} &= \int \frac{d^D k}{(2\pi)^D} \frac{1, k_\mu, k_\mu k_\nu, \ldots}{P_1 P_2} \ , \\ C_{0,\mu,\mu\nu,\ldots} &= \int \frac{d^D k}{(2\pi)^D} \frac{1, k_\mu, k_\mu k_\nu, \ldots}{P_1 P_2 P_3} \ , \\ D_{0,\mu,\mu\nu,\ldots} &= \int \frac{d^D k}{(2\pi)^D} \frac{1, k_\mu, k_\mu k_\nu, \ldots}{P_1 P_2 P_3 P_4} \ , \end{aligned} \tag{2.7}$$

where the indexes refer to the tensor structure of the integral (with 0 for scalars) and can be further generalized to include more propagators. The propagators P_i are given by:

$$P_i = ((k - a_i)^2 - m_i^2 + i\lambda) \quad , \tag{2.8}$$

where a_i are vectors related to the momenta entering the loop. While we could shift the integration variable to absorb one of the a_i in every integration we prefer to leave it in a more symmetric form. The propagators in equation 2.8 include an infinitesimal complex term $i\lambda$ added to avoid poles whenever $(k - a_i)^2 = m_i^2$. We might sometimes omit this term but its presence should be implied. Integrals of type B_0, C_0, and D_0 are referred as bubbles, triangles and boxes, and their topologies are shown in fig. 2.2.

2.2.1 Momentum Integration

In this section we will discuss the steps required to perform a scalar momentum integration in diagrams involving two, three and four propagators, with the final result being a parameter integration. These integrals are very important not only by themselves, but also because they are used to express more complicated integrations. As we will see later in this

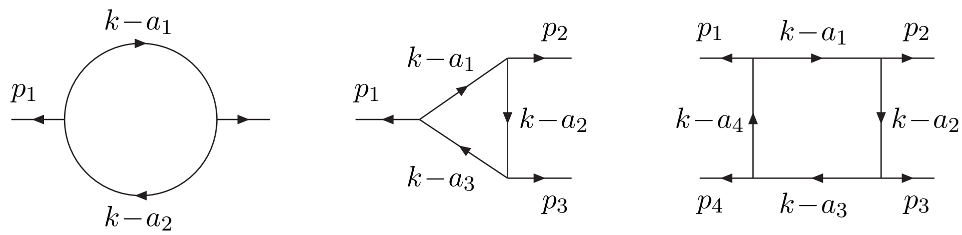


Figure 2.2: One-loop topology for bubbles (B_0), triangles (C_0), and boxes (D_0). Arrows show momentum direction. Arrows show momentum direction, and four-momentum is conserved in every vertex.

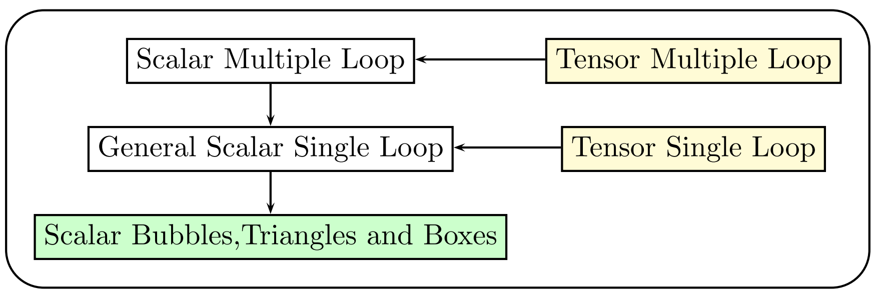


Figure 2.3: Simplification path for momentum integrations. Arrows indicate the origin can be expressed in terms of integrals of the destination type.

chapter, tensor integrals can always be reduced to scalar ones (section 2.3), and one-loop scalar integrations involving more than four propagators can be expressed in terms of bubbles, triangles and boxes (subsection 2.2.2). Furthermore, scalar multi-loop calculations are always performed using single-loop calculations as building blocks (section 2.4). In practice, these are the only momentum integrations that are ever performed.

In order to illustrate the required steps for the momentum integration we use a C_0 scalar triangle loop integration (fig. 2.2), given by:

$$C_0 = \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left((k-a_1)^2 - m_1^2\right)^{\nu_1} \left((k-a_2)^2 - m_2^2\right)^{\nu_2} \left((k-a_3)^2 - m_3^2\right)^{\nu_3}} \,, \tag{2.9}$$

where the powers in the propagators ν_i can be different from one if this integration is arising from the reduction from tensor integrals (in which case it might involve an integer larger than one (section 2.3)), or if this loop is a block within more loops (in which case it would involve an infinitesimal ε (subsection 2.4.1) or even a complex integration variable (subsection 2.4.2)).

Parametrization. The first step is called parametrization and will replace the denominators P_i arising from the propagators with expressions more suitable for momentum integration. The cost of this step is the introduction of new parameter integrations. There are two types of parametrizations, introduced by Schwinger and Feynman. The Schwinger parametrization utilizes the exponential function in the form:

$$\frac{1}{P^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty dx x^{\nu-1} \exp(-xP) \ , \tag{2.10}$$

while the parametrization introduced by Feynman, in its most general form, is given by:

$$\frac{1}{P_1^{\nu_1} P_2^{\nu_2} \dots P_n^{\nu_n}} = \int_0^1 dx_1 \dots dx_n \ \delta\Big(\sum x_i - 1\Big) \frac{\prod x_i^{\nu_i - 1}}{(\sum x_i P_i)^{\sum \nu_i}} \frac{\Gamma(\nu_1 + \dots + \nu_n)}{\Gamma(\nu_1) \dots \Gamma(\nu_n)} \quad , \tag{2.11}$$

where $\Gamma(x)$ is the gamma function. The definition and a few properties of the gamma function can be found in appendix B.

The Feynman parametrization is used to replace the product of propagators with a single polynomial in k, and will be applied in the following. Equation 2.9 becomes:

$$\frac{\Gamma(\nu_{123})}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)} \int \frac{d^D k}{(2\pi)^D} \times$$
$$\int_0^1 \frac{dx_1 \ dx_2 \ dx_3 \ \delta(1 - x_1 - x_2 - x_3) \ x_1^{\nu_1 - 1} \ x_2^{\nu_2 - 1} \ x_3^{\nu_3 - 1}}{\Big[x_1 \left((k - a_1)^2 - m_1^2\right) + x_2 \left((k - a_2)^2 - m_2^2\right) + x_3 \left((k - a_3)^2 - m_3^2\right)\Big]^{\nu_{123}}} \ , \tag{2.12}$$

Momentum Shifting. Next the integration momentum in equation 2.12 is shifted in order to complete the square in the denominator. Defining:

$$p^\mu = k^\mu - x_1 \ a_1^\mu - x_2 \ a_2^\mu - x_3 \ a_3^\mu \tag{2.13}$$

and

$$\Delta = -x_1 x_2 \ p_2^2 - x_1 x_3 \ p_1^2 - x_2 x_3 \ p_3^2 + \left(x_1 \ m_1^2 + x_2 \ m_2^2 + x_3 \ m_3^2\right) \tag{2.14}$$

we can re-express the integrand's denominator as:

$$x_1 \left((k - a_1)^2 - m_1^2\right) + x_2 \left((k - a_2)^2 - m_2^2\right) + x_3 \left((k - a_3)^2 - m_3^2\right) \equiv p^2 - \Delta \ , \tag{2.15}$$

where p_1, p_2, and p_3 are the momenta leaving the loop, as shown in figure 2.2, and we have used

$$
\begin{aligned}
a_1 \cdot a_2 &= \frac{a_1^2 + a_2^2 - p_2^2}{2} \,, \\
a_1 \cdot a_3 &= \frac{a_1^2 + a_3^2 - p_1^2}{2} \,, \\
a_1 \cdot a_2 &= \frac{a_2^2 + a_3^2 - p_3^2}{2} \,,
\end{aligned}
\tag{2.16}
$$

to perform the simplification. The equivalence sign was used in equation 2.15 because we used the Dirac delta condition $x_1 + x_2 + x_3 = 1$ which is valid only inside the integration.

Finally this brings equation 2.12 to the form:

$$
\begin{aligned}
&\frac{\Gamma(\nu_{123})}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)} \int \frac{d^D p}{(2\pi)^D} \times \\
&\int_0^1 \frac{dx_1\ dx_2\ dx_3\ \delta(1 - x_1 - x_2 - x_3)\ \ x_1^{\nu_1 - 1}\ x_2^{\nu_2 - 1}\ x_3^{\nu_3 - 1}}{(p^2 - \Delta)^{\nu_{123}}} \,.
\end{aligned}
\tag{2.17}
$$

The same procedure can be done in loops with a different number of external legs. For bubbles (fig. 2.2) we have:

$$
\begin{aligned}
p^\mu &= k^\mu - x_1\ a_1^\mu - x_2\ a_2^\mu \,, \\
\Delta &= -x_1 x_2\ p_1^2 + \left(x_1\ m_1^2 + x_2\ m_2^2\right) \,,
\end{aligned}
\tag{2.18}
$$

and for boxes (fig. 2.2):

$$
\begin{aligned}
p^\mu &= k^\mu - x_1\ a_1^\mu - x_2\ a_2^\mu - x_3\ a_3^\mu - x_4\ a_4^\mu \,, \\
\Delta &= -x_1 x_2\ p_2^2 - x_1 x_3\ \ (p_2 + p_3)^2 - x_1 x_4\ p_1^2 - x_2 x_3\ p_3^2 \\
&\quad -x_2 x_4\ \ (p_1 + p_2)^2 - x_3 x_4\ p_4^2 + \left(x_1\ m_1^2 + x_2\ m_2^2 + x_3\ m_3^2 + x_4\ m_4^2\right) \,,
\end{aligned}
\tag{2.19}
$$

where the momenta p_1, p_2, p_3, and p_4 are defined in figure 2.2.

Wick Rotation. After shifting the momentum integration variable we perform a Wick rotation, which will turn Minkowski metric into Euclidean.

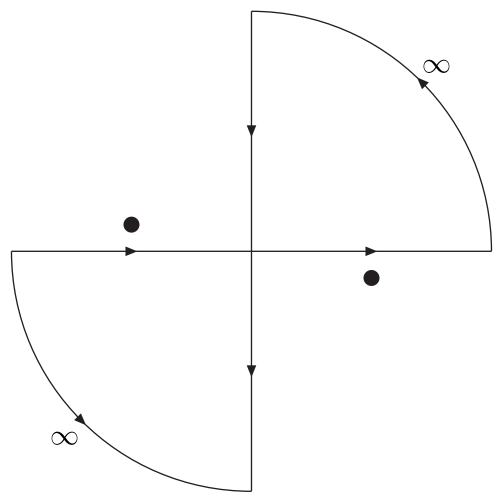


Figure 2.4: Contour for the p^0 integration in the complex plane showing poles shifted outside.

Using the fact that the integrand falls off rapidly enough at infinity, we complete the p_0 integration contour as shown in figure 2.4. Since the poles have been shifted outside of the contour this integration yields:

$$\oint dp_0 \; f(p_0) = 0 \; , \tag{2.20}$$

which can be rewritten as

$$\int_{-\infty}^{\infty} dp_0 \; f(p_0) = -\int_{i\infty}^{-i\infty} dp_0 \; f(p_0) \; . \tag{2.21}$$

By making a change of integration variable,

$$\begin{aligned} p_0 &= iK_0 \quad , \\ p_{1,2,3} &= K_{1,2,3} \quad , \end{aligned} \tag{2.22}$$

we finally obtain

$$\int d^D p \; f(p^2) = i \int d^D K \; f(-K^2) \quad , \tag{2.23}$$

which lead equation 2.17 to the form

$$i\ (-1)^{-\nu_{123}} \frac{\Gamma(\nu_{123})}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)} \int \frac{d^D K}{(2\pi)^D} \times$$
$$\int_0^1 \frac{dx_1\ dx_2\ dx_3\ \delta(1-x_1-x_2-x_3)\ x_1^{\nu_1-1}\ x_2^{\nu_2-1}\ x_3^{\nu_3-1}}{(K^2+\Delta)^{\nu_{123}}} \ . \tag{2.24}$$

Angular Integration. Now that we are in Euclidean metric and all K dependence is through K^2 we can easily perform the angular integration using spherical coordinates. In integer D dimensions the equations relating spherical coordinates to Cartesian are given by:

$$\begin{aligned} K_0 &= K\cos\theta_1\ , \\ K_1 &= K\sin\theta_1\cos\theta_2\ , \\ &\dots \\ K_{D-2} &= K\sin\theta_1\dots\sin\theta_{D-2}\cos\theta_{D-1}\ , \\ K_{D-1} &= K\sin\theta_1\dots\sin\theta_{D-2}\sin\theta_{D-1}\ , \end{aligned} \tag{2.25}$$

where K is the radial variable, θ_{D-1} is an azimuthal angle (ranging from 0 to 2π radians), and all other θ_i are polar angles (ranging from 0 to π radians).

The momentum integration in equation 2.24 is given by:

$$\int \frac{d^D K}{(2\pi)^{\frac{D}{2}}} \frac{1}{(K^2+\Delta)^{\nu_{123}}} \tag{2.26}$$

where

$$d^D K = K^{D-1} dK \prod_{i=1}^{D-1} \sin^{D-1-i}\theta_i d\theta_i \quad . \tag{2.27}$$

The angular integration gives:

$$\int_0^\pi d\theta_1 \sin^{D-2}\theta_1 \dots \int_0^\pi d\theta_{D-2}\sin\theta_{D-2} \int_0^{2\pi} d\theta_{D-1} = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \ . \tag{2.28}$$

While the last few equations were performed using D as an integer dimension, the right side of equation 2.28 is valid for any D by analytical continuation.

Finally, eq. 2.24 becomes:

$$\frac{2\;i\;(-1)^{-\nu_{123}}}{(4\pi)^{\frac{D}{2}}\Gamma(\frac{D}{2})}\frac{\Gamma(\nu_{123})}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)}\int dK\;K^{D-1}\times$$
$$\int_0^1 \frac{dx_1\;dx_2\;dx_3\;\delta(1-x_1-x_2-x_3)\;\,x_1^{\nu_1-1}\;\,x_2^{\nu_2-1}\;\,x_3^{\nu_3-1}}{(K^2+\Delta)^{\nu_{123}}}\;. \tag{2.29}$$

Radial Integration. Ignoring for the moment the prefactors and the parameter integration in equation 2.29, the last part of the momentum integration, the radial part, is given by:

$$\int \frac{dK\;K^{D-1}}{(K^2+\Delta)^{\nu_{123}}}\quad , \tag{2.30}$$

which can be rewritten as

$$\frac{\Delta^{\frac{D}{2}-\nu_{123}}}{2}\int_0^\infty dz\frac{z^{\frac{D-2}{2}}}{(1+z)^{\nu_{123}}}\quad , \tag{2.31}$$

with the transformation $z=\frac{K^2}{\Delta}$. The integral in equation 2.31 is one of the definitions of the beta function (appendix B, equation B.15), and it evaluates to:

$$\frac{\Delta^{\frac{D}{2}-\nu_{123}}}{2}\frac{\Gamma\left(\frac{D}{2}\right)\Gamma\left(\nu_{123}-\frac{D}{2}\right)}{\Gamma(\nu_{123})}\;. \tag{2.32}$$

Finally, this brings equation 2.29 to the form:

$$C_0(\nu_1,\nu_2,\nu_3)=\frac{i\;(-1)^{\nu_{123}}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma\left(\nu_{123}-\frac{D}{2}\right)}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)}\times$$
$$\int_0^1 \frac{dx_1\;dx_2\;dx_3\;\delta(1-x_1-x_2-x_3)\;\,x_1^{\nu_1-1}\;\,x_2^{\nu_2-1}\;\,x_3^{\nu_3-1}}{\Delta^{\nu_{123}-\frac{D}{2}}}\;, \tag{2.33}$$

which concludes the momentum integration, leaving the parameter integrals to be dealt with. The reasoning for performing all these steps only to substitute an integration over momenta for another one over Feynman parameters is justified by the fact that the latter involves multiple integrals over scalars instead of D-dimension vectors.

The sequence of steps required to obtain equation 2.33 are systematic and can be used in different situations. The same techniques may be applied to integrals involving powers of k^2 in the numerators, and also to calculations involving bubbles or boxes. These results can be summarized in the following set of equations:

$$\begin{aligned}
\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2-\Delta)^n} &= \frac{(-1)^n\, i}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(n-\frac{D}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{D}{2}} \quad , \\
\int \frac{d^D k}{(2\pi)^D} \frac{k^2}{(k^2-\Delta)^n} &= \frac{(-1)^{n-1}\, i}{(4\pi)^{\frac{D}{2}}} \frac{D}{2} \frac{\Gamma\left(n-\frac{D}{2}-1\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{D}{2}-1} \quad , \\
\int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^2}{(k^2-\Delta)^n} &= \frac{(-1)^n\, i}{(4\pi)^{\frac{D}{2}}} \frac{D(D+2)}{4} \frac{\Gamma\left(n-\frac{D}{2}-2\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{D}{2}-2} \quad .
\end{aligned} \tag{2.34}$$

It is interesting to notice that, apart from multiplicative constants, the addition of a k^2 in the numerator is equivalent to a change $D \to D+2$, a feature that can be used to simplify equations.

2.2.2 Reduction to Bubbles, Triangles and Boxes

The reduction of scalar single-loop integrals with more than four propagators to bubbles, triangles, and boxes comes as a consequence of the fact that one cannot have more than four linearly independent vectors in a four dimensional space, and this holds true with even in non-integer dimensionality, $4-2\varepsilon$, with some corrections. To illustrate this we present identities for massless integrals. A general one-loop integral is given by:

$$I_n = \int \frac{d^D k}{(2\pi)^D} \frac{1}{P_1 P_2 \dots P_n} \tag{2.35}$$

with $P_1 = k^2$ and $P_i = (k-a_i)^2$ for $i = 2, \dots, n$. We define:

$$S_{ij} = (a_i - a_j)^2 \ . \tag{2.36}$$

A pentagon (one-loop scalar integral with five propagators) can be written as:

$$I_5 = -2\varepsilon B I_5^{6-2\varepsilon} - \sum_{i=1}^{5} b_i I_4^{(i)} \ , \tag{2.37}$$

where $B = \sum_i b_i$, $b_i = \sum_j S_{ij}^{-1}$, $I_5^{6-2\varepsilon}$ represents the same integral in $(6-2\varepsilon)$ dimensions while $I_4^{(i)}$ are box integrals built from I_5 by removing the i-th propagator. Since $I_5^{6-2\varepsilon}$ is finite, the first term is infinitesimal and can be discarded.

In the same way, hexagons (one-loop scalar integral with six propagators) can be expressed as a sum of pentagon ones by:

$$I_6 = -\sum_{i=1}^{6} b_i I_5^{(i)} \ , \tag{2.38}$$

while loops with more propagators can be reduced iteratively as:

$$I_n = -\sum_{i=1}^{n} r_i I_{n-1}^{(i)} \ , \tag{2.39}$$

where the coefficients r_i are dependent on the momenta a_i.

While the identities presented in this subsection are for the massless case, the procedure is also possible for the massive one. Since in this work we will not be applying these reductions directly, we will not go into further details.

2.3 Tensor Integrals

We started this chapter with an example of a loop diagram which led to equation 2.6, an integral that, when expressed in terms of the notation introduced in equations 2.7, involves C_μ and $C_{\mu\nu}$ tensor integrals. We have mentioned that tensor integrals can be reduced to scalar ones. One of the first such methods developed for one-loop integrals is called Passarino-Veltman reduction and will be discussed in subsection 2.3.1. A more general approach applicable to multi-loop calculations will be presented in subsection 2.3.2.

2.3.1 Passarino-Veltman Reduction

In the Passarino-Veltman reduction we express a tensor integral as a linear combination of all fundamental tensors available with a given number of indices. For a single index only p_i^μ is available, for two indexes we have $p_i^\mu p_j^\nu$ and $g^{\mu\nu}$, and so on. This allows us to write, for example:

$$C^\mu = c_1 \ p_2^\mu + c_2 \ p_3^\mu \ , \tag{2.40}$$

$$C^{\mu\nu} = c_3 \ g^{\mu\nu} + c_4 \ p_2^\mu p_2^\nu + c_5 \ p_3^\mu p_3^\nu + c_6 \ (p_2^\mu p_3^\nu + p_3^\mu p_2^\nu) \ . \tag{2.41}$$

The c_i coefficients are obtained by contracting each equation with the corresponding tensor structures on the r.h.s., and solving the obtained system of equations for the coefficients.

By expressing the dot product between momenta using equations such as:

$$k \cdot p_2 = \frac{1}{2}\left(\left((k-a_1)^2 - m_1^2\right) - \left((k-a_2)^2 - m_2^2\right) + \left(m_1^2 - m_2^2 - a_1^2 + a_2^2\right)\right) \quad , \qquad (2.42)$$

where $(a_2 - a_1) = p_2$, all c_i are expressed as functions of tensor integrals of lower rank. By iterating the procedure, all coefficients end up being expressed in terms of scalar integrals only, hence the term "reduction".

The Passarino-Veltman procedure may display numerical instabilities and so alternative methods have been developed [22, 23, 24]. Furthermore, it cannot be applied to multi-loop integrals as there is no equivalent to equation 2.42 available. In that case a more general approach must be taken.

2.3.2 Generalized Reduction

For diagrams involving multiple loops or when the Passarino-Veltman reduction is not suitable due to numerical instability when calculating the inverse Gram determinants, a more general approach is used. In this case, the initial steps performed in the scalar integral are followed up to completing the square of the momentum in the denominator of the integrand. At this point, the numerator will involve a polynomial in terms of the momentum integration variable. Monomials of odd power will cancel out when integrated over the whole space, leaving only even terms to be dealt with. Due to symmetry these can be expressed in terms of scalar integrals according to:

$$\begin{aligned} \int d^Dk\ k^\mu k^\nu\ f(k^2) &= \frac{g^{\mu\nu}}{D}\int d^Dk\ k^2\ f(k^2)\ , \qquad (2.43) \\ \int d^Dk\ k^\mu k^\nu k^\rho k^\sigma\ f(k^2) &= \frac{(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})}{D(D+2)}\int d^Dk\ \left(k^2\right)^2\ f(k^2)\ , \end{aligned}$$

and similar ones for higher-rank integrals.

Equations 2.43 combined with 2.34 give some general results for tensor integrals, such as:

$$\int \frac{d^Dk}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2-\Delta)^n} = \frac{(-1)^{n-1}}{(4\pi)^{\frac{D}{2}}} \frac{i\, g^{\mu\nu}}{2} \frac{\Gamma\big(n-\frac{D}{2}-1\big)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{D}{2}-1}, \tag{2.44}$$

$$\int \frac{d^Dk}{(2\pi)^D} \frac{k^\mu k^\nu k^\rho k^\sigma}{(k^2-\Delta)^n} = \frac{(-1)^n}{(4\pi)^{\frac{D}{2}}} \frac{i\, \Gamma\big(n-\frac{D}{2}-2\big)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{D}{2}-2} \times \frac{1}{4}\left(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) .$$

It is important to mention that while we suggested the same initial steps performed on scalar integrals can be performed on tensor integrals, one difference occurs when we complete the square of the momentum variable in the denominator. This variable shifting will introduce parameter variables in the numerator wherever there is a momentum variable. These factors can be moved out of the momentum integration and left to be dealt with only when the parameters integration is performed. However, with the use of an intermediate Schwinger parametrization, it possible to relate integrals with parameters in the numerator to integrals that have higher powers of the propagator. As an example, we can look at a B^μ integration:

$$\int \frac{d^Dk}{(2\pi)^D} \frac{k^\mu}{\Big((k-a_1)^2-m_1^2\Big)^{\nu_1}\Big((k-a_2)^2-m_2^2\Big)^{\nu_2}} \tag{2.45}$$

$$= \int \frac{d^Dk\; k^\mu}{(2\pi)^D} \int_0^\infty dx\; \frac{x_1^{\nu_1-1}\; x_2^{\nu_2-1}}{\Gamma(\nu_1)\Gamma(\nu_2)} \exp\Big(-x_1\Big((k-a_1)^2-m_1^2\Big) - x_2\Big((k-a_2)^2-m_2^2\Big)\Big)$$

$$= \int \frac{d^Dp\;\left(p^\mu + x_1 a_1^\mu + x_2 a_2^\mu\right)}{(2\pi)^D} \int_0^\infty dx\; \frac{x_1^{\nu_1-1}\; x_2^{\nu_2-1}}{\Gamma(\nu_1)\Gamma(\nu_2)} \exp\left(-p^2+\Delta\right) .$$

If we focus only on the second term (involving $x_1 a_1^\mu$) we have:

$$\int \frac{d^Dp\;(x_1 a_1^\mu)}{(2\pi)^D} \int_0^\infty dx\; \frac{x_1^{\nu_1-1}\; x_2^{\nu_2-1}}{\Gamma(\nu_1)\Gamma(\nu_2)} \exp\left(-p^2+\Delta\right) \tag{2.46}$$

$$= a_1^\mu \int \frac{d^Dp}{(2\pi)^D} \int_0^\infty dx\; \frac{x_1^{\nu_1}\; x_2^{\nu_2-1}}{\Gamma(\nu_1)\Gamma(\nu_2)} \exp\left(-p^2+\Delta\right)$$

$$= a_1^\mu \int \frac{d^Dk}{(2\pi)^D} \frac{1}{\Big((k-a_1)^2-m_1^2\Big)^{\nu_1+1}\Big((k-a_2)^2-m_2^2\Big)^{\nu_2}}$$

which brings us to a scalar integration with shifted power in one propagator. The normal steps using Feynman parametrization can be resumed.

2.4 Scalar Multi-loop Integrals

Now that we know how to approach one-loop scalar integrals and how tensor integrals can always be reduced to scalar ones, we can focus on multi-loop scalar integrals. We only need to consider loops with two, three, and four legs (as discussed in subsection 2.2.2); we will allow even dimensions higher or equal to four (subsection 2.2.1); moreover, propagators will in general have powers different from one (section 2.3.2).

Since we do not have a specific process in mind, and we want to have a better understanding of loop calculations as a whole, it is useful to consider every possible topology for a given number of loops. We focus on two loop topologies, since it is the simplest case yet already very challenging. We ignore diagrams that lead to integrations that factorize, since these will simply involve products of single-loop integrations, and focus on those that do not factorize.

As a procedure for finding all possible nonfactorizable topologies with a given number of loops we use a geometrical approach; we simply glue individual loops together by making them share at least one side (propagator) with each other, with the number of external legs coming out of each vertex not being relevant since they do not affect the integration. This can lead to very complicated topologies. Even if we focus just on two-loop diagrams, we obtain altogether six planar diagrams (figure 2.5) and a non-planar one (figure 2.6). Planar diagrams arise when any two loops share only one propagator while non-planar diagrams have more than one propagator shared between two loops. Naturally, when more loops are involved, the number and complexity of the topologies increases dramatically, making it desirable to obtain a systematic approach that uses single-loop calculations as building blocks.

In the next two subsections we will discuss two different approaches to dealing with multi-loop integrations and in particular two-loop integrations. While these methods differ in how to perform the second momentum integration, the first integration is performed as if we were dealing with a single loop and follows the steps discussed in previous sections.

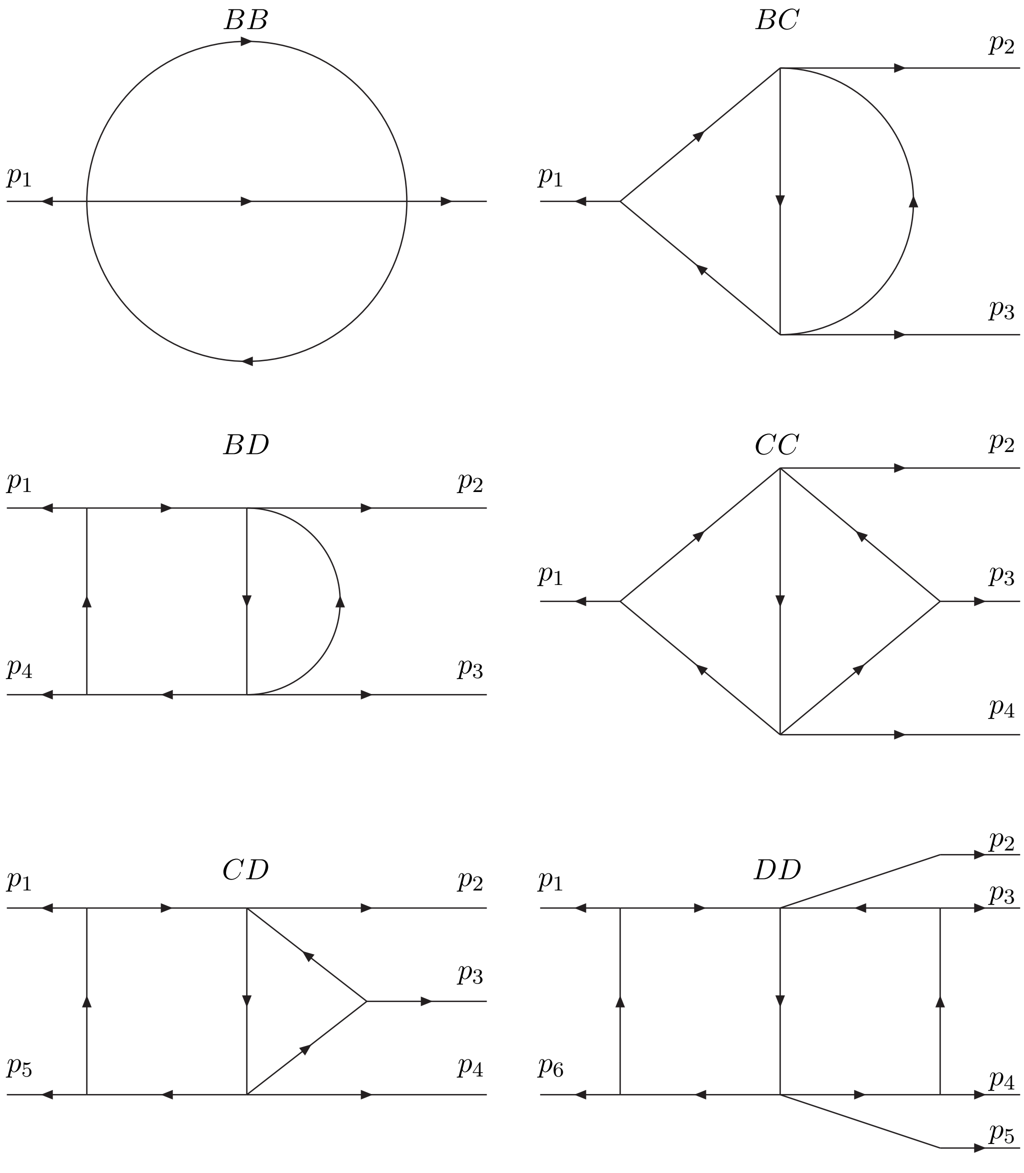


Figure 2.5: Planar two-loop topologies involving bubbles, triangles, and boxes as building blocks. Arrows show momentum direction. Conservation of four-momentum implies the sum of all outgoing momenta is zero. The nomenclature for each diagram refers to the building blocks used: BB for bubble/bubble, BC for bubble/triangle, BD for bubble/box and so on.

DD'

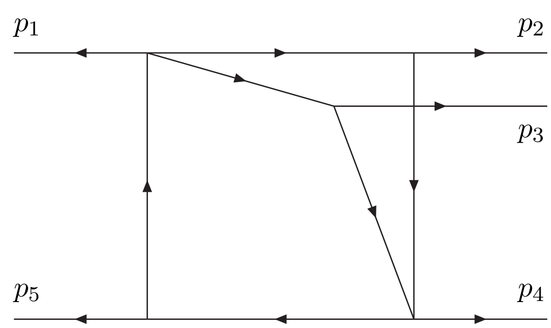


Figure 2.6: Non-planar two-loop topology. Arrows show momentum direction.

Let us look at a CC integration as an example (see figure 2.5):

$$
\begin{aligned}
CC &= \int \frac{d^D k_2}{(2\pi)^D} \frac{1}{\left((k_2+p_2)^2 - m_4^2\right)^{\nu_4} \left((k_2+p_{23})^2 - m_5^2\right)^{\nu_5}} \\
&\times \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{\left(k_1^2 - m_1^2\right)^{\nu_1} \left((k_1+k_2)^2 - m_2^2\right)^{\nu_2} \left((k_1+p_1)^2 - m_3^2\right)^{\nu_3}} \ .
\end{aligned}
\tag{2.47}
$$

Using equation 2.33 for the k_1 integration we obtain:

$$
\begin{aligned}
&\frac{i\ (-1)^{\nu_{123}}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(\nu_{123} - \frac{D}{2}\right)}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)} \int \frac{d^D k_2}{(2\pi)^D} \frac{1}{\left((k_2+p_2)^2 - m_4^2\right)^{\nu_4} \left((k_2+p_{23})^2 - m_5^2\right)^{\nu_5}} \\
\times &\int_0^1 \frac{dx_1\ dx_2\ dx_3\ \delta(1 - x_1 - x_2 - x_3)\ x_1^{\nu_1 - 1}\ x_2^{\nu_2 - 1}\ x_3^{\nu_3 - 1}}{\Delta_1^{\nu_{123} - \frac{D}{2}}} \ ,
\end{aligned}
\tag{2.48}
$$

where Δ_1 depends on k_2 and is given by:

$$
\Delta_1 = -x_1 x_2\ k_2^2 - x_1 x_3\ p_1^2 - x_2 x_3\ (k_2 - p_1)^2 + \left(x_1\ m_1^2 + x_2\ m_2^2 + x_3\ m_3^2\right) \quad . \tag{2.49}
$$

The next steps will depend on which approach is taken.

2.4.1 Traditional Approach

The traditional approach is to re-express Δ_1 as an artificial propagator in the k_2 integration. Using:

$$\begin{aligned} c_6 &= -x_2\,(x_1+x_3)\ , \\ a_6 &= \frac{-x_3\ p_1}{(x_1+x_3)}\ , \\ m_6^2 &= \frac{x_1\ m_1^2+x_2\ m_2^2+x_3\ m_3^2}{x_2\,(x_1+x_3)} - \frac{x_1 x_3 p_1^2}{x_2\,(x_1+x_3)^2} \end{aligned} \tag{2.50}$$

equation 2.49 can be rewritten as:

$$\Delta_1 \equiv c_6\left((k_2+a_6)^2-m_6^2\right)\ , \tag{2.51}$$

where the equivalence sign appears because we used the delta condition $x_1+x_2+x_3=1$ which is valid only inside the integration.

With this transformation the k_2 integration in equation 2.48 becomes:

$$\int \frac{d^D k_2}{(2\pi)^D} \frac{1}{\left((k_2+p_2)^2-m_4^2\right)^{\nu_4}\left((k_2+p_{23})^2-m_5^2\right)^{\nu_5}\left((k_2+a_6)^2-m_6^2\right)^{\nu_{123}-\frac{D}{2}}}\ , \tag{2.52}$$

which is identical to a normal single-loop integration with the exception of a non-integer power of the propagator involving an infinitesimal (since $D=2m-2\varepsilon$), which is characteristic of this reduction. This integration can be performed using one-loop techniques and eq. 2.48 becomes:

$$\begin{aligned} &\frac{-(-1)^{\nu_{45}-\frac{D}{2}}}{(4\pi)^D}\frac{\Gamma\left(\nu_{12345}-\frac{D}{2}\right)}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)\,\Gamma(\nu_4)\,\Gamma(\nu_5)} \\ \times &\int_0^1 \frac{dx_1\ dx_2\ dx_3\ \delta(1-x_1-x_2-x_3)\ x_1^{\nu_1-1}\ x_2^{\nu_2-1}\ x_3^{\nu_3-1}}{\left(-x_2\,(x_1+x_3)\right)^{\nu_{123}-\frac{D}{2}}} \\ \times &\int_0^1 \frac{dx_4\ dx_5\ dx_6\ \delta(1-x_4-x_5-x_6)\ x_4^{\nu_4-1}\ x_5^{\nu_5-1}\ x_6^{\nu_{123}-\frac{D}{2}-1}}{\left(-x_4x_5p_3^2-x_4x_6\,(a_6-p_2)^2-x_5x_6\,(p_{23}-a_6)^2+\left(x_4m_4^2+x_5m_5^2+x_6m_6^2\right)\right)^{\nu_{12345}-D}} \end{aligned} \tag{2.53}$$

where a_6 and m_6 are functions of x_1, x_2, and x_3. This concludes the momentum integrations, and we are left to deal with the Feynman parameter integration.

The parameter integrations are in general not easy to perform, and so in chapter 3 we will discuss how to systematize the procedure by performing a Taylor expansion of the integrand, leading to expressions in terms of concatenated sums. At that point, we will use the Z-Sum approach to express these in terms of multiple polylogarithms.

The steps used in this subsection apply not only to the CC integration but for all cases, including involving more loops. Irrespective of the topology of each loop, there will always be a transformation similar to equations 2.50 that expresses Δ_{i-1} as an artificial propagator in the i-th loop integration, which can then be performed using standard single-loop techniques. The final result will be a parameter integration with one set of variables and a Dirac delta for every loop, and one or more denominator factors involving polynomials of the parameter variables. While the denominator for a single loop is always at most quadratic, for more loops it could involve higher powers depending on how it is factored out. Naturally, the more complicated the initial diagram, the more complicated will be the integrand in the scalar integration that follows.

2.4.2 Mellin-Barnes Splitting

Here we describe an alternative to the traditional approach discussed in the last subsection, a method we will refer to as "Mellin-Barnes splitting" [14, 13]. While before we expressed Δ_1 (see equations 2.48 and 2.49) in terms of a single artificial propagator, now we will use the inverse Mellin-Barnes (MB) transformation [25]:

$$\frac{1}{(A_1+A_2)^\nu} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma\ A_1^\sigma\ A_2^{-\nu-\sigma}\ \frac{\Gamma(-\sigma)\Gamma(\nu+\sigma)}{\Gamma(\nu)} \quad , \tag{2.54}$$

to express it in terms of several factors, where all k_2 dependence appears as already present propagators, or as new artificial ones. These propagators can then be merged into the second momentum integration which is performed as usual.

Let us illustrate the method with an example, the massless CC diagram (fig. 2.5) with two of the external legs removed by setting:

$$p_2 = p_4 = 0 \quad \text{and} \quad p_3 = -p_1\ . \tag{2.55}$$

The purpose of this simplification is that Δ_1 can be completely expressed in terms of existing propagators without the need to introduce new artificial ones. We start by applying the Mellin-Barnes transformation repeatedly on the denominator of equation 2.48 and obtain:

$$\begin{aligned}
&\Delta_1^{-\nu_{123}+\frac{D}{2}} = \\
&= \left[-x_1x_2\ k_2^2 - x_1x_3\ p_1^2 - x_2x_3\ \left(k_2-p_1\right)^2\right]^{-\nu_{123}+\frac{D}{2}} \\
&= \frac{-1}{(2\pi)^2}\int_{-i\infty}^{+i\infty} d\sigma_1 \int_{-i\infty}^{+i\infty} d\sigma_2 \frac{\Gamma(-\sigma_1)\Gamma(-\sigma_2)\Gamma(\nu_{123}-\frac{D}{2}+\sigma_1+\sigma_2)}{\Gamma(\nu_{123}-\frac{D}{2})} \\
&\times \frac{\left[-x_1x_2\left(k_2^2\right)\right]^{\sigma_1}\left[-x_2x_3\left(k_2-p_1\right)^2\right]^{\sigma_2}}{\left[-x_1x_3p_1^2\right]^{\nu_{123}-\frac{D}{2}+\sigma_1+\sigma_2}} \ . \qquad (2.56)
\end{aligned}$$

Note that all k_2 dependence now appears as $\left(k_2^2\right)^{\sigma_1}$ and $\left(\left(k_2-p_1\right)^2\right)^{\sigma_2}$ and can be merged with already existing propagators in the k_2 integration, which is performed as if it were a single-loop bubble:

$$\begin{aligned}
&\int \frac{d^Dk_2}{(2\pi)^D}\frac{1}{\left(k_2^2\right)^{\nu_4-\sigma_1}\left(\left(k_2-p_1\right)^2\right)^{\nu_5-\sigma_2}} = \frac{(-1)^{\frac{D}{2}}\ i}{(4\pi)^{\frac{D}{2}}\left(p_1^2\right)^{\nu_{45}-\sigma_1-\sigma_2-\frac{D}{2}}} \\
&\times \frac{\Gamma(\nu_{45}-\sigma_1-\sigma_2-\frac{D}{2})}{\Gamma(\nu_4-\sigma_1)\Gamma(\nu_5-\sigma_2)}\int_0^1 dx_4\ x_4^{\sigma_2-\nu_5+\frac{D}{2}-1}(1-x_4)^{\sigma_1-\nu_4+\frac{D}{2}-1} \ . \qquad (2.57)
\end{aligned}$$

Applying equations 2.56 and 2.57 on 2.48 we obtain:

$$\begin{aligned}
&\frac{(-1)^D}{(4\pi)^D(2\pi)^2(p_1^2)^{\nu_{12345}-D}}\int_{-i\infty}^{+i\infty} d\sigma_1 \int_{-i\infty}^{+i\infty} d\sigma_2 \qquad (2.58)\\
&\times \frac{\Gamma(-\sigma_1)\ \Gamma(-\sigma_2)\ \Gamma(\sigma_1+\sigma_2+\nu_{123}-\frac{D}{2})\ \Gamma(-\sigma_1-\sigma_2+\nu_{45}-\frac{D}{2})}{\Gamma(\nu_1)\ \Gamma(\nu_2)\ \Gamma(\nu_3)\ \Gamma(\nu_4-\sigma_1)\ \Gamma(\nu_5-\sigma_2)} \\
&\times \int_0^1 dx_1\ dx_2\ dx_3\ \delta(1-x_1-x_2-x_3)\ x_1^{-\sigma_2-\nu_{23}+\frac{D}{2}-1} x_2^{\sigma_1+\sigma_2+\nu_2-1} x_3^{-\sigma_1-\nu_{12}+\frac{D}{2}-1} \\
&\times \int_0^1 dx_4\ x_4^{\sigma_2-\nu_5+\frac{D}{2}-1}(1-x_4)^{\sigma_1-\nu_4+\frac{D}{2}-1} \ .
\end{aligned}$$

In the next chapter we will discuss how to perform a parameter integration such as the above one in more detail, using a change of variables and the beta function. For now we simply state the result as:

$$\frac{(-1)^D}{(4\pi)^D(2\pi)^2(p_1^2)^{\nu_{12345}-D}}\int_{-i\infty}^{+i\infty}d\sigma_1\int_{-i\infty}^{+i\infty}d\sigma_2 \tag{2.59}$$
$$\times\quad\frac{\Gamma(-\sigma_1)\Gamma(-\sigma_2)\Gamma(\sigma_1+\sigma_2+\nu_{123}-\frac{D}{2})\Gamma(-\sigma_1-\sigma_2+\nu_{45}-\frac{D}{2})}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4-\sigma_1)\Gamma(\nu_5-\sigma_2)}$$
$$\times\quad\frac{\Gamma(-\sigma_2-\nu_{23}+\frac{D}{2})\Gamma(\sigma_1+\sigma_2+\nu_2)}{\Gamma(\sigma_1-\nu_3+\frac{D}{2})}\frac{\Gamma(\sigma_1-\nu_3+\frac{D}{2})\Gamma(-\sigma_1-\nu_{12}+\frac{D}{2})}{\Gamma(-\nu_{123}+D)}$$
$$\times\quad\frac{\Gamma(\sigma_2-\nu_5+\frac{D}{2})\Gamma(\sigma_1-\nu_4+\frac{D}{2})}{\Gamma(\sigma_1+\sigma_2-\nu_{45}+D)}\ .$$

This completes both momentum and parameter integrations, and we are left with complex integrals. In order to perform these we need to complete the integration contour at infinity and use the residue theorem to obtain a sum over the poles in the complex plane. The first step is only valid if the integrand vanishes at infinity so that the initial integration equals the final one over the closed contour, a requirement that creates the condition of convergence for the final summation. This will not always be possible, and so care must be taken when choosing the order over which the complex integrations are performed. All poles will arise from gamma functions involving the complex variables as arguments, with residue given by:

$$res_z(\Gamma(z))(-k)=\frac{(-1)^k}{k!}\ ,\qquad k\in\mathbb{N}\ . \tag{2.60}$$

Once the summations are obtained we would proceed to use the Z-Sum approach to try to express these in terms of multiple polylogarithms, but that is the topic of the next chapters.

In our example, we removed two external legs and set all masses to zero for the sake of simplicity. The method however is still applicable without these conditions, but will be slightly more extensive. If the first loop contains massive propagators we would require more applications of the MB transformation in equation 2.56, leading to extra complex integrations and consequently more infinite sums (and conditions for convergence), although all steps remain the same. If the second loop contains massive propagators then we would need to expand Δ_1 in terms of these massive propagators, and to do so we would possibly need to add and subtract masses to Δ_1, which again would lead to extra summations in the end. Finally, if we had not removed two external legs it would have been impossible to express Δ_1 in terms of only existing propagators, meaning we would have to add new artificial ones, making the second momentum integration more complicated. The procedure will not require new artificial propagators whenever one of the loops is a bubble with at least one shared vertex without external legs or a triangle or box where both shared vertices do not have external legs.

The main advantage of the Mellin-Barnes splitting method over the traditional one is the fact that it disentangles the parameter integrations arising from the first loop from the ones arising from the second loop, leading directly to simple beta functions. This is because it does not create a denominator that involves both set of variables like it happens with the traditional approach (see equation 2.53). Furthermore, under the right conditions, it reduces the complexity of the second momentum integration, as it happened in our example where the second triangle loop became a bubble. This comes because we were able to express Δ_1 completely in terms of existing propagators of the second loop.

In section 3.2 we will discuss the inverse Mellin-Barnes transformation in more detail and show that under some conditions it produces the same result as a simple Taylor series. That could raise the question whether it is possible to perform the method described in this section using the latter so as to avoid having to deal with complex integrations. In all cases the answer is no, it is not possible since it would always lead to divergent summations and the reason is that, while both the Mellin-Barnes transformation and Taylor series produce similar (but not identical) expansions, the condition for convergence is enforced at different moments. For the Taylor series it must be observed immediately as the expansion is performed (before the parameter integration) while with Mellin-Barnes only when the contour is completed. Since we only do so after the parameter integration, the condition does not need to be enforced over a whole range of integration variables, making only the series originating from Mellin-Barnes convergent.

CHAPTER 3

PARAMETER INTEGRATION AND EXPANSIONS

In the last chapter we covered all major steps for performing the momentum integrations arising in Feynman loop diagrams. While multi-loop calculations performed with MB splitting lead directly to concatenated sums through the use of the residue theorem, single or multi-loop calculations performed by the traditional method resulted in a more complicated parameter integration. In the first few sections of this chapter we will discuss two different approaches to performing this integration by expanding the integrand, first using Taylor series, and then using Mellin-Barnes transformations. After this step all methods lead to concatenated sums with a very typical structure. The second part of this chapter will be devoted to discussing how to use the algebraic properties of the Z-Sums to attempt to express these summations in terms of polylogarithms or if necessary multiple polylogarithms.

We start by observing that in general the parameter integration for a single loop is, apart from overall constants, given by:

$$\int_0^1 \frac{dx_1 \ \ldots \ dx_n \ \delta(1 - \sum_i x_i) \ \prod_j x_j^{a_j}}{\left(\sum_{k=1}^{\xi} \ c_k \ x_1^{p_{k,1}} \ \ldots \ x_n^{p_{k,n}}\right)^d} \ , \tag{3.1}$$

where ξ is the number of terms in the denominator, and the number of integration variables depend on the number of propagators in the loop. For one-loop cases, c_i depends only on kinematical invariants (that is, masses and dot products of external momenta), however if this integration is part of a multi-loop diagram they will also depend on variables from other parameter integrations. The powers $p_{i,j}$ take values of zero, one, or two.

Change of Variables. Our first step is to apply the delta function condition in equation 3.1 but, for the sake of systematicity, we would like to keep all integration ranges unaltered (between 0 and 1). When only two parameter integrations are involved this is trivial. For three integrations (triangles) and four integrations (boxes) we introduce the following changes of variables:

Triangles:

$$\begin{aligned} x_1 &= u_1\, u_2 \quad , \\ x_2 &= (1-u_1)\, u_2 \quad , \\ x_3 &= (1-u_2) \quad , \\ |J| &= u_2 \quad ; \end{aligned} \tag{3.2}$$

Boxes:

$$\begin{aligned} x_1 &= u_1\, u_2 \quad , \\ x_2 &= (1-u_1)\, u_2 \quad , \\ x_3 &= (1-u_2)\, u_3 \quad , \\ x_4 &= (1-u_2)\,(1-u_3) \quad , \\ |J| &= u_2\,(1-u_2) \quad , \end{aligned} \tag{3.3}$$

where J is the Jacobian.

The transformations above are not unique, for example a rearrangement of pair assignments (that is, an exchange of x_i on the left side of the equations while keeping the right side unaltered) or changes of the form $u \to (1-u)$ also accomplish the desired objectives. The most convenient choice will vary from diagram to diagram, with some choices leading to simpler expressions. Other more exotic transformations exist but would introduce step functions or absolute values into the integrand making the problem unnecessarily complicated and so will not be considered.

Equation 3.1 now becomes:

$$\int_0^1 \frac{du_1 \ \ldots \ du_{n-1} \ \prod_i u_i^{a'_{2i-1}}(1-u_i)^{a'_{2i}}}{\left(\sum_{k=1}^{\xi'} \ c'_k \ u_1^{p'_{k,1}} \ (1-u_1)^{p'_{k,2}} \ \ldots \ u_{n-1}^{p'_{k,2n-1}} \ (1-u_{n-1})^{p'_{k,2n}}\right)^d} \ . \tag{3.4}$$

Parameter Integration. For very simple diagrams, where most invariants are equal to zero, the denominator in equation 3.4 will contain only one term, in which case all parameter integrations are beta functions:

$$\int_0^1 u^a(1-u)^b du = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} = B(a+1, b+1) \ , \tag{3.5}$$

valid for $Re(a) > -1$ and $Re(b) > -1$. Properties of the beta function can be found in Appendix B. More complicated diagrams will however involve several terms in the denominator and would not immediately lead to an integration as simple as a beta function. While we could treat every diagram individually and try to perform each specific integral, for the sake of systematicity we would like to manipulate equation 3.4 so that we always reduce it to beta function integrations. In order to do that we will expand the integrand, either by a Taylor series or with the use of Mellin-Barnes transformations, and switch the order of sums and integrations, after which all integrations will again be of the form in equation 3.5. There are conditions we will need to verify. First, all expansions will need to be convergent, but we also need absolute convergence for the sum reordering. In most cases we deal with, if the former is observed so will the latter.

3.1 Taylor Expansion of the Integrand

In this section we will discuss expanding the integrand in equation 3.1 using a simple Taylor series, which will involve a denominator expansion and in many cases a subsequent numerator expansion.

Denominator Expansion. If we factor out one of the terms in the denominator we get:

$$\begin{aligned}
&\left(\sum_{k=1}^{\xi'} c'_k u_1^{p'_{k,1}} (1-u_1)^{p'_{k,2}} \dots u_{n-1}^{p'_{k,2n-1}} (1-u_{n-1})^{p'_{k,2n}}\right)^{-d} \\
= \;& \left(c'_1 u_1^{p'_{1,1}} (1-u_1)^{p'_{1,2}} \dots u_{n-1}^{p'_{1,2n-1}} (1-u_{n-1})^{p'_{1,2n}}\right)^{-d} \\
\times \;& \left(1 - \sum_{k=1}^{\xi'-1} c''_k u_1^{p''_{k,1}} (1-u_1)^{p''_{k,2}} \dots u_{n-1}^{p''_{k,2n-1}} (1-u_{n-1})^{p''_{k,2n}}\right)^{-d} \\
= \;& F^{-d} (1-\alpha)^{-d} \,, \qquad (3.6)
\end{aligned}$$

where F can be absorbed by the factors in the numerator. This allows us to perform a denominator expansion according to:

$$(1-\alpha)^{-d} = \frac{1}{\Gamma(d)} \sum_{i=1}^{\infty} \frac{\Gamma(i+d)}{\Gamma(i+1)} \, \alpha^i \,, \qquad (3.7)$$

valid for $d > 0$ and guaranteed to be convergent for $|\alpha| < 1$. Since this condition must be observed for the whole range of parameter integrations, it imposes conditions on both the power of the integration variables p'' and on the coefficients c''.

All p'' must be non-negative since $u^{p''}$ and $(1-u)^{p''}$ diverge with negative p'' when $u \to 0$ and $u \to 1$, respectively [1]. That means we must be careful when choosing the term to factor out in equation 3.6, selecting one that leaves all powers in the remaining terms non-negative.

Regarding the coefficients c'', the condition $|\alpha| < 1$ must be valid for some choice of values even if it implies unphysical invariants. Physically meaningful results can be regained by analytical continuation once the summation has been expressed in terms of closed form functions. Regulators may be added if there is no range of values that make a given expansion well defined, and can be removed once the final expression has been obtained.

Sometimes it is not possible to immediately factor out a term, meaning we would not obtain $(1-\alpha)^{-d}$ but instead $(f(u)-\alpha)^{-d}$, which cannot yet be expanded. In such cases,

[1] Whenever subscripts on u, p'', c, etc, are not required for full understanding of the argument we will omit them for clarity, but they are implied.

there are a few simple tricks that allow to obtain a legal expansion. We may add and subtract a constant which will eventually cancel out by the end of the calculation, as in:

$$c_1\ u_1 + c_2\ u_2 = T - T + c_1\ u_1 + c_2\ u_2 \rightarrow T\left(1 - \frac{c_3\ T - c_1\ u_1 - c_2\ u_2}{T}\right) \quad , \tag{3.8}$$

where we added c_3 as a regulator, or expand individual terms using trivial substitutions like:

$$\begin{aligned} (1-u) &\rightarrow (1)-(u) \\ (u) &\rightarrow (1)-(1-u) \\ (1) &\rightarrow (u)+(1-u)\ , \end{aligned} \tag{3.9}$$

where the parentheses are being used to emphasize objects that will be expanded as a single term, as in the example:

$$c_1\ u_1 + c_2\ u_2 = c_1\left[1-(1-u_1)\right] + c_2\ u_2 \rightarrow c_1\left(1 - c_3\ (1-u_1) + \frac{c_2}{c_1}u_2\right) = c_1(1-\alpha)\ . \tag{3.10}$$

Note that we added an extra degree of freedom c_3 to ensure $|\alpha| < 1$. With these tricks it is always possible to perform a valid denominator expansion for any diagram.

Numerator Expansion. Going back to equations 3.6 and 3.7, for cases where α contains several terms the integration will still not be in the form of a beta function as desired, and so α^i itself must be expanded. In this case we use the binomial expansion:

$$\alpha^i = (a+b)^i = \sum_{j=0}^{\infty} \binom{i}{j}\ a^{i-j}\ b^j\ . \tag{3.11}$$

This sum is convergent for $|\frac{b}{a}| < 1$ if i is not an integer. In our case i will always be an integer, which truncates the sum since $\binom{i}{j} = 0$ if $j > i$, so the sum is convergent for any a and b. The binomial expansion can be used repeatedly if α contains several terms, leading to several concatenated sums.

In principle there are two steps in evaluating conditions for all the expansions discussed. First we verify the convergence condition when performing the denominator expansion. After the numerator expansion, we test absolute convergence (in order to perform the reordering). Evaluating convergence of concatenated sums is not trivial; fortunately, it is not necessary, since the binomial expansion has a finite number of terms and is always well defined, such that it does not alter the convergence. Both tests can be performed when expanding the denominator, and in practice, if enough degrees of freedom (non zero invariants) are present, absolute convergence is observed whenever the series is convergent.

3.1.1 Alternative Expansion Choices

While we will always be only performing denominator and numerator Taylor expansions as given by equations 3.7 and 3.11, there are several choices that need to be made before the expansion is performed that will ultimately change the final form of the summation. These include the change of parameter variables, the type of factorization, and the specific way the denominator is expressed and ordered. Different choices lead to different summations which in principle could be related at the summation level if they share a range of convergence, but because these sum operations can be quite complicated and relations among different sums may not be simple, it is easier to consider all options on every step and study how these choices affect the form of the summations obtained.

Change of Variables. There are a few different ways of performing the change of variables, some of which might lead to summations that are easier to deal with. It is difficult to decide a priori which choice will be the best for a given calculation and a brute force approach is a good option since the total number of variable changes is small.

Form of the Denominator. While a polynomial is uniquely defined when expressed only in terms of u, in our case we have the option of using terms involving powers of both u and $(1-u)$, leading to an infinite number of different but equivalent expressions which can be obtained by using the simple substitutions given in equation 3.9. If we are to stick to expressions with a specific number of terms there will only be a finite number of options, and for the most part it is desirable to keep the denominator with the least number of terms since each extra term ultimately leads to an extra concatenated sum, possibly generating a more complicated expression (although in some cases a summation with more concatenated sums might be easier to deal with than one with less). Once the specific form of the polynomial is chosen the ordering of terms must also be considered since it will affect the final form of the summations. Generally the guideline for making these choices is given by what arguments we would prefer to have in the gamma functions arising from the beta function (equation 3.5). This will become more clear as we discuss the general expression for the expansion in subsection 3.1.2.

Factorization of the Denominator. Some diagrams will lead to denominator expressions that can be factorized before performing the expansion. Let us consider for example two different situations:

$$\begin{aligned} \Delta &= \left(1 - c_1 \, u - c_2 \, u^2\right)^{-d} \\ \text{Expansion} &\rightarrow \frac{1}{\Gamma(d)} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{\Gamma(i+d)}{\Gamma(i-j+1)\Gamma(j+1)} \, c_1^{i-j} \, c_2^{j} \, u^{i+j} \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \Delta &= (1 - a_1 \, u)^{-d} \, (1 - a_2 \, u)^{-d} \\ \text{Expansion} &\rightarrow \frac{1}{\Gamma(d)^2} \sum_{i=0}^{\infty} \frac{\Gamma(i+d)}{\Gamma(i+1)} \, a_1^{i} \sum_{j=0}^{\infty} \frac{\Gamma(j+d)}{\Gamma(j+1)} \, a_2^{j} \, u^{i+j} \; . \end{aligned} \tag{3.13}$$

If we set:

$$\begin{aligned} c_1 &= (a_1 + a_2) \quad , \\ c_2 &= a_1 \, a_2 \quad , \end{aligned} \tag{3.14}$$

both series represent the same function, however the second is better to work with since it leads directly to Z-Sums, as we will see in section 3.4.

While this example is very simple and without even considering the denominator factorization it would be possible to obtain the second series from the first one, in general rearranging concatenated sums is far less intuitive than evaluating possible factorization options. With this in mind, before performing the denominator expansion we try to consider every possible factorization and then evaluate the summations originating from them. Normally the best option will be the one that factorizes the most.

Before finishing this subsection, it is important to note that in some cases dealing with summation operations like reordering, splitting or merging of terms, mixing of coefficient, etc, is unavoidable, and for such situations we will use the operations discussed in Appendix C.

3.1.2 General Expression

Now we can combine all steps discussed previously and obtain a general expression for single loops integrals involving concatenated sums. We start by rewriting equation 3.4 as:

$$\int_0^1 \frac{\prod_{l=1}^{n} du_l \; u_l^{a_{(2l-1)}} (1-u_l)^{a_{(2l)}}}{\prod_{k=1}^{\gamma} \left(1 - \sum_{t_k=1}^{\beta_k} c_{(k,t_k)} \prod_{s=1}^{n} u_s^{p_{(k,t_k,2s-1)}} (1-u_s)^{p_{(k,t_k,2s)}}\right)^d} \; , \tag{3.15}$$

where we considered a product in the denominator for cases where it factorizes (with γ being the total number of factors) and dropped all prime superscripts for simplicity. We proceed with the denominator expansion:

$$\frac{1}{\Gamma(d)^\gamma} \int_0^1 \prod_{l=1}^{n} du_l \; u_l^{a_{(2l-1)}} (1-u_l)^{a_{(2l)}} \tag{3.16}$$
$$\times \prod_{k=1}^{\gamma} \sum_{i_{(k,0)}=0}^{\infty} \frac{\Gamma(i_{(k,0)}+d)}{\Gamma(i_{(k,0)}+1)} \left(\sum_{t_k=1}^{\beta_k} c_{(k,t_k)} \prod_{s=1}^{n} u_s^{p_{(k,t_k,2s-1)}} (1-u_s)^{p_{(k,t_k,2s)}} \right)^{i_{(k,0)}} \; ,$$

followed by the numerator expansion:

$$\frac{1}{\Gamma(d)^\gamma} \int_0^1 \prod_{l=1}^{n} du_l \; u_l^{a_{(2l-1)}} (1-u_l)^{a_{(2l)}} \prod_{k=1}^{\gamma} \sum_{i_{(k,0)}=0}^{\infty} \frac{\Gamma(i_{(k,0)}+d)}{\Gamma(i_{(k,0)}+1)}$$
$$\times \sum_{i_{(k,1)}=0}^{i_{(k,0)}} \cdots \sum_{i_{(k,\beta_k-1)}=0}^{i_{(k,\beta_k-2)}} \left(i_{(k,0)} - i_{(k,1)}, \; \ldots \; , i_{(k,\beta_k-1)} - i_{(k,\beta_k)}\right)! \tag{3.17}$$
$$\times \prod_{t_k=1}^{\beta_k} \left(c_{(k,t_k)} \prod_{s=1}^{n} u_s^{p_{(k,t_k,2s-1)}} (1-u_s)^{p_{(k,t_k,2s)}} \right)^{i_{(k,t_k-1)} - i_{(k,t_k)}} \; ,$$

where $i_{(k,\beta_k)} = 0$ was used to shorten the expression and we used the multinomial notation:

$$(a_1, \; \ldots \; , a_n)! \; = \; \frac{\Gamma(a_1 + \; \ldots \; + a_n + 1)}{\Gamma(a_1+1) \; \ldots \; \Gamma(a_n+1)} \; . \tag{3.18}$$

The multinomial is simply the product of the binomial coefficient for each numerator expansion. Equation 3.17 is valid even when a numerator expansion is not necessary, with the corresponding summation collapsing to a single term. At this point all u dependence comes from powers of u and $(1-u)$ in the numerator, and so the parameter integration is a beta function (eq. 3.5):

$$\frac{1}{\Gamma(d)^{\gamma}} \prod_{k=1}^{\gamma} \sum_{i_{(k,0)}=0}^{\infty} \frac{\Gamma(i_{(k,0)}+d)}{\Gamma(i_{(k,0)}+1)} \sum_{i_{(k,1)}=0}^{i_{(k,0)}} \cdots \sum_{i_{(k,\beta_k-1)}=0}^{i_{(k,\beta_k-2)}} \tag{3.19}$$

$$\times \quad \left(i_{(k,0)}-i_{(k,1)},\ \ldots\ ,i_{(k,\beta_k-1)}-i_{(k,\beta_k)}\right)! \prod_{t_k=1}^{\beta_k} c_{(k,t_k)}^{i_{(k,t_k-1)}-i_{(k,t_k)}} \left[\prod_{l=1}^{n} \frac{\Gamma(g_{(2l-1)})\Gamma(g_{(2l)})}{\Gamma(g_{(2l-1)}+g_{(2l)})}\right]^{\frac{1}{\gamma}}$$

with

$$\begin{aligned} g_{(\ell)} &= a_{(\ell)} + \sum_{q=1}^{\gamma} \sum_{j_q=1}^{\beta_q} p_{(q,j_q,\ell)}(i_{(q,j_q-1)} - i_{(q,j_q)}) + 1 \\ &= a_{(\ell)} + \sum_{q=1}^{\gamma} \left(p_{(q,1,\ell)} i_{(q,0)} + \sum_{j_q=1}^{\beta_q-1} i_{(q,j_q)} (p_{(q,j_q+1,\ell)} - p_{(q,j_q,\ell)}) \right) + 1 \end{aligned} \tag{3.20}$$

Equation 3.19 is the main result of this section and should be well understood. The k product comes from the number of factors in the denominator, and from each factor we get one sum from the denominator expansion ($i_{(k,0)}$) and several other from the numerator expansions ($i_{(k,1)}$ to $i_{(k,\beta_k-1)}$, $\beta_k - 1$ sums coming from a numerator with β_k terms). All invariants appear only through the $c_{(k,t_k)}$ coefficients. The number of beta functions will be two and three for triangle and box integrations, respectively, and originate from the parameter integrations

The general structure of the summation, that is, how many individual sums exist and how they are concatenated, does not depend specifically on the diagram itself but on how we chose to factorize and express the integrand's denominator. The main difference between different diagrams will always appear in the arguments of the gamma functions originating from the parameter integration. Equations 3.19 and 3.20 are very general and consequently not very transparent, and so we explicitly write a special case in order to clarify the discussion. For a single triangle ($n = 2$) with one denominator factor ($\gamma = 1$) including three terms ($\beta_1 = 3$) we get:

$$\int_0^1 \frac{du_1\ du_2\ u_1^{a_1}\ (1-u_1)^{a_2}\ u_2^{a_3}\ (1-u_2)^{a_4}}{\left(1-\sum_{t=1}^{3} c_t u_1^{p(t,1)}(1-u_1)^{p(t,2)}u_2^{p(t,3)}(1-u_2)^{p(t,4)}\right)^d} \tag{3.21}$$
$$= \frac{1}{\Gamma(d)}\sum_{i_0=0}^{\infty}\frac{\Gamma(i_0+d)}{\Gamma(i_0+1)}\sum_{i_1=0}^{i_0}\binom{i_0}{i_1}\sum_{i_2=0}^{i_1}\binom{i_1}{i_2}c_1^{i_0-i_1}c_2^{i_1-i_2}c_3^{i_2}\frac{\Gamma(g_1)\Gamma(g_2)}{\Gamma(g_1+g_2)}\frac{\Gamma(g_3)\Gamma(g_4)}{\Gamma(g_3+g_4)}$$

with

$$g_\ell = a_\ell + p_{(1,\ell)}i_0 + i_1(p_{(2,\ell)} - p_{(1,\ell)}) + i_2(p_{(3,\ell)} - p_{(2,\ell)}) + 1 \quad , \tag{3.22}$$

where arguments of the gamma functions are composed of two parts, a constant term a_ℓ and another dependent on the summation variables.

Constant Term. The constant term a_ℓ comes from powers of u and $(1-u)$ in the numerator of the integration in 3.15. These powers have two possible origins. First, they come from the numerator factors introduced by the Feynman parametrization and altered by the reparametrization ($x_i \to u_i$). For example, for the triangle under the transformation given by equations 3.2 we get

$$x_1^{\nu_1-1}\ x_2^{\nu_2-1}\ x_3^{\nu_3-1}\ \to\ u_1^{\nu_1-1}\ (1-u_1)^{\nu_2-1}\ u_2^{\nu_{12}-1}\ (1-u_2)^{\nu_3-1}\ , \tag{3.23}$$

where we also included a u_2 factor from the Jacobian. Second, powers of u and $(1-u)$ come from the overall term factored out in the denominator, the F in eq. 3.6. Again, using a triangle example:

$$\frac{1}{\Delta^d} = \frac{1}{F^d\,(1-\alpha)^d} = \frac{1}{\left(f_0 u_1^{f_1}(1-u_1)^{f_2}u_2^{f_3}(1-u_2)^{f_4}\right)^d(1-\alpha)^d}$$
$$= \frac{f_0^{-d}\ u_1^{-df_1}\ (1-u_1)^{-df_2}\ u_2^{-df_3}\ (1-u_2)^{-df_4}}{(1-\alpha)^d}\ . \tag{3.24}$$

Combining equations 3.23 and 3.24 we can explicitly write what a_ℓ would be for the triangle under consideration:

$$\begin{aligned} a_1 &= \nu_1 - d\ f_1 - 1 \quad , \\ a_2 &= \nu_2 - d\ f_2 - 1 \quad , \\ a_3 &= \nu_{12} - d\ f_3 - 1 \quad , \\ a_4 &= \nu_3 - d\ f_4 - 1 \quad . \end{aligned} \tag{3.25}$$

Note that $d = 2m - 2\varepsilon$, where $2m$ is the dimensionality of momentum space while ε is an infinitesimal. Hence gamma functions (with a non-zero f) will need to be expanded in powers of ε. Since the only way to obtain a pole in $1/\varepsilon$ is from gamma functions in the numerator (see eqs. B.11 and B.12), this means that diagrams with poles will necessarily have powers of u factoring out from the denominator, consequently leaving a simpler polynomial to be expanded.

Variable Term. The part of the gamma function argument that depends on summation variables is the most important as it is the one that will define the entanglement between all summations. The p factors multiplying the summation variables i arise from the powers of u and $(1-u)$ in the denominator of 3.15, and so it becomes clear why re-expressing or changing the order of terms in the denominator ultimately changes the form of the final summation. This will be the guideline for how to express the denominator before expansion: we try to identify which arguments within the gamma functions lead to a simpler summation and then try to organize the denominator to achieve that result.

This concludes the Taylor expansion, and the next step is to expand the summation in powers of ε. This will require the expansion of gamma functions, a topic that will be discussed in section 3.4. However before we get there we will first discuss how to obtain expansions of the parameter integrand by using the Mellin-Barnes transformation.

3.2 Mellin-Barnes Summations

An alternative way to express loop calculations in terms of concatenated sums is given by the inverse Mellin-Barnes transformation [25]. This should not be confused with MB splitting (subsection 2.4.2), where the transformation is used before the momentum integrations. In this case we use it after all momentum integrations have been performed in order to expand the denominator in the parameter integration (see equation 3.15).

3.2.1 Comparison of Taylor and Mellin-Barnes Expansions

First we would like to gain a better understanding of differences and similarities between Taylor and Mellin-Barnes expansions of a denominator. In order to compare both procedures we use a very simple but general example, the function $(A+B)^{-c}$. The Taylor expansion is straightforward and we get:

$$\frac{1}{(A+B)^c} = \frac{1}{A^c\left(1+\frac{B}{A}\right)^c} = \frac{1}{A^c\,\Gamma(c)}\sum_{n=0}^{\infty}\frac{\Gamma(n+c)}{\Gamma(n+1)}\left(\frac{-B}{A}\right)^n \quad , \tag{3.26}$$

where the condition for convergence $|B/A| < 1$ must be enforced the moment the expansion is written.

The Mellin-Barnes procedure is a bit longer. First we express the function in terms of a complex integration:

$$\frac{1}{(A+B)^c} = \frac{1}{2\pi i}\frac{1}{A^c\,\Gamma(c)}\int_{-i\infty}^{i\infty}\Gamma(z+c)\Gamma(-z)\left(\frac{B}{A}\right)^z \tag{3.27}$$

In order to perform the integration we complete the contour at complex infinity and use the residue theorem to obtain the final summation, but for this procedure to be correct the integrand must vanish at complex infinity (so that the initial contour integration matches the final one). This will impose conditions on the integrand, namely the product of gamma functions and the monomial $\left(\frac{B}{A}\right)^z$, which ultimately will ensure convergence of the final summation.

To evaluate the product of gamma functions at infinity we use the asymptotic expansion valid for $|z| \to \infty$ and $Arg(z) < \pi$:

$$\log(\Gamma(z+c)) \approx \left(z+c-\frac{1}{2}\right)\log(z) - z + \frac{1}{2}\log(2\pi) + \sum_{k=2}^{\infty}\frac{(-1)^k B_k(c)}{k(k-1)z^{k-1}} \ , \tag{3.28}$$

where B_k are Bernoulli polynomials. Ignoring vanishing terms and assuming $Arg(z) \neq \pi$ and $Arg(-z) \neq \pi$ we obtain:

$$\begin{aligned}
& \log(\Gamma(z+c)\Gamma(-z)) && (3.29)\\
= \ & \log(\Gamma(z+c)) + \log(\Gamma(-z)) + 2\pi i\left\lfloor\frac{\pi - \arg(\Gamma(z+c)) - \arg(\Gamma(-z))}{2\pi}\right\rfloor \\
\approx \ & \left[\left(z+c-\frac{1}{2}\right)\log(z) - z + \frac{1}{2}\log(2\pi)\right] + \left[\left(-z-\frac{1}{2}\right)\log(-z) + z + \frac{1}{2}\log(2\pi)\right] + O(z^0) \\
\approx \ & \left[\left(z+c-\frac{1}{2}\right)\log(z)\right] + \left[\left(-z-\frac{1}{2}\right)(\log(z)+i\pi)\right] + O(z^0) \\
\approx \ & (c-1)\log(z) - i\pi z + O(z^0) \ , \qquad |z| \to \infty && (3.30)
\end{aligned}$$

When we exponentiate only the first term is relevant and we obtain:

$$\Gamma(z+c)\,\Gamma(-z) \approx z^{c-1}\ , \qquad |z| \to \infty \tag{3.31}$$

which is divergent. However, since the monomial $\left(\frac{B}{A}\right)^z$ is dominant the full integrand will vanish as long as $|B/A| < 1$.

With conditions well understood, we proceed by completing the contour and using the residue theorem to perform the integration. Poles will arise only from gamma functions (eq. B.8) in the numerator, and for our example we obtain:

$$\begin{aligned}\frac{1}{2\pi i}\frac{1}{A^c\Gamma(c)}\oint \Gamma(z+c)\Gamma(-z)\left(\frac{B}{A}\right)^z &= \frac{1}{A^c\Gamma(c)}\sum_{n=0}^{\infty}\Gamma(n+c)\frac{(-1)^n}{n!}\left(\frac{B}{A}\right)^n \\ = \frac{1}{A^c\ \Gamma(c)}\sum_{n=0}^{\infty}\frac{\Gamma(n+c)}{\Gamma(n+1)}\left(\frac{-B}{A}\right)^n\ , \end{aligned}\tag{3.32}$$

which matches equation 3.26. While both procedures reached the same final result, the main difference is on the moment the convergence condition must be imposed. Comparing equations 3.26 and 3.27, we notice that both express the initial function in terms of powers of B/A, but only the Taylor case requires already at that moment a condition to be valid. That means the Mellin-Barnes approach is more flexible when we introduce new intermediate operations like a parameter integration, and in some cases it will allow for valid calculations that could not be performed with a Taylor expansion. In the example above no extra operations were introduced which explains why both procedures lead to identical results.

In order to make the difference more obvious, let's consider the following integral:

$$\int_0^1 \frac{du}{(a-b\ u)^2}\ . \tag{3.33}$$

If we factorize the denominator as $a^2(1-(b/a)u)^2$ there are no surprises, both Taylor and MB are valid and lead to the same result:

$$\begin{aligned}\int_0^1 \frac{du}{a^2\left(1-\frac{b\ u}{a}\right)^2} &= \frac{1}{a^2}\sum_{i=0}^{\infty}\frac{\Gamma(i+2)}{\Gamma(2)\Gamma(i+1)}\int_0^1 du\left(\frac{b\ u}{a}\right)^i \\ = \frac{1}{a^2}\sum_{i=0}^{\infty}\frac{\Gamma(i+2)}{\Gamma(2)\Gamma(i+1)}\left(\frac{b}{a}\right)^i\frac{\Gamma(i+1)}{\Gamma(i+2)} &= \frac{1}{a^2}\sum_{i=0}^{\infty}\left(\frac{b}{a}\right)^i = \frac{1}{a(a-b)}\end{aligned}\tag{3.34}$$

and

$$\int_0^1 \frac{du}{a^2\left(1-\frac{b\ u}{a}\right)^2} = \frac{1}{2\pi i a^2}\int_{-i\infty}^{i\infty} dz \frac{\Gamma(-z)\Gamma(z+2)}{\Gamma(2)} \int_0^1 du \left(\frac{-b\ u}{a}\right)^z$$
$$= \frac{1}{2\pi i a^2}\int_{-i\infty}^{i\infty} dz \frac{\Gamma(-z)\Gamma(z+2)}{\Gamma(2)} \left(\frac{-b}{a}\right)^z \frac{\Gamma(z+1)}{\Gamma(z+2)}$$
$$= \frac{1}{a^2}\sum_{i=0}^{\infty}\Gamma(i+1)\left(\frac{-b}{a}\right)^i \frac{(-1)^i}{i!} = \frac{1}{a^2}\sum_{i=0}^{\infty}\left(\frac{b}{a}\right)^i = \frac{1}{a(a-b)} \ . \qquad (3.35)$$

The interesting part appears when we factorize the denominator as $(b\ u)^2\left(1-\frac{a}{b\ u}\right)^2$. The Taylor expansion fails because the convergence condition $|\frac{a}{b\ u}| < 1$ cannot be ensured for the full range of u integration:

$$\int_0^1 \frac{du}{(b\ u)^2\left(1-\frac{a}{b\ u}\right)^2} \neq \frac{1}{b^2}\int_0^1 du \sum_{i=0}^{\infty} \frac{\Gamma(i+2)}{\Gamma(2)\Gamma(i+1)}\left(\frac{a}{b}\right)^i \left(\frac{1}{u}\right)^{i+2} \ , \qquad (WRONG) \quad (3.36)$$

however with MB we obtain:

$$\int_0^1 \frac{du}{(b\ u)^2\left(1-\frac{a}{b\ u}\right)^2} = \frac{1}{2\pi i}\frac{1}{b^2}\int_{-i\infty}^{i\infty} dz \frac{\Gamma(-z)\Gamma(z+2)}{\Gamma(2)} \int_0^1 du \left(\frac{-a}{b}\right)^z u^{-z-2}$$
$$= \frac{1}{2\pi i}\frac{1}{b^2}\int_{-i\infty}^{i\infty} dz \frac{\Gamma(-z)\Gamma(z+2)}{\Gamma(2)} \left(\frac{-a}{b}\right)^z \frac{\Gamma(-z-1)}{\Gamma(-z)}$$
$$= \frac{1}{2\pi i}\frac{1}{b^2}\int_{-i\infty}^{i\infty} dz \Gamma(-z-1)\Gamma(z+2) \left(\frac{-a}{b}\right)^z$$
$$= \frac{1}{b^2}\sum_{n=-1}^{\infty} \frac{(-1)^{n+1}}{(n+1)!}\Gamma(z+2)\left(\frac{-a}{b}\right)^n \qquad \text{(Condition has been imposed)}$$
$$= \frac{-1}{b^2}\sum_{n=-1}^{\infty}\left(\frac{a}{b}\right)^n = \frac{-1}{b^2}\frac{-b^2}{a(a-b)} = \frac{1}{a(a-b)} \ , \qquad (3.37)$$

which is the right answer. Mellin-Barnes succeeds where Taylor fails because the former delays the imposition of a condition until after the parameter integration has been performed, and it is much easier to verify a condition for a single point than over a whole range of variables. When used in this form, one could think of Mellin-Barnes as some sort of "Taylor series operator" which will give a valid expansion of a function yet to be written. In this sense, for these kind of integrals the Mellin-Barnes expansion is a generalization of the Taylor expansion, however the latter is simpler and so in this work we will try to determine the applicability and limitations of Taylor expansions to loop integrals, leaving the general study of Mellin-Barnes expansion to a later date.

3.3 General Simplifications

Before proceeding to the expansion of gamma functions in terms of Z-Sums, we would like to discuss some simplifications which will be useful when introducing properties of these functions.

Partial Fractioning. Whenever we encounter a product of the form:

$$\frac{1}{(ai+b)^{m_1}}\ \frac{1}{(ci+d)^{m_2}}\ , \tag{3.38}$$

where a, b, c and d are constants or summation variables other than i, we will use partial fractioning to reduce it to factors with a single denominator dependent on i. In most general form it is given by:

$$\frac{1}{d_1}\ \frac{1}{d_2} = \frac{1}{c_1\ d_1 + c_2\ d_2}\left(\frac{c_2}{d_1} + \frac{c_1}{d_2}\right)\ , \tag{3.39}$$

with $(c_1\ d_1 + c_2\ d_2) \neq 0$. If $(c_1\ d_1 + c_2\ d_2) = 0$ no partial fractioning is required since the second factor would be a multiple of the first. The most commonly used special cases are:

$$\frac{1}{i+b}\ \frac{1}{i+d} = \frac{1}{d-b}\left(\frac{1}{i+b} - \frac{1}{i+d}\right)\ , \tag{3.40}$$

$$\frac{1}{i+b}\ \frac{1}{n-i+d} = \frac{1}{n+d+b}\left(\frac{1}{i+b} + \frac{1}{n-i+d}\right)\ , \tag{3.41}$$

$$\frac{1}{i+b}\ \frac{1}{2i+d} = \frac{1}{d-2b}\left(\frac{1}{i+b} - \frac{2}{2i+d}\right)\ . \tag{3.42}$$

Repeated applications of equation 3.40 on 3.38 will eventually lead to a sum involving single factors in i as given by:

$$\begin{aligned}
&\frac{1}{(ai+b)^{m_1}}\ \frac{1}{(ci+d)^{m_2}} = \\
&= \left(\frac{a}{ad-cb}\right)^{m_2} \sum_{j=1}^{m_1} \binom{m_1+m_2-j-1}{m_2-1} \left(\frac{-c}{ad-cb}\right)^{m_1-j} \frac{1}{(ai+b)^j} \\
&+ \left(\frac{-c}{ad-cb}\right)^{m_1} \sum_{j=1}^{m_2} \binom{m_1+m_2-j-1}{m_1-1} \left(\frac{a}{ad-cb}\right)^{m_2-j} \frac{1}{(ci+d)^j}\ .
\end{aligned} \tag{3.43}$$

Ratios of Gamma Functions. Quite often we will have to deal with ratios of gamma functions, which may appear directly from the parameter integration as in:

$$\frac{\Gamma(i+a)\Gamma(b)}{\Gamma(i+a+b)}\ , \tag{3.44}$$

with a and b being constants, or as a beta function that matches the summation variables of the inverse of a binomial coefficient:

$$\binom{i}{j}\frac{\Gamma(i-j+a)\Gamma(j+b)}{\Gamma(i+a+b)} = \frac{\Gamma(i+1)}{\Gamma(i+a+b)}\frac{\Gamma(i-j+a)}{\Gamma(i-j+1)}\frac{\Gamma(j+b)}{\Gamma(j+1)} \; . \tag{3.45}$$

Whenever we encounter a ratio of gamma functions as in 3.44, with only two gamma functions involving the same summation variables plus a constant gamma function, we will refer to it as a pair of gammas. To simplify these pairs of gamma we make use of the identity:

$$\Gamma(z+1) = z\Gamma(z) \; . \tag{3.46}$$

Combining equations 3.46 and 3.39 we can write an expression for full partial fractioning of a pair of gamma functions:

$$\frac{\Gamma(i+a)}{\Gamma(i+b)} = \frac{1}{(i+b-1)(i+b-2)\dots(i+a)} = \frac{1}{\Gamma(b-a)}\sum_{j=0}^{b-a-1}\binom{b-a-1}{j}\frac{(-1)^j}{i+a+j} \; , \tag{3.47}$$

with $(b-a)$ a positive integer. This is a special case of equation B.16.

For the case of $(b-a)$ being a negative integer we can write it in terms of a polynomial in i:

$$\frac{\Gamma(i+a)}{\Gamma(i+b)} = (i+a-1)(i+a-2)\dots(i+b) = \sum_{j=0}^{a-b}\kappa_{a-b-j}(b,\dots,a-1)\; i^j \; , \tag{3.48}$$

where the kappa function $\kappa_s(t)$ is the sum of products of every subset of t with s elements, defined in appendix B, section B.3.

Using partial fractioning it is also possible to find identities that mix the arguments of two pairs of gamma. For example:

$$\begin{aligned} &\frac{\Gamma(n_1)}{\Gamma(n_1+a)}\frac{\Gamma(n_2)}{\Gamma(n_2+b)} \\ = \; &\sum_{k=1}^{a}\binom{a+b-k-1}{a-k}\frac{\Gamma(n_1)}{\Gamma(n_1+k)}\frac{\Gamma(n_1+n_2+k-1)}{\Gamma(n_1+n_2+a+b-1)} \\ + \; &\sum_{k=1}^{b}\binom{a+b-k-1}{b-k}\frac{\Gamma(n_2)}{\Gamma(n_2+k)}\frac{\Gamma(n_1+n_2+k-1)}{\Gamma(n_1+n_2+a+b-1)} \quad , \end{aligned} \tag{3.49}$$

$$\begin{aligned}
&\frac{\Gamma(n_1)}{\Gamma(n_1+a)}\frac{\Gamma(n_2)}{\Gamma(n_2+b)} \\
= \quad & (-1)^b \sum_{k=1}^{a} \binom{a+b-k-1}{a-k} \frac{\Gamma(n_1)}{\Gamma(n_1+k)} \frac{\Gamma(n_1-n_2-b+k)}{\Gamma(n_1-n_2+a)} \\
+ \quad & (-1)^b \sum_{k=1}^{b} \binom{a+b-k-1}{b-k} (-1)^k \frac{\Gamma(n_2)}{\Gamma(n_2+k)} \frac{\Gamma(n_1-n_2-b+1)}{\Gamma(n_1-n_2+a+k-1)} \quad .
\end{aligned} \tag{3.50}$$

where a and b are natural numbers. These identities can be useful in obtaining the result for an unknown integral from two other known integrations.

Lowering and Raising Operators. In some cases we need to perform sums involving powers of the summation variable in the numerator. Sometimes it will be useful to remove these factors by using derivatives, as in:

$$\sum_{i=1}^{n} i^m x^i f(i) = \left(x \frac{d}{dx} \right)^m \left(\sum_{i=1}^{n} x^i f(i) \right) \quad , \tag{3.51}$$

Similarly, when the i factor is in the denominator we could use:

$$\sum_{i=1}^{n} \frac{x^i f(i)}{i^m} = \underbrace{\int \frac{dx'}{x'} \dots \int \frac{dx'}{x'}}_{m} \left(\sum_{i=1}^{n} (x')^i f(i) \right) \quad . \tag{3.52}$$

In case $f(i)$ has a x dependence we can always make the substitution $x^i \to (x_0 x)^i$ (but not in $f(i)$), perform the derivative or integration with respect to x_0 and and ultimately take the limit $x_0 \to 1$. We call these lowering and raising operators:

$$\begin{aligned}
(x^-) \cdot f(x) &= x \frac{d}{dx} f(x) \quad , \\
(x^+) \cdot f(x) &= \int_0^x \frac{dx'}{x'} f(x') \quad , \\
(x^+)^m \cdot 1 &= \frac{1}{m!} \ln^m(x) \quad .
\end{aligned} \tag{3.53}$$

3.4 Z-Sums and their Properties

In the first two sections of this chapter we obtained a general expression for a loop calculation in terms of concatenated sums and noted that the dependence on the infinitesimal ε (originating from $d = 2m - 2\varepsilon$) appears only inside gamma functions. We would like to obtain the full result in terms of a power series in ε, including poles for divergent diagrams, and so we expand the gamma function according to:

$$\begin{aligned}\Gamma(n+\epsilon) &= \theta(n>0)\ \Gamma(1+\epsilon)\ \Gamma(n)\ \sum_{i=0}^{n-1} \epsilon^i\ Z_i(n-1) \\ &+ \theta(n\leq 0)\ \frac{\Gamma(1+\epsilon)\ (-1)^n}{\epsilon\ \Gamma(1-n)}\ \sum_{i=0}^{\infty} \epsilon^i\ S_i(-n)\end{aligned} \tag{3.54}$$

and

$$\begin{aligned}\frac{1}{\Gamma(n+\epsilon)} &= \theta(n>0)\ \frac{1}{\Gamma(1+\epsilon)\ \Gamma(n)} \sum_{i=0}^{\infty} (-\epsilon)^i\ S_i(n-1) \\ &+ \theta(n\leq 0)\ \frac{\epsilon\ \Gamma(1-n)\ (-1)^n}{\Gamma(1+\epsilon)} \sum_{i=0}^{-n} (-\epsilon)^i\ Z_i(-n)\end{aligned} \tag{3.55}$$

where n is an integer, θ is a boolean step function (appendix A), and Z_i and S_i are called Euler-Zagier sum and harmonic sum (subsection 3.4.6), respectively. These functions are special cases of Z-Sums and S-Sums [10, 9, 27], defined by:

$$Z(n) = \begin{cases} 1 & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases} \tag{3.56}$$

$$Z(n; m_1, \ldots, m_k; x_1, \ldots, x_k) = \sum_{n \geq i_1 > i_2 > \ldots > i_k > 0} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}} \tag{3.57}$$

and

$$S(n) = \begin{cases} 1 & \text{if } n > 0, \\ 0 & \text{if } n \leq 0. \end{cases} \tag{3.58}$$

$$S(n; m_1, \ldots, m_k; x_1, \ldots, x_k) = \sum_{n \geq i_1 \geq i_2 \geq \ldots \geq i_k \geq 1} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}}\ , \tag{3.59}$$

where k is called "depth" and $(m_1 + \ldots + m_k)$ is the "weight". Z-Sums satisfy a Hopf algebra, with properties like multiplication, conjugation, and convolution. These operations will be discussed in this section at the summation level, and used for a systematic reduction of concatenated sums.

3.4.1 Conversion

S-Sums are very similar to Z-Sums and differ only by the summation limits. They are introduced as sometimes it is more convenient to use the former than the latter. The conversion between both can be done iteratively by:

$$\begin{aligned} S(n;m_1,\ldots;x_1,\ldots) &= \sum_{i_1=1}^{n} \frac{x_1^{i_1}}{i_1^{m_1}} \sum_{i_2=1}^{i_1-1} \frac{x_2^{i_2}}{i_2^{m_2}} S(i_2;m_3,\ldots;x_3\ldots) \\ &+ S(n;m_1+m_2,m_3,\ldots;x_1x_2,x_3\ldots) \\ Z(n;m_1,\ldots;x_1,\ldots) &= \sum_{i_1=1}^{n} \frac{x_1^{i_1}}{i_1^{m_1}} \sum_{i_2=1}^{i_1} \frac{x_2^{i_2-1}}{i_2^{m_2}} Z(i_2-1;m_3,\ldots;x_3\ldots) \\ &- Z(n;m_1+m_2,m_3,\ldots;x_1x_2,x_3\ldots)\ , \end{aligned} \tag{3.60}$$

where the Z-Sum or S-Sum is obtained once the depth of the innermost sum goes to zero.

3.4.2 Upper Limit Operations

Sometimes it is necessary to shift or double the upper limit on a Z-Sum or S-Sum. We use the following algorithms.

Shifting. Shifting of the upper limit can be done iteratively using:

$$\begin{aligned} Z(n+c;m_1,\ldots) &= Z(n-1;m_1,\ldots) + \sum_{j=0}^{c} \frac{x_1^{n+j}}{(n+j)^{m_1}} Z(n+j-1;m_2,\ldots)\ , \qquad (3.61) \\ Z(n-c;m_1,\ldots) &= Z(n-1;m_1,\ldots) - \sum_{j=1}^{c-1} \frac{x_1^{n-j}}{(n-j)^{m_1}} Z(n-j-1;m_2,\ldots)\ , \end{aligned}$$

and

$$\begin{aligned} S(n+c;m_1,\ldots) &= S(n;m_1,\ldots) + \sum_{j=1}^{c} \frac{x_1^{n+j}}{(n+j)^{m_1}} S(n+j;m_2,\ldots)\ , \qquad (3.62) \\ S(n-c;m_1,\ldots) &= S(n;m_1,\ldots) - \sum_{j=0}^{c-1} \frac{x_1^{n-j}}{(n-j)^{m_1}} S(n-j;m_2,\ldots)\ , \end{aligned}$$

where c is an integer. Repeated applications will reduce the depth of the Z-Sum or S-Sum on the right to zero. It is also possible to write the full algorithms which are given by:

$$\begin{aligned} Z(n+c;m_1,\ldots) &= Z(n-1;m_1,\ldots) + Z(n-1;m_2,\ldots)\sum_{j_1=0}^{c}\frac{x_1^{n+j_1}}{(n+j_1)^{m_1}} \\ &+ Z(n-1;m_3,\ldots)\sum_{j_1=0}^{c}\frac{x_1^{n+j_1}}{(n+j_1)^{m_1}}\sum_{j_2=0}^{j_1-1}\frac{x_2^{n+j_2}}{(n+j_2)^{m_2}} + \ldots \\ &+ \sum_{j_1=0}^{c}\frac{x_1^{n+j_1}}{(n+j_1)^{m_1}}\sum_{j_2=0}^{j_1-1}\frac{x_2^{n+j_2}}{(n+j_2)^{m_2}}\cdots\sum_{j_k=0}^{j_{k-1}-1}\frac{x_k^{n+j_k}}{(n+j_k)^{m_k}} \quad , \end{aligned} \tag{3.63}$$

$$\begin{aligned} Z(n-c;m_1,\ldots) &= Z(n-1;m_1,\ldots) - Z(n-1;m_2,\ldots)\sum_{j_1=1}^{c-1}\frac{x_1^{n-j_1}}{(n-j_1)^{m_1}} \\ &+ Z(n-1;m_3,\ldots)\sum_{j_1=1}^{c-1}\frac{x_1^{n-j_1}}{(n-j_1)^{m_1}}\sum_{j_2=1}^{j_1}\frac{x_2^{n-j_2}}{(n-j_2)^{m_2}} - \ldots \\ &+ (-1)^k\sum_{j_1=1}^{c-1}\frac{x_1^{n-j_1}}{(n-j_1)^{m_1}}\sum_{j_2=1}^{j_1}\frac{x_2^{n-j_2}}{(n-j_2)^{m_2}}\cdots\sum_{j_k=1}^{j_{k-1}}\frac{x_k^{n-j_k}}{(n-j_k)^{m_k}} \quad , \end{aligned} \tag{3.64}$$

$$\begin{aligned} S(n+c;m_1,\ldots) &= S(n;m_1,\ldots) + S(n;m_2,\ldots)\sum_{j_1=1}^{c}\frac{x_1^{n+j_1}}{(n+j_1)^{m_1}} \\ &+ S(n;m_3,\ldots)\sum_{j_1=1}^{c}\frac{x_1^{n+j_1}}{(n+j_1)^{m_1}}\sum_{j_2=1}^{j_1}\frac{x_2^{n+j_2}}{(n+j_2)^{m_2}} + \ldots \\ &+ \sum_{j_1=1}^{c}\frac{x_1^{n+j_1}}{(n+j_1)^{m_1}}\sum_{j_2=1}^{j_1}\frac{x_2^{n+j_2}}{(n+j_2)^{m_2}}\cdots\sum_{j_k=1}^{j_{k-1}}\frac{x_k^{n+j_k}}{(n+j_k)^{m_k}} \quad , \end{aligned} \tag{3.65}$$

$$\begin{aligned} S(n-c;m_1,\ldots) &= S(n;m_1,\ldots) - S(n;m_2,\ldots)\sum_{j_1=0}^{c-1}\frac{x_1^{n-j_1}}{(n-j_1)^{m_1}} \\ &+ S(n;m_3,\ldots)\sum_{j_1=0}^{c-1}\frac{x_1^{n-j_1}}{(n-j_1)^{m_1}}\sum_{j_2=0}^{j_1-1}\frac{x_2^{n-j_2}}{(n-j_2)^{m_2}} - \ldots \\ &+ (-1)^k\sum_{j_1=0}^{c-1}\frac{x_1^{n-j_1}}{(n-j_1)^{m_1}}\sum_{j_2=0}^{j_1-1}\frac{x_2^{n-j_2}}{(n-j_2)^{m_2}}\cdots\sum_{j_k=0}^{j_{k-1}-1}\frac{x_k^{n-j_k}}{(n-j_k)^{m_k}} \quad . \end{aligned} \tag{3.66}$$

Note that these reduction can be performed regardless of c being a "bounded number" or a summation variable[2]. Usually partial fractioning will be used to express the result in terms of a single factor $\frac{1}{(n\pm j)^m}$.

Limit Doubling. The purpose of this algorithm is to express $Z(n;\ldots)$ in terms of $Z(2n;\ldots)$, and for that we use equation C.5. When the depth is one we have:

$$\begin{aligned} Z(n;m;x) &= \sum_{i=1}^{n} \frac{x^i}{i^m} = 2^m \sum_{i=1}^{n} \frac{(\sqrt{x})^{2i}}{(2i)^m} = 2^{m-1} \left(\sum_{i=1}^{2n} \frac{(\sqrt{x})^i}{i^m} + \sum_{i=1}^{2n} \frac{(-\sqrt{x})^i}{i^m} \right) \\ &= 2^{m-1} \left(Z(2n;m;\sqrt{x}) + Z(2n;m;-\sqrt{x}) \right) \quad . \end{aligned} \tag{3.67}$$

If the depth is higher than one we use:

$$\begin{aligned} Z(n;m_1,\ldots;x_1,\ldots) = \sum_{i=1}^{n} \frac{x_1^i}{i^{m_1}} Z(i-1;\ldots) = 2^{m_1} \sum_{i=1}^{n} \frac{(\sqrt{x_1})^{2i}}{(2i)^{m_1}} Z(i-1;\ldots) \\ = 2^{m_1-1} \left(\sum_{i=1}^{2n} \frac{(\sqrt{x_1})^i}{i^{m_1}} Z(\lfloor\frac{i}{2}\rfloor - 1;\ldots) + \sum_{i=1}^{2n} \frac{(-\sqrt{x_1})^i}{i^{m_1}} Z(\lfloor\frac{i}{2}\rfloor - 1;\ldots) \right) \quad . \end{aligned} \tag{3.68}$$

The internal sum limit can be doubled recursively until the depth is zero, and we obtain a result in terms of $Z(2n;\ldots)$. Note that only the terms with even i correspond to the initial equation, while the odd i terms simply cancel out. That means we could even use the incorrect substitution $2(\lfloor\frac{i}{2}\rfloor - 1) \to (i-2)$ and obtain the right answer as long as we apply it systematically. This substitution is of course not necessary but simplifies the process.

This procedure can be generalized to arbitrary multipliers using roots of unity (see equations C.6 and C.7, appendix C) but since we are only concerned with parameter integrations involving quadratic denominators equations 3.67 and 3.68 will be sufficient.

[2] We define bounded numbers as involving integers and variables from summations with a finite number of terms. Furthermore, when we mention "summation variable" without further explanation it usually means a variable from a sum with an infinite number of terms. We will be using this definition repeatedly from now on.

3.4.3 Multiplication

As a consequence of equation C.11, Z-Sums with the same upper limit have a product operation defined iteratively by:

$$\begin{aligned}
& Z(n;m_1,\ldots,m_k;x_1,\ldots,x_k)\times Z(n;m'_1,\ldots,m'_l;x'_1,\ldots,x'_l) \qquad (3.69)\\
= & \sum_{i_1=1}^{n}\frac{x_1^{i_1}}{i_1^{m_1}}Z(i_1-1;m_2,\ldots,m_k;x_2,\ldots,x_k)\times Z(i_1-1;m'_1,\ldots,m'_l;x'_1,\ldots,x'_l)\\
+ & \sum_{i_2=1}^{n}\frac{{x'_1}^{i_2}}{i_2^{m'_1}}Z(i_2-1;m_1,\ldots,m_k;x_1,\ldots,x_k)\times Z(i_2-1;m'_2,\ldots,m'_l;x'_2,\ldots,x'_l)\\
+ & \sum_{i=1}^{n}\frac{(x_1x'_1)^{i}}{i_1^{m_1+m'_1}}Z(i-1;m_2,\ldots,m_k;x_2,\ldots,x_k)\times Z(i-1;m'_2,\ldots,m'_l;x'_2,\ldots,x'_l)\ .
\end{aligned}$$

After repeated applications one of the Z-Sums will be of zero depth. S-Sums have a similar identity:

$$\begin{aligned}
& S(n;m_1,\ldots,m_k;x_1,\ldots,x_k)\times S(n;m'_1,\ldots,m'_l;x'_1,\ldots,x'_l) \qquad (3.70)\\
= & \sum_{i_1=1}^{n}\frac{x_1^{i_1}}{i_1^{m_1}}S(i_1;m_2,\ldots,m_k;x_2,\ldots,x_k)\times S(i_1;m'_1,\ldots,m'_l;x'_1,\ldots,x'_l)\\
+ & \sum_{i_2=1}^{n}\frac{{x'_1}^{i_2}}{i_2^{m'_1}}S(i_2;m_1,\ldots,m_k;x_1,\ldots,x_k)\times S(i_2;m'_2,\ldots,m'_l;x'_2,\ldots,x'_l)\\
- & \sum_{i=1}^{n}\frac{(x_1x'_1)^{i}}{i_1^{m_1+m'_1}}S(i;m_2,\ldots,m_k;x_2,\ldots,x_k)\times S(i;m'_2,\ldots,m'_l;x'_2,\ldots,x'_l)\ .
\end{aligned}$$

While the multiplication operation requires summations with the same upper limit, if necessary we can always synchronize the summation limits by using the operations discussed in subsection 3.4.2, meaning any product of Z and S-Sums can always be simplified using the multiplication operation.

3.4.4 Convolution

Whenever a summation $\sum_{i=1}^{n-1}$ involves both $Z(i;\ldots)$ and $Z(n-i;\ldots)$ we use the convolution operation to decrease the depth of the latter all the way to zero and obtain a simpler expression. The general function to be treated is given by:

$$\sum_{i=1}^{n-1} f(i)\ Z(n-i+c;m_1,\ldots)\ , \qquad (3.71)$$

where $f(i)$ will involve $Z(i;\ldots)$.

We start by synchronizing the sub-sum to obtain:

$$\begin{aligned}
&\sum_{i=1}^{n-1} f(i)\ Z(n-i-1;m_1,\ldots) \\
=\ &\sum_{i=1}^{n-1} f(i) \sum_{j=1}^{n-i-1} \frac{{x_1}^j}{j^{m_1}}\ Z(j-1;m_2,\ldots) \\
=\ &\sum_{j=1}^{n-1} \left(\sum_{i=1}^{j-1} f(i) \frac{{x_1}^{j-i}}{(j-i)^{m_1}}\ Z(j-i-1;m_2,\ldots) \right) ,
\end{aligned} \tag{3.72}$$

The internal sum in the final result is of the same form as the starting point, but with reduced depth. Repeated applications will eventually bring it to zero, concluding the recursion.

In order to obtain equation 3.72 we performed a simple shifting and reordering of terms as given by:

$$\sum_{i=1}^{n-1} \sum_{j=1}^{n-i-1} f(i,j) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n-1} f(i,j-i) = \sum_{j=2}^{n-1} \sum_{i=1}^{j-1} f(i,j-i) = \sum_{j=1}^{n-1} \sum_{i=1}^{j-1} f(i,j-i) \quad . \tag{3.73}$$

The procedure also works when dealing with S-Sums, in which case the reduction is given by:

$$\begin{aligned}
&\sum_{i=1}^{n} f(i)\ S(n-i+c;m_1,\ldots) \to \sum_{i=1}^{n} f(i)\ S(n-i;m_1,\ldots) \\
&= \sum_{i=1}^{n} f(i) \sum_{j=1}^{n-i} \frac{{x_1}^j}{j^{m_1}}\ S(j;m_2,\ldots) = \sum_{j=1}^{n} \left(\sum_{i=1}^{j-1} f(i) \frac{{x_1}^{j-i}}{(j-i)^{m_1}}\ S(j-i;m_2,\ldots) \right) .
\end{aligned} \tag{3.74}$$

While equations 3.72 and 3.74 are general and valid for any $f(i)$, our intention in using them is to remove the dependence on $Z(n-i;\ldots)$ and so we will always require that $f(i)$ itself has no unremovable $(n-i)$ dependence.

3.4.5 Conjugation

The Conjugation operation enables the following reduction:

$$-\sum_{i=1}^{n} \binom{n}{i} (-1)^i \frac{x^i}{i^m} S(i;\ldots;\ldots) \to S(n;\ldots;\ldots) \ . \tag{3.75}$$

In order to perform the reduction first we define B-Sums according to:

$$B(n;N;m_1,\ldots,m_k;x_1,\ldots,x_k) = \sum_{i_1=n+1}^{N}\sum_{i_2=i_1+1}^{N}\cdots\sum_{i_k=i_{k-1}+1}^{N}\frac{x_1^{i_1}}{i_1^{m_1}}\frac{x_2^{i_2}}{i_2^{m_2}}\cdots\frac{x_k^{i_k}}{i_k^{m_k}}\ . \tag{3.76}$$

S-Sums are now converted to B-Sums using:

$$\begin{aligned} S(i;m_1,\ldots) &= S(N;m_1,\ldots) - S(N;m_2,\ldots)\sum_{i_1=i+1}^{N}\frac{x_1^{i_1}}{i_1^{m_1}} \\ &+S(N;m_3,\ldots)\sum_{i_1=i+1}^{N}\sum_{i_2=i_1+1}^{N}\frac{x_1^{i_1}}{i_1^{m_1}}\frac{x_2^{i_2}}{i_2^{m_2}} \\ &-\ldots+(-1)^k\sum_{i_1=i+1}^{N}\cdots\sum_{i_k=i_{k-1}+1}^{N}\frac{x_1^{i_1}}{i_1^{m_1}}\cdots\frac{x_k^{i_k}}{i_k^{m_k}}\ , \end{aligned} \tag{3.77}$$

where N is an arbitrary variable that eventually will be taken to infinity. Equation 3.77 is obtained by repeated applications of:

$$\sum_{i=1}^{n}\sum_{j=1}^{i}f(i,j) = \sum_{i=1}^{N}\sum_{j=1}^{i}f(i,j) - \sum_{i=n+1}^{N}\sum_{j=1}^{N}f(i,j) + \sum_{i=n+1}^{N}\sum_{j=i+1}^{N}f(i,j)\ . \tag{3.78}$$

Using raising operators (equations 3.53) we can write:

$$\begin{aligned} &-\sum_{i=1}^{n}\binom{n}{i}(-1)^i\frac{x_0^i}{i^{m_0}}\sum_{i_1=i+1}^{N}\cdots\sum_{i_k=i_{k-1}+1}^{N}\frac{x_1^{i_1}}{i_1^{m_1}}\cdots\frac{x_k^{i_k}}{i_k^{m_k}} \\ =\ &-(x_k^+)^{m_k}\ldots(x_0^+)^{m_0}\sum_{i=1}^{n}\binom{n}{i}(-1)^i x_0^i\sum_{i=i+1}^{N}\cdots\sum_{i_k=i_{k-1}+1}^{N}x_1^{i_1}\ldots x_k^{i_k} \\ =\ &(x_k^+)^{m_k}\ldots(x_0^+)^{m_0}\frac{x_k}{1-x_k}\frac{x_{k-1}x_k}{1-x_{k-1}x_k}\cdots\frac{x_1\ldots x_k}{1-x_1\ldots x_k}\left[1-(1-x_0x_1\ldots x_k)^n\right]\ , \end{aligned} \tag{3.79}$$

where in the last equality we used:

$$\sum_{i=n+1}^{N}x^i = \frac{x}{1-x}x^n - \frac{x}{1-x}x^N \to \frac{x}{1-x}x^n \tag{3.80}$$

and

$$\sum_{i=1}^{n}\binom{n}{i}x^i = -1+(1+x)^n\ . \tag{3.81}$$

The second term in equation 3.80 was discarded since it leads to vanishing contributions once we take the limit $N\to\infty$.

Now we use iteratively the integration identities:

$$(x^+)[1-(1-ax)^n] = \sum_{i=1}^{n} \frac{1}{i} \left[1-(1-ax)^i\right] \quad , \tag{3.82}$$

$$\begin{aligned}(x^+)\frac{ax}{1-ax}[1-(1-abx)^n] &= -(1-b)^n \sum_{i=1}^{n} \frac{1}{i} \left(\frac{1}{1-b}\right)^i \left[1-(1-abx)^i\right] \\ &\quad + [1-(1-b)^n] \sum_{i=1}^{N} \frac{(ax)^i}{i} + (x^+)\frac{ax}{1-ax}(ax)^N [1-(1-b)^n] \quad ,\end{aligned} \tag{3.83}$$

$$(x^+)\frac{ax}{1-ax}[1-(1-ax)^n] = -\frac{1}{n}[1-(1-ax)^n] + \sum_{i=1}^{N} \frac{(ax)^i}{i} + (x^+)\frac{ax}{1-ax}(ax)^N \quad , \tag{3.84}$$

which can be easily proven starting from the r.h.s.

After all (x^+) operators have been applied the result will depend on $B(n;N;\ldots)$ and $S(n;\ldots)$ only. The former can be converted back to S-Sums by using:

$$\begin{aligned}&\sum_{i_1=n+1}^{N} \cdots \sum_{i_k=i_{k-1}+1}^{N} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}} = \\ =\ & (-1)^k S(n;m_1,\ldots) - (-1)^k S(N;m_1,\ldots) + (-1)^k S(N;m_2,\ldots) \sum_{i_1=n+1}^{N} \frac{x_1^{i_1}}{i_1^{m_1}} \\ & - \ldots + (-1)^k S(N;m_k;x_k) \sum_{i_1=n+1}^{N} \cdots \sum_{i_{k-1}=i_{k-2}+1}^{N} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_{k-1}^{i_{k-1}}}{i_{k-1}^{m_{k-1}}} \ ,\end{aligned} \tag{3.85}$$

after which we apply the limit $N \to \infty$ and conclude the algorithm by obtaining an expression strictly in terms of $S(n;\ldots)$ and $S(\infty;\ldots)$, as desired.

3.4.6 Special Cases

Z and S-Sums are very general functions and have several special cases of interest which will be discussed in this subsection.

Euler-Zagier and Harmonic Sums. An important special case is obtained for Z-Sums when $x_1 = \ldots = x_k = 1$ and are called Euler-Zagier sums [29]:

$$Z_{m_1,\ldots,m_k}(n) = Z(n;m_1,\ldots,m_k;1,\ldots,1) \ . \tag{3.86}$$

Under the same condition, S-Sums reduce to Harmonic Sums:

$$S_{m_1,\ldots,m_k}(n) = S(n; m_1, \ldots, m_k; 1, \ldots, 1) \; . \tag{3.87}$$

The Z and S-Sums that appear on the gamma function expansion (equations 3.54 and 3.55) are special cases of Euler-Zagier and Harmonic Sums with $m_1 = \ldots = m_k = 1$:

$$\begin{aligned} Z_p(n) &= Z_{\underbrace{1,\ldots,1}_{\text{p}}}(n) \quad , \\ S_p(n) &= S_{\underbrace{1,\ldots,1}_{\text{p}}}(n) \quad . \end{aligned} \tag{3.88}$$

These functions satisfy the identity:

$$\left(\sum_{i}^{n-1} \varepsilon^i Z_i(n-1) \right)^{-1} = \sum_{i}^{\infty} (-\varepsilon)^i S_i(n-1) \quad , \tag{3.89}$$

which was used to obtain equation 3.55 from 3.54.

Multiple Polylogarithms. When the outer summation limit goes to infinity we obtain multiple polylogarithms of Goncharov [30] (note the reverse argument convention):

$$Li_{m_k,\ldots,m_1}(x_k, \ldots, x_1) = Z(\infty; m_1, \ldots, m_k; x_1, \ldots, x_k) \; . \tag{3.90}$$

Multiple polylogarithms generalize several special cases. These include harmonic polylogarithms of Remiddi-Vermaseren [31]:

$$H_{m_1,\ldots,m_k}(x) = Li_{m_k,\ldots,m_1}(\underbrace{1,\ldots,1}_{\text{k-1}}, x) \; , \tag{3.91}$$

Nielsen's generalized polylogarithms [32]:

$$S_{n,p}(x) = Li_{\underbrace{1,\ldots,1}_{\text{p-1}},n+1}(\underbrace{1,\ldots,1}_{\text{p-1}}, x) \; , \tag{3.92}$$

and classical polylogarithms $Li_n(x)$ (section B.4 and [33, 34]).

Multiple ζ-Values. When the conditions for multiple polylogarithms and Euler-Zagier sums are combined ($n = \infty$ and $x_1 = \ldots = x_k = 1$) we obtain Multiple ζ-Values:

$$\zeta(m_k, \ldots, m_1) = Z(\infty; m_1, \ldots, m_k; 1, \ldots, 1) \tag{3.93}$$

Now that we know these important special cases, the reason for choosing the approach discussed in this thesis for calculating loop integrations should become more clear. Z-Sums are important because they connect Euler-Zagier and Harmonic sums with multiple polylogarithms. The former arise naturally from the expansion of gamma functions while the latter are used to express loop integrations in terms of closed form functions; while physical one loop diagrams are known to be expressed in terms of only classical polylogarithms, non-physical single-loop and multi-loop diagrams require the more general multiple polylogarithms.

Let us for a moment pay attention to the mathematical building blocks that occur in the summations obtained from the Taylor series expansion of a loop integral. They are basically the following: concatenated sums and binomial coefficients obtained from the denominator and numerator expansions, denominators involving summation variables arising from the simplification of gamma pairs, and finally Euler-Zagier and Harmonic sums obtained from the expansion of gamma functions. All of these building blocks are also present in the operations from the Z-Sum machinery, that is, multiplication, convolution, and conjugation, and so the expectation is that, at least for some cases, the sums obtained from the Taylor series will immediately match the Z-Sum machinery leading to a systematic reduction method that ultimately will lead to multiple polylogarithms (since the outer summation always goes to infinity).

While this is a good hypothesis that will turn out to be correct in some cases, for others the procedure will not be so seamless. For these cases the parameter integration will lead to gamma functions with complicated arguments that mix several of the summation variables in combinations that do not match the Z-Sum machinery, leading to the necessity of new intermediate steps in the calculation. In the next chapter we will go through a thorough survey of all calculations in order to explore these possibilities and hopefully advance our understanding of the problem. Before that, we list in the next section all known summations with a systematic algorithm for reduction to Z-Sums.

3.5 Seamless Reduction Algorithms

Now we use the Z-Sum operations discussed in the last section to discuss algorithms for iteratively reducing sums obtained from a Taylor series (or either Mellin-Barnes methods) into Z-Sums, starting from the innermost all the way to the outer sums. We split the algorithms into four main types (A, B, C, and D) according to their content. A and B will not involve the binomial coefficient, unlike C and D. A and C will have the dependence on the summation variable i appear strictly as i or $(n-i)$, while B and D will necessarily use both (n denotes the upper limit on the summation $\sum_{i=1}^{n}$). As a consequence of these properties the reduction algorithms will require different Z-Sum operations for each type. A will use only multiplication, B will use multiplication and convolution, C will require multiplication and conjugation, and finally D will involve all three operations. Some of these algorithms will be further divided into different cases with minor variations. Whenever concatenated sums can be reduced exclusively with these algorithms we will refer it as a "seamless reduction".

A few of the required reduction steps will not be explicitly discussed but are implied. For example, if an expression involves both Z-Sums and S-Sums we will first convert all into the same type, the choice depending on the algorithm type to be used (Z-Sums for A and B, S-Sums for C and D). We will only consider summations involving $Z(i;\ldots)$ or $Z(2i;\ldots)$, but not the products $Z(i;\ldots)Z(i;\ldots)$, $Z(i;\ldots)Z(2i;\ldots)$ or $Z(2i;\ldots)Z(2i;\ldots)$ since these can always be easily reduced using the upper limit operations on Z-Sums and the multiplication operation. Similarly, whenever there is a product $\frac{1}{ai+b}\frac{1}{ci+d}$ we automatically use partial fractioning to obtain a single factor. Finally, for summations where all i dependence appears as $(n-i)$ we will invert the summation order by performing the substitution $i \to (n-i)$. We will only be discussing algorithms that might appear from a parameter integration involving a quadratic denominator, that is, factors multiplying summation variables can only take values of one and two. The reduction techniques for each algorithm are usually not unique, and we will be presenting the simplest ones. Their systematicity make them well suited for computational implementation [35, 36, 37]. Table 3.1 lists all summations types being considered.

Table 3.1: Summations with a systematic reduction algorithm.

A1	$\sum_{i=1}^{n} \frac{x^i}{(i+b_1)^m} Z(i+d_1; m_1, \ldots)$
A2	$\sum_{i=1}^{n} \frac{x^i}{(2i+b_1)^m} Z(2i+d_1; m_1, \ldots)$
A3	$\sum_{i=1}^{n} \frac{x^i}{(i+b_1)^m} Z(2i+d_1; m_1, \ldots)$
A4	$\sum_{i=1}^{n} \frac{x^i}{(2i+b_1)^m} Z(i+d_1; m_1, \ldots)$
B1	$\sum_{i=1}^{n-1} \frac{x^i}{[a_1 i+b_1]^{m_0}} \; Z(i+d_1; m_1, \ldots) \; \frac{y^{n-i}}{[a_2(n-i)+b_2]^{m_0'}} \; Z(n-i+d_2; m_1', \ldots)$
B2	$\sum_{i=1}^{n-1} \frac{x^i}{[2i+b_1]^{m_0}} \; Z(2i+d_1; m_1, \ldots) \; \frac{y^{n-i}}{[2(n-i)+b_2]^{m_0'}} \; Z(2(n-i)+d_2; m_1', \ldots)$
B3	$\sum_{i=1}^{n-1} \frac{x^i}{[a_1 i+b_1]^{m_0}} \; Z(c_1 i+d_1; m_1, \ldots) \; \frac{y^{n-i}}{[a_2(n-i)+b_2]^{m_0'}} \; Z(c_2(n-i)+d_2; m_1', \ldots)$
C	$\sum_{i=1}^{n} \binom{n}{i} \frac{x^i}{(i+b_1)^m} S(i+d_1; \ldots; \ldots)$
D	$\sum_{i=1}^{n-1} \binom{n}{i} \frac{x^i}{(i+b_1)^{m_0}} S(i+d_1; m_1, \ldots) \frac{y^{n-i}}{(n-i+b_2)^{m_0'}} S(n-i+d_2; m_1', \ldots)$

Type A1.

$$\sum_{i=1}^{n} \frac{x^i}{(i \pm b)^m} Z(i \pm d; m_1, \ldots) \quad . \tag{3.94}$$

Equation 3.94 almost immediately matches the Z-Sum definition were it not for the constants b and d, which for now are assumed to be bounded numbers. If the depth k of the inner sum is zero and the offset positive ($b > 0$) we get:

$$\sum_{i=1}^{n} \frac{x^i}{(i+b)^m} = \frac{1}{x^b} \left(\sum_{i=1}^{n+b} \frac{x^i}{i^m} - \sum_{i=1}^{b} \frac{x^i}{i^m} \right) = \frac{1}{x^b} \left(Z(n+b, m, x) - Z(b, m, x) \right) \quad . \tag{3.95}$$

Similarly, with negative offset ($n < b$) we use:

$$\sum_{i=1}^{n} \frac{x^i}{(i-b)^m} = \frac{x^b}{(-1)^m}\left(\sum_{i=1}^{b-1}\frac{(1/x)^i}{i^m} - \sum_{i=1}^{b-n-1}\frac{(1/x)^i}{i^m}\right) \tag{3.96}$$
$$= \frac{x^b}{(-1)^m}\left(Z(b-1,m,1/x) - Z(b-n-1,m,1/x)\right) \quad .$$

If the depth is not zero, first we synchronize the sub-sum using equations 3.63 and 3.64. If the offset is negative ($n < b$) we express it in terms of a B1 sum with positive offset (yet to be discussed):

$$\sum_{i=1}^{n} \frac{x^i}{(i-b)^m} Z(i-1;m_1,\ldots) = \frac{x^{n+1}}{(-1)^m}\sum_{i=1}^{n}\frac{(1/x)^i}{(i+b-n+1)^m} Z(n-i;m_1,\ldots) \quad . \tag{3.97}$$

For positive offset ($b > 0$) we use:

$$\sum_{i=1}^{n} \frac{x^i}{(i+b)^m} Z(i-1;m_1,\ldots) = \frac{x^n}{(n+b)^m} Z(n-1;m_1,\ldots) + \tag{3.98}$$
$$+\frac{1}{x}\sum_{i=1}^{n}\frac{x^i}{(i+b-1)^m} Z(i-1;m_1,\ldots) - \sum_{i=1}^{n-1}\frac{x^i}{(i+b)^m}\frac{x_1^i}{i^{m_1}} Z(i-1;m_2,\ldots) \quad .$$

Repeated use of equation 3.98 will bring the offset or depth to zero, and some use of partial fractioning will be required.

Type A2.

$$\sum_{i=1}^{n} \frac{x^i}{(2i+b)^m} Z(2i+d;m_1,\ldots) \tag{3.99}$$
$$= \frac{1}{2}\left(\sum_{i=1}^{2n}\frac{(\sqrt{x})^i}{(i+b)^m} Z(i+d;m_1,\ldots) + \sum_{i=1}^{2n}\frac{(-\sqrt{x})^i}{(i+b)^m} Z(i+d;m_1,\ldots)\right) \quad .$$

The summations on the right are of type A1. The appearance of a square root in the argument and the outer limit multiplied by two is typical of a sum obtained from the integration of a denominator containing a quadratic polynomial.

Type A3.

$$\sum_{i=1}^{n} \frac{x^i}{(i+b)^m} Z(2i+d;m_1,\ldots) = 2^m \sum_{i=1}^{n}\frac{x^i}{(2i+2b)^m} Z(2i+d;m_1,\ldots) \quad . \tag{3.100}$$

The expression on the right is of type A2.

Type A4.

$$\sum_{i=1}^{n} \frac{x^i}{(2i+b)^m} Z(i+d; m_1, \ldots) \rightarrow \sum_{i=1}^{n} \frac{x^i}{(2i+b)^m} Z(2i+2d; m_1, \ldots) \quad . \tag{3.101}$$

In this case we double the limit in the Z-Sum and obtain a sum of type A2.

Type B1.

$$\sum_{i=1}^{n-1} \frac{x^i}{[a_1 i + b_1]^{m_0}} \; Z(i+d_1; m_1, \ldots) \; \frac{y^{n-i}}{[a_2(n-i)+b_2]^{m'_0}} \; Z(n-i+d_2; m'_1, \ldots) \rightarrow Z(\ldots) \quad . \tag{3.102}$$

We apply partial fractioning on $\frac{x^i}{[a_1 i+b_1]^{m_0}} \frac{y^{n-i}}{[a_2(n-i)+b_2]^{m'_0}}$ followed by a summation inversion $i \rightarrow (n-i)$ on summations involving $\frac{1}{[a'_2(n-i)+b'_2]}$, leading to expressions of the form:

$$\sum_{i=1}^{n-1} \frac{x^i}{[a'_1 i + b'_1]^{m_0}} \; Z(i+d'_1; m_1, \ldots) \; Z(n-i+d'_2; m'_1, \ldots) \quad . \tag{3.103}$$

At this point we use the definition of convolution (equation 3.71) with:

$$f(i) = \frac{x^i}{[a'_1 i + b'_1]^{m_0}} \; Z(i+d'_1; m_1, \ldots) \; . \tag{3.104}$$

Repeated application will reduce the depth of the sub-sum, leading to one of the summations of type A.

Type B2.

$$\begin{aligned} & \sum_{i=1}^{n-1} \frac{x^i}{[2i+b_1]^{m_0}} \; Z(2i+d_1; m_1, \ldots) \; \frac{y^{n-i}}{[2(n-i)+b_2]^{m'_0}} \; Z(2(n-i)+d_2; m'_1, \ldots) \\ = \; & \frac{1}{2} \sum_{i=1}^{2n-2} \frac{(\sqrt{x})^i}{[i+b_1]^{m_0}} \; Z(i+d_1; m_1, \ldots) \; \frac{(\sqrt{y})^{2n-i}}{[2n-i+b_2]^{m'_0}} \; Z(2n-i+d_2; m'_1, \ldots) \\ + \; & \frac{1}{2} \sum_{i=1}^{2n-2} \frac{(-\sqrt{x})^i}{[i+b_1]^{m_0}} \; Z(i+d_1; m_1, \ldots) \; \frac{(\sqrt{y})^{2n-i}}{[2n-i+b_2]^{m'_0}} \; Z(2n-i+d_2; m'_1, \ldots) \quad . \end{aligned} \tag{3.105}$$

The summations on the right hand side are of type B1.

Type B3.

$$\sum_{i=1}^{n-1} \frac{x^i}{[a_1 i + b_1]^{m_0}} \; Z(c_1 i + d_1; m_1, \ldots) \; \frac{y^{n-i}}{[a_2(n-i) + b_2]^{m_0'}} \; Z(c_2(n-i) + d_2; m_1', \ldots) \to Z(\ldots) \; . \tag{3.106}$$

For type B3 we consider all cases with variables a_1, a_2, c_1, and c_2 taking values of one or two, with either or both c_1 and c_2 different than one. It can be reduced to type B2 by applying the substitutions below as necessary:

$$\begin{aligned} \frac{x^i}{[i + b_1]^{m_0}} &= 2^{m_0} \frac{x^i}{[2i + 2b_1]^{m_0}} \quad , \\ \frac{y^{n-i}}{[n - i + b_2]^{m_0'}} &= 2^{m_0'} \frac{y^{n-i}}{[2(n-i) + 2b_2]^{m_0'}} \quad , \\ Z(i + d_1; m_1, \ldots) &\to Z(2i + 2d_1; m_1, \ldots) \quad , \\ Z(n - i + d_2; m_1', \ldots) &\to Z(2(n-i) + 2d_2; m_1', \ldots) \quad . \end{aligned} \tag{3.107}$$

Type C.

$$\sum_{i=1}^{n} \binom{n}{i} \frac{x^i}{(i+b)^m} S(i + d; \ldots) \to S(n; \ldots) \quad . \tag{3.108}$$

We start by synchronizing the sub-sum and applying partial fractioning if necessary to obtain:

$$\sum_{i=1}^{n} \binom{n}{i} \frac{x^i}{(i+b')^m} S(i; \ldots) \quad . \tag{3.109}$$

The offset b' can be removed by using:

$$\sum_{i=1}^{n} \binom{n}{i} \frac{x^i}{(i+b')^m} S(i; \ldots) = \frac{1}{x} \frac{1}{n+1} \sum_{i=1}^{n+1} \binom{n+1}{i} \frac{x^i}{(i + b' - 1)^m} i S(i-1; \ldots) \quad . \tag{3.110}$$

After repeated iterations and, if necessary, application of the lowering operator to remove factors of i in the numerator we obtain:

$$\sum_{i=1}^{n} \binom{n}{i} \frac{x^i}{i^m} S(i; \ldots; \ldots) \quad , \tag{3.111}$$

which can be reduced using the conjugation operation.

TYPE D.

$$\sum_{i=1}^{n-1}\binom{n}{i}\frac{x^i}{(i+b_1)^{m_0}}S(i+d_1;m_1,\ldots)\frac{y^{n-i}}{(n-i+b_2)^{m'_0}}S(n-i+d_2;m'_1,\ldots)\to S(n-1;\ldots)\ . \tag{3.112}$$

We start by removing the $\frac{1}{(n-i+b_2)}$ dependence by applying partial fractioning and $i\to(n-i)$ changes whenever necessary. This leads to equations of the form:

$$\sum_{i=1}^{n-1}\binom{n}{i}\frac{x^i}{(i+b'_1)^{m_0}}S(i+d'_1;m_1,\ldots)S(n-i+d'_2;m'_1,\ldots)\ , \tag{3.113}$$

after which the convolution is used (equation 3.74) with:

$$f(i)=\binom{n}{i}\frac{x^i}{(i+b'_1)^{m_0}}S(i+d'_1;m_1,\ldots)\ . \tag{3.114}$$

Repeated applications will bring the depth of one sub-sum to zero, reducing the summation to type C.

3.6 Gamma Expansion for Taylor Case

Now that we understand Z-Sums and their properties, we proceed with the expansion of gamma functions (eqs. 3.54 and 3.55) in the general expression of the Taylor series (eq. 3.19). Since these expansions involve conditional theta functions, they will lead to several summations with different upper and lower limits. In this section we study how the expansion of several gamma functions in the same expression affects the procedure. While it would not be very hard to discuss the following step in most general form, for the sake of clarity we deal only with the example involving one factor in the denominator (eq. 3.21). We start by performing the Taylor expansion and parameter integration:

$$\int_0^1\frac{du_1\ du_2\ u_1^{a_1+b_1\varepsilon-1}(1-u_1)^{a_2+b_2\varepsilon-1}\ u_2^{a_3+b_3\varepsilon-1}(1-u_2)^{a_4+b_4\varepsilon-1}}{\left(1-\sum_{t=1}^{\beta}c_t\ u_1^{p(t,1)}\ (1-u_1)^{p(t,2)}\ u_2^{p(t,3)}\ (1-u_2)^{p(t,4)}\right)^{a_0+b_0\varepsilon}} \tag{3.115}$$

$$\begin{aligned}=&\sum_{i_0=0}^{\infty}\frac{\Gamma(i_0+a_0+b_0\varepsilon)}{\Gamma(i_0+1)\Gamma(a_0+b_0\varepsilon)}\sum_{i_1=0}^{i_0}\cdots\sum_{i_{\beta-1}=0}^{i_{\beta-2}}(i_0-i_1,\ldots,i_{\beta-1}-i_\beta)!\ c_1^{i_0-i_1}\ldots c_{\beta-1}^{i_{\beta-2}-i_{\beta-1}}c_\beta^{i_{\beta-1}}\\ \times&\ \frac{\Gamma(n_1+a_1+b_1\varepsilon)\Gamma(n_2+a_2+b_2\varepsilon)}{\Gamma(n_{12}+a_{12}+b_{12}\varepsilon)}\frac{\Gamma(n_3+a_3+b_3\varepsilon)\Gamma(n_4+a_4+b_4\varepsilon)}{\Gamma(n_{34}+a_{34}+b_{34}\varepsilon)}\ ,\end{aligned}$$

with

$$n_r = p_{(1,r)} i_0 + \sum_{t=1}^{\beta-1} \left(p_{(t+1,r)} - p_{(t,r)} \right) i_t \quad . \tag{3.116}$$

Equation 3.115 is similar to 3.21 with the substitutions $d \to a_0 + b_0\varepsilon$, $a_\ell \to a_\ell + b_\ell\varepsilon$ ($\ell = 1, \ldots, 4$), while the total number of terms in the denominator is kept arbitrary (β).

Next we expand all gamma functions that involve summation variables and the infinitesimal parameter. We do not expand other functions dependent on ε since those are not involved in the operations of the Z-Sum machinery and so would unnecessarily complicate the expressions. Because of this, the final result will be in terms of an unexpanded factor dependent on ε, multiplied by a power series.

To shorten the notation, we rewrite eqs. 3.54 and 3.55 as:

$$\begin{aligned} \Gamma(n + b\varepsilon) &= \sum_{j=0}^{\infty} \varepsilon^j g_j(n, b) \quad , \\ \frac{1}{\Gamma(n + b\varepsilon)} &= \sum_{j=0}^{\infty} \varepsilon^j h_j(n, b) \quad , \end{aligned} \tag{3.117}$$

with

$$\begin{aligned} g_j(n,b) &= \theta(n > 0) g_j^+(n,b) + \theta(n \le 0) g_j^-(n,b) \quad , \\ h_j(n,b) &= \theta(n > 0) h_j^+(n,b) + \theta(n \le 0) h_j^-(n,b) \quad , \\ g_j^+(n,b) &= \Gamma(1 + b\varepsilon)\Gamma(n) b^j Z_j(n-1) \quad , \\ g_j^-(n,b) &= \frac{\Gamma(1 + b\varepsilon)(-1)^n}{b\varepsilon\Gamma(1-n)} b^j S_j(-n) \quad , \\ h_j^+(n,b) &= \frac{1}{\Gamma(1 + b\varepsilon)\Gamma(n)} (-b)^j S_j(n-1) \quad , \\ h_j^-(n,b) &= \frac{b\varepsilon\Gamma(1-n)(-1)^n}{\Gamma(1 + b\varepsilon)} (-b)^j Z_j(-n) \quad . \end{aligned} \tag{3.118}$$

The definitions above are equivalent to the ones given previously when one takes into account the fact that $Z_i(n) = S_i(n) = 0$ if $i > n$. This explains why we may write equations 3.117 with all upper summation limits set at infinity.

We proceed with the expansions of gamma functions and reorder the obtained expression in powers of ε:

$$
\begin{aligned}
&\Gamma(i_0+a_0+b_0\varepsilon)\frac{\Gamma(n_1+a_1+b_1\varepsilon)\Gamma(n_2+a_2+b_2\varepsilon)}{\Gamma(n_{12}+a_{12}+b_{12}\varepsilon)}\frac{\Gamma(n_3+a_3+b_3\varepsilon)\Gamma(n_4+a_4+b_4\varepsilon)}{\Gamma(n_{34}+a_{34}+b_{34}\varepsilon)}\\
&=\sum_{j_0=0}^{\infty}\varepsilon^{j_0}g_{j_0}(i_0+a_0,b_0)\\
&\times\sum_{j_1=0}^{\infty}\varepsilon^{j_1}g_{j_1}(n_1+a_1,b_1)\sum_{j_2=0}^{\infty}\varepsilon^{j_2}g_{j_2}(n_2+a_2,b_2)\sum_{j_3=0}^{\infty}\varepsilon^{j_3}h_{j_3}(n_{12}+a_{12},b_{12})\\
&\times\sum_{j_4=0}^{\infty}\varepsilon^{j_4}g_{j_4}(n_3+a_3,b_3)\sum_{j_5=0}^{\infty}\varepsilon^{j_5}g_{j_5}(n_4+a_4,b_4)\sum_{j_6=0}^{\infty}\varepsilon^{j_6}h_{j_6}(n_{34}+a_{34},b_{34})\\
&=\sum_{j_0=0}^{\infty}\varepsilon^{j_0}\sum_{j_1=0}^{j_0}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}\sum_{j_4=0}^{j_3}\sum_{j_5=0}^{j_4}\sum_{j_6=0}^{j_5}g_{j_0-j_1}(i_0+a_0,b_0)\\
&\times g_{j_1-j_2}(n_1+a_1,b_1)\;g_{j_2-j_3}(n_2+a_2,b_2)\;h_{j_3-j_4}(n_{12}+a_{12},b_{12})\\
&\times g_{j_4-j_5}(n_3+a_3,b_3)\;g_{j_5-j_6}(n_4+a_4,b_4)\;h_{j_6}(n_{34}+a_{34},b_{34})\quad .
\end{aligned}
\tag{3.119}
$$

Applying eq. 3.119 in eq. 3.115, and switching the order of summations on the i and j variables, we obtain sums of the form:

$$
\begin{aligned}
&\sum_{i_0=0}^{\infty}\sum_{i_1=0}^{i_0}\cdots\sum_{i_{\beta-1}=0}^{i_{\beta-2}}(i_0-i_1,\ldots,i_{\beta-1}-i_\beta)!\;c_1^{i_0-i_1}\ldots c_{\beta-1}^{i_{\beta-2}-i_{\beta-1}}c_\beta^{i_{\beta-1}}\frac{g_{k_0}(i_0+a_0,b_0)}{\Gamma(i_0+1)}\\
&\times g_{k_1}(n_1+a_1,b_1)\;g_{k_2}(n_2+a_2,b_2)\;h_{k_3}(n_{12}+a_{12},b_{12})\\
&\times g_{k_4}(n_3+a_3,b_3)\;g_{k_5}(n_4+a_4,b_4)\;h_{k_6}(n_{34}+a_{34},b_{34})\;,
\end{aligned}
$$

where the outer j summations have been omitted since they will be ultimately truncated to include only the desired powers in ε. The variables k are functions of j, but are seen here as constants. The coefficients g and h must be expanded in terms of positive and negative parts leading to:

$$\sum_{i_0=0}^{\infty}\sum_{i_1=0}^{i_0}\dots\sum_{i_{\beta-1}=0}^{i_{\beta-2}} (i_0-i_1,\dots,i_{\beta-1}-i_\beta)!\; c_1^{i_0-i_1}\dots c_{\beta-1}^{i_{\beta-2}-i_{\beta-1}} c_\beta^{i_{\beta-1}} \frac{g^+_{k_0}(i_0+a_0,b_0)}{\Gamma(i_0+1)}$$
$$\times\Big(\theta(n_1+a_1>0)g^+_{k_1}(n_1+a_1,b_1)+\theta(n_1+a_1\le 0)g^-_{k_1}(n_1+a_1,b_1)\Big)$$
$$\times\Big(\theta(n_2+a_2>0)g^+_{k_2}(n_2+a_2,b_2)+\theta(n_2+a_2\le 0)g^-_{k_2}(n_2+a_2,b_2)\Big) \quad (3.120)$$
$$\times\Big(\theta(n_{12}+a_{12}>0)h^+_{k_3}(n_{12}+a_{12},b_{12})+\theta(n_{12}+a_{12}\le 0)h^-_{k_3}(n_{12}+a_{12},b_{12})\Big)$$
$$\times\Big(\theta(n_3+a_3>0)g^+_{k_4}(n_3+a_3,b_3)+\theta(n_3+a_3\le 0)g^-_{k_4}(n_3+a_3,b_3)\Big)$$
$$\times\Big(\theta(n_4+a_4>0)g^+_{k_5}(n_4+a_4,b_4)+\theta(n_4+a_4\le 0)g^-_{k_5}(n_4+a_4,b_4)\Big)$$
$$\times\Big(\theta(n_{34}+a_{34}>0)h^+_{k_6}(n_{34}+a_{34},b_{34})+\theta(n_{34}+a_{34}\le 0)h^-_{k_6}(n_{34}+a_{34},b_{34})\Big) \quad .$$

When the product of theta functions is expanded we arrive at a large number of summations with different conditions imposed by theta functions. Depending on the values of n_k and a_k these restrictions will generate summations with altered concatenation. Generally speaking, for positive c, factors of the type $\theta(ci+d\le 0)$ and $\theta(-ci+d>0)$ will impose restrictions on the upper limit of the i summation, while $\theta(ci+d>0)$ and $\theta(-ci+d\le 0)$ will alter the lower limit. After shifting summation variables and synchronizing binomial coefficients if necessary, one should be able to arrive at sums with proper concatenation, thus allowing for the use of the Z-Sum machinery. For example:

$$\begin{aligned}\sum_{i=0}^{n}\theta(i\le a)f(i) &= \theta(n\le a)\sum_{i=0}^{n}f(i)+\theta(0\le a<n)\sum_{i=0}^{a}f(i)\\ \sum_{i=0}^{n}\theta(i\ge n-a)f(i) &= \theta(n\le a)\sum_{i=0}^{n}f(i)+\theta(n>a)\sum_{i=0}^{a}f(i+n-a)\\ \sum_{i=0}^{n}\theta(i\ge a)f(i) &= \theta(a\le 0)\sum_{i=0}^{n}f(i)+\theta(0<a\le n)\sum_{i=0}^{n-a}f(i+a)\ ,\end{aligned} \quad (3.121)$$

where a is a bounded number. Summations of the type $\sum_{i=0}^{a}$ are truncated sums and need no further computation, while $\sum_{i=0}^{n-a}$ should lead to $Z(n-a,\dots)$. If $f(i)$ contains a binomial coefficient then it must be synchronized with the upper summation limit so that we may apply the Z-Sum machinery. For example:

$$\sum_{i=0}^{n+a} f(i+b) = \sum_{i=0}^{n+a} \binom{n}{i+b} f'(i+b) = \tag{3.122}$$
$$= \sum_{i=0}^{n+a} \binom{n+a}{i} \frac{\Gamma(n+1)}{\Gamma(n+a+1)} \frac{\Gamma(i+1)}{\Gamma(i+b+1)} \frac{\Gamma(n-i+a+1)}{\Gamma(n-i-b+1)} f'(i+a) \to Z(n+a,\ldots)$$

After all valid sums have been found, the functions $g^{\pm}$ and $h^{\pm}$ may be substituted according to equations 3.118 and the Z-Sum machinery applied, if possible.

This concludes the expansions and also this chapter. In the next one we will cover specific calculations in order to find out how often a seamless reduction to Z-Sums is applicable to actual loop integrals.

CHAPTER 4

TRIANGLES IN QUANTUM FIELD THEORY

In the last chapter we covered all steps necessary to perform the parameter integration arising from loop calculations with the end result being expressed in terms of concatenated summations involving Z-Sums. While our discussion so far has been always very general, now we focus on individual examples to obtain a better understanding on the procedure and specifically find out how often the reduction can be applied seamlessly.

When trying to understand which diagrams lead to summations with seamless reduction we find that two different approaches are possible, each one with its own advantages and disadvantages. The most obvious choice is to simply choose a specific diagram and try to manipulate the integral and subsequent summation in order to obtain an expression with seamless reduction. This procedure is quite difficult since for each case it requires different manipulations, making it not very systematic. The second choice is to start from all possible summations with seamless reduction and walk our way backwards to the integrand that would have led to such an expression. The advantage of this approach is that it is fully systematic, however not all integrals obtained will necessarily be related to a loop integration. Furthermore, while this will in principle lead to all calculations with seamless reduction, unlike the first approach, it gives no information on integrals that do not lead to seamless summations.

We see both approaches as complementary. Initially we applied the forward approach to known triangle loop integrations, developing extensive computer code using both FORM and Mathematica programming languages. These tools allowed for the study and understanding of the best ways in performing the Taylor expansion of the parameter integration, leading to the results presented in the previous chapters. The experience gained from those calculations ultimately lead to very general, analytical expression applicable to a generic parameter integration (eq. 3.19), which allowed for a clear understanding on the form of the concatenated sums to be expected. Using this expression, we were able to apply the backwards approach, leading to a systematic procedure that resulted in obtaining all concatenated sums with seamless reductions that could appear from a loop-like integration. This was again performed by developing computational code using Mathematica, and the results will be discussed in section 4.1.

4.1 Obtaining Denominator Factorization from Allowed Summations

In this section we will use a reverse approach, starting from summations that can be seamlessly reduced to Z-Sums and working backwards from beta functions to the parametric integrations that would have generated it. In order to consider all possibilities, we use each individual sum from the algorithms previously discussed as building blocks of the full concatenated summation, and evaluate every combination. Not all summations will be of interest. Some generate integrands that would never occur in a loop calculation, and so are disregarded. We start the procedure by listing all building blocks, as shown in table 4.1.

The variable x in table 4.1 is defined by:

$$x_1^{i_1} \dots x_s^{i_s} = c_1^{i_1} \dots c_s^{i_s} \times [p_1, p_2] \dots [p_{2b-1}, p_{2b}] \ , \tag{4.1}$$

where s is the number of sums after all building blocks have been put together, b is the desired number of parameter integrations and the product $[p_1, p_2] \dots [p_{2b-1}, p_{2b}]$ consists of a combination of any of the brackets allowed in each building block.

While there are other building block sums that lead to a seamless reduction, for example:

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor}{i} x^i, \quad \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} f(\lfloor \frac{n}{2} \rfloor, i) \ x^i, \quad \sum_{i=0}^{\lfloor \frac{q-n}{2} \rfloor} \binom{\lfloor \frac{q-n}{2} \rfloor}{i} x^i, \quad \sum_{i=0}^{\lfloor \frac{q-n}{2} \rfloor} f(\lfloor \frac{q-n}{2} \rfloor, i) \ x^i \tag{4.2}$$

Table 4.1: Single sums used as building blocks. See equation 4.1 for the relation between x and $[p_1, p_2]$. In the last two rows q represents the upper limit in the hidden n sum. $f(j,i)$ stands for any of the following: 1, $\frac{\Gamma(i+d)}{\Gamma(i+1)\Gamma(d)}$, $\frac{\Gamma(j-i+d)}{\Gamma(j-i+1)\Gamma(d)}$ or $\frac{\Gamma(j-i+d)}{\Gamma(j-i+1)\Gamma(d)}\frac{\Gamma(i+d)}{\Gamma(i+1)\Gamma(d)}$.

Sum	$[p_1,p_2] = \frac{\Gamma(p_1+a_1)\Gamma(p_2+a_2)}{\Gamma(p_1+p_2+a_{12})} = \int_0^1 du\ u^{p_1+a_1-1}\ (1-u)^{p_2+a_2-1}$
$\sum_{i=0}^{\infty} \frac{\Gamma(i+d)}{\Gamma(i+1)\Gamma(d)} x^i$	$[0,0],\ [i,0],\ [2i,0]$
$\sum_{i=0}^{n} \binom{n}{i}\ x^i$	$[0,0],\ [i,0],\ [n-i,0],\ [i,n-i]$
$\sum_{i=0}^{n} f(n,i)\ x^i$	$[0,0],\ [i,0],\ [2i,0],\ [n-i,0],\ [2n-2i,0]$
$\sum_{i=0}^{q-n} \binom{q-n}{i}\ x^i$	$[0,0],\ [i,0],\ [q-n-i,0],\ [i,q-n-i]$
$\sum_{i=0}^{q-n} f(q-n,i)\ x^i$	$[0,0],\ [i,0],\ [2i,0],\ [q-n-i,0],\ [2q-2n-2i,0]$
$\sum_{i=0}^{2n} \binom{2n}{i}\ x^i$	$[0,0],\ [i,0],\ [2n-i,0],\ [i,2n-i]$
$\sum_{i=0}^{2n} f(2n,i)\ x^i$	$[0,0],\ [i,0],\ [2i,0],\ [2n-i,0],\ [4n-2i,0]$
$\sum_{i=0}^{2q-2n} \binom{2q-2n}{i}\ x^i$	$[0,0],\ [i,0],\ [2q-2n-i,0],\ [i,2q-2n-i]$
$\sum_{i=0}^{2q-2n} f(2q-2n,i)\ x^i$	$[0,0],\ [i,0],\ [2i,0],\ [2q-2n-i,0],\ [4q-4n-2i,0]$

they do not lead to a desirable integrand and so are not included in table 4.1.

In order to make the approach more clear, let us take a look at one example. First we pick two individual sums:

$$\sum_{i=0}^{\infty} \frac{\Gamma(i+d)}{\Gamma(i+1)\Gamma(d)} x_1^i \quad \text{and} \quad \sum_{j=0}^{i} \binom{i}{j}\ x_2^j\ , \tag{4.3}$$

then we choose a combination of allowed $[p_1, p_2]$ representing beta functions, for example $[2i, 0]$ and $[j, 0]$ (two since we are discussing triangle integrations), as prescribed in table

4.1, leading to:

$$\begin{aligned}
& \sum_{i=0}^{\infty} \frac{\Gamma(i+d)}{\Gamma(i+1)\Gamma(d)} c_1^i \frac{\Gamma(2i+a_1+1)\Gamma(a_2+1)}{\Gamma(2i+a_{12}+2)} \sum_{j=0}^{i} \binom{i}{j} c_2^j \frac{\Gamma(j+a_3+1)\Gamma(a_4+1)}{\Gamma(j+a_{34}+2)} \\
= & \sum_{i=0}^{\infty} \frac{\Gamma(i+d)}{\Gamma(i+1)\Gamma(d)} c_1^i \sum_{j=0}^{i} \binom{i}{j} c_2^j \int_0^1 du \; u_1^{2i+a_1} (1-u_1)^{a_2} u_2^{j+a_3} (1-u_2)^{a_4} \\
= & \int_0^1 du \; u_1^{a_1} (1-u_1)^{a_2} u_2^{a_3} (1-u_2)^{a_4} \sum_{i=0}^{\infty} \frac{\Gamma(i+d)}{\Gamma(i+1)\Gamma(d)} \left(c_1 \; u_1^2\right)^i \sum_{j=0}^{i} \binom{i}{j} \left(c_2 \; u_2\right)^j \\
= & \int_0^1 \frac{du \; u_1^{a_1} (1-u_1)^{a_2} u_2^{a_3} (1-u_2)^{a_4}}{\left(1 - c_1 \; u_1^2 \left(1 + c_2 \; u_2\right)\right)^d} \quad .
\end{aligned} \tag{4.4}$$

Note that since there are enough degrees of freedom (c_1 and c_2) the expansion is convergent.

Going back to the building blocks presented in table 4.1, we now would like to consider every possible way to concatenate them. This can be a bit confusing, and so a useful method is to visualize them using rooted trees as a guideline. A rooted tree is composed of knots and the links between these knots. A connected rooted tree has a single root on top. A knot may be linked to several lower knots, but only one upper one. We start by finding all connected rooted trees including up to four knots, and from these obtain all concatenated sums. Table 4.2 shows the procedure up to four sums.

Now we need to also consider trees with disconnected branches, equivalent to summations involving more than one sum with infinity as the upper limit. This can be done by combining connected trees with the correct number of knots, as shown in table 4.3 (with up to four knots or sums):

Following this procedure, we obtain the results presented in table 4.4. These results may be used to compare with the parameter integration obtained from specific diagrams.

Table 4.2: All fully concatenated summations (up to four sums) using connected rooted trees as a guideline. The terms in parentheses represent the upper limit on summations from second to last, with the outer summation going to infinity. For example: $(i, 2j, k) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{2j} \sum_{l=0}^{k}$.

	$()$
	(i) $(2i)$
	(i,i) $(i,2i)$ $(2i,2i)$
	(i,j) $(i,2j)$ $(2i,j)$ $(2i,2j)$
	(i,i,i) $(i,i,2i)$ $(i,2i,2i)$ $(2i,2i,2i)$
	(i,i,k) $(i,i,2k)$ $(i,2i,k)$ $(i,2i,2k)$ $(2i,i,k)$ $(2i,i,2k)$ $(2i,2i,k)$ $(2i,2i,2k)$
	(i,j,j) $(i,j,i-j)$ $(i,j,2j)$ $(i,j,2i-2j)$ $(2i,j,j)$ $(2i,j,2i-j)$ $(2i,j,2j)$ $(2i,j,4i-2j)$
	(i,j,k) $(i,j,2k)$ $(i,2j,k)$ $(i,2j,2k)$ $(2i,j,k)$ $(2i,j,2k)$ $(2i,2j,k)$ $(2i,2j,2k)$

Table 4.3: All rooted trees up to four knots built by joining connected rooted trees.

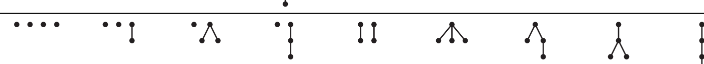
#Sums	Rooted Trees
1	
2	
3	
4	

Table 4.4: Parameter integrations leading to summations with seamless reduction. The integration form is given by: $\int \frac{du_1\ du_2\ u_1^{a_1-1}\ (1-u_1)^{a_2-1}\ u_2^{a_3-1}\ (1-u_2)^{a_4-1}}{D}$, where D is the denominator presented in each row.

#	Integrand's Denominator (D) / Summation obtained from expanding the integration
1.	$(1-c_1u_2)^d$ $\sum_{i_1=0}^{\infty} \frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)} \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(a_{12})} \frac{\Gamma(a_4)\Gamma(i_1+a_3)}{\Gamma(i_1+a_{34})} c_1^{i_1}$
2.	$\left(1-c_1u_2^2\right)^d$ $\sum_{i_1=0}^{\infty} \frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)} \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(a_{12})} \frac{\Gamma(a_4)\Gamma(2i_1+a_3)}{\Gamma(2i_1+a_{34})} c_1^{i_1}$
3.	$(1-c_1u_1u_2)^d$ $\sum_{i_1=0}^{\infty} \frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)} \frac{\Gamma(a_2)\Gamma(i_1+a_1)}{\Gamma(i_1+a_{12})} \frac{\Gamma(a_4)\Gamma(i_1+a_3)}{\Gamma(i_1+a_{34})} c_1^{i_1}$
4.	$\left(1-c_1u_1u_2^2\right)^d$ $\sum_{i_1=0}^{\infty} \frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)} \frac{\Gamma(a_2)\Gamma(i_1+a_1)}{\Gamma(i_1+a_{12})} \frac{\Gamma(a_4)\Gamma(2i_1+a_3)}{\Gamma(2i_1+a_{34})} c_1^{i_1}$
5.	$\left(1-c_1u_1^2u_2^2\right)^d$ $\sum_{i_1=0}^{\infty} \frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)} \frac{\Gamma(a_2)\Gamma(2i_1+a_1)}{\Gamma(2i_1+a_{12})} \frac{\Gamma(a_4)\Gamma(2i_1+a_3)}{\Gamma(2i_1+a_{34})} c_1^{i_1}$
6.	$\left((1-c_1u_1)(1-c_2u_2)\right)^d$ $\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)} \frac{\Gamma(i_2+d)}{\Gamma(d)\Gamma(i_2+1)} \frac{\Gamma(a_2)\Gamma(i_1+a_1)}{\Gamma(i_1+a_{12})} \frac{\Gamma(a_4)\Gamma(i_2+a_3)}{\Gamma(i_2+a_{34})} c_1^{i_1} c_2^{i_2}$
7.	$\left((1-c_1u_1)\left(1-c_2u_2^2\right)\right)^d$ $\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)} \frac{\Gamma(i_2+d)}{\Gamma(d)\Gamma(i_2+1)} \frac{\Gamma(a_2)\Gamma(i_1+a_1)}{\Gamma(i_1+a_{12})} \frac{\Gamma(a_4)\Gamma(2i_2+a_3)}{\Gamma(2i_2+a_{34})} c_1^{i_1} c_2^{i_2}$
8.	$\left(\left(1-c_1u_1^2\right)\left(1-c_2u_2^2\right)\right)^d$ $\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)} \frac{\Gamma(i_2+d)}{\Gamma(d)\Gamma(i_2+1)} \frac{\Gamma(a_2)\Gamma(2i_1+a_1)}{\Gamma(2i_1+a_{12})} \frac{\Gamma(a_4)\Gamma(2i_2+a_3)}{\Gamma(2i_2+a_{34})} c_1^{i_1} c_2^{i_2}$
9.	$\left((1-c_1u_2)(1-c_1c_2u_2)\right)^d$ $\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{i_1} \frac{\Gamma(i_1-i_2+d)}{\Gamma(d)\Gamma(i_1-i_2+1)} \frac{\Gamma(i_2+d)}{\Gamma(d)\Gamma(i_2+1)} \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(a_{12})} \frac{\Gamma(a_4)\Gamma(i_1+a_3)}{\Gamma(i_1+a_{34})} c_1^{i_1} c_2^{i_2}$
10.	$\left((1-c_1u_1u_2)(1-c_1c_2u_1u_2)\right)^d$ $\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{i_1} \frac{\Gamma(i_1-i_2+d)}{\Gamma(d)\Gamma(i_1-i_2+1)} \frac{\Gamma(i_2+d)}{\Gamma(d)\Gamma(i_2+1)} \frac{\Gamma(a_2)\Gamma(i_1+a_1)}{\Gamma(i_1+a_{12})} \frac{\Gamma(a_4)\Gamma(i_1+a_3)}{\Gamma(i_1+a_{34})} c_1^{i_1} c_2^{i_2}$
11.	$\left((1-c_1u_1)(1-c_1c_2u_1u_2)\right)^d$ $\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{i_1} \frac{\Gamma(i_1-i_2+d)}{\Gamma(d)\Gamma(i_1-i_2+1)} \frac{\Gamma(i_2+d)}{\Gamma(d)\Gamma(i_2+1)} \frac{\Gamma(a_2)\Gamma(i_1+a_1)}{\Gamma(i_1+a_{12})} \frac{\Gamma(a_4)\Gamma(i_2+a_3)}{\Gamma(i_2+a_{34})} c_1^{i_1} c_2^{i_2}$
12.	$\left((1-c_1u_1)\left(1-c_1c_2u_1u_2^2\right)\right)^d$ $\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{i_1} \frac{\Gamma(i_1-i_2+d)}{\Gamma(d)\Gamma(i_1-i_2+1)} \frac{\Gamma(i_2+d)}{\Gamma(d)\Gamma(i_2+1)} \frac{\Gamma(a_2)\Gamma(i_1+a_1)}{\Gamma(i_1+a_{12})} \frac{\Gamma(a_4)\Gamma(2i_2+a_3)}{\Gamma(2i_2+a_{34})} c_1^{i_1} c_2^{i_2}$
13.	$\left(1-c_1\left(1+\frac{c_2u_1}{u_2}\right)u_2\right)^d$ $\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{i_1} \frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)} \binom{i_1}{i_2} \frac{\Gamma(a_4)\Gamma(i_1-i_2+a_3)}{\Gamma(i_1-i_2+a_{34})} \frac{\Gamma(a_2)\Gamma(i_2+a_1)}{\Gamma(i_2+a_{12})} c_1^{i_1} c_2^{i_2}$
14.	$\left(1-c_1\left(1+\frac{c_2u_1(1-u_2)}{u_2}\right)u_2\right)^d$ $\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{i_1} \frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)} \binom{i_1}{i_2} \frac{\Gamma(i_1-i_2+a_3)\Gamma(i_2+a_4)}{\Gamma(i_1+a_{34})} \frac{\Gamma(a_2)\Gamma(i_2+a_1)}{\Gamma(i_2+a_{12})} c_1^{i_1} c_2^{i_2}$
15.	$(1-c_1(1+c_2u_1)u_2)^d$ $\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{i_1} \frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)} \binom{i_1}{i_2} \frac{\Gamma(a_4)\Gamma(i_1+a_3)}{\Gamma(i_1+a_{34})} \frac{\Gamma(a_2)\Gamma(i_2+a_1)}{\Gamma(i_2+a_{12})} c_1^{i_1} c_2^{i_2}$

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Table 4.4 continued

#	Integrand's Denominator (D)	Summation obtained from expanding the integration
16.	$\left(1-c_1\left(1+c_2u_1\right)u_2^2\right)^d$	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{i_1}\frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)}\binom{i_1}{i_2}\frac{\Gamma(a_4)\Gamma(2i_1+a_3)}{\Gamma(2i_1+a_{34})}\frac{\Gamma(a_2)\Gamma(i_2+a_1)}{\Gamma(i_2+a_{12})}c_1^{i_1}c_2^{i_2}$
17.	$\left(1-c_1u_1\left(1+\frac{c_2(1-u_1)u_2^2}{u_1}\right)\right)^d$	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{i_1}\frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)}\binom{i_1}{i_2}\frac{\Gamma(i_1-i_2+a_1)\Gamma(i_2+a_2)}{\Gamma(i_1+a_{12})}\frac{\Gamma(a_4)\Gamma(2i_2+a_3)}{\Gamma(2i_2+a_{34})}c_1^{i_1}c_2^{i_2}$
18.	$\left(1-c_1u_1\left(1+\frac{c_2(1-u_1)}{u_1u_2^2}\right)u_2^2\right)^d$	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{i_1}\frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)}\binom{i_1}{i_2}\frac{\Gamma(a_4)\Gamma(2i_1-2i_2+a_3)}{\Gamma(a_{34}+2(i_1-i_2))}\frac{\Gamma(i_1-i_2+a_1)\Gamma(i_2+a_2)}{\Gamma(i_1+a_{12})}c_1^{i_1}c_2^{i_2}$
19.	$\left(1-c_1\left(1+\frac{c_2u_1}{u_2}\right)^2u_2^2\right)^d$	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{2i_1}\frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)}\binom{2i_1}{i_2}\frac{\Gamma(a_4)\Gamma(2i_1-i_2+a_3)}{\Gamma(2i_1-i_2+a_{34})}\frac{\Gamma(a_2)\Gamma(i_2+a_1)}{\Gamma(i_2+a_{12})}c_1^{i_1}c_2^{i_2}$
20.	$\left(1-c_1\left(1+\frac{c_2u_1(1-u_2)}{u_2}\right)^2u_2^2\right)^d$	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{2i_1}\frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)}\binom{2i_1}{i_2}\frac{\Gamma(2i_1-i_2+a_3)\Gamma(i_2+a_4)}{\Gamma(2i_1+a_{34})}\frac{\Gamma(a_2)\Gamma(i_2+a_1)}{\Gamma(i_2+a_{12})}c_1^{i_1}c_2^{i_2}$
21.	$\left(1-c_1\left(1+c_2u_1\right)^2u_2\right)^d$	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{2i_1}\frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)}\binom{2i_1}{i_2}\frac{\Gamma(a_4)\Gamma(i_1+a_3)}{\Gamma(i_1+a_{34})}\frac{\Gamma(a_2)\Gamma(i_2+a_1)}{\Gamma(i_2+a_{12})}c_1^{i_1}c_2^{i_2}$
22.	$\left(1-c_1\left(1+c_2u_1\right)^2u_2^2\right)^d$	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{2i_1}\frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)}\binom{2i_1}{i_2}\frac{\Gamma(a_4)\Gamma(2i_1+a_3)}{\Gamma(2i_1+a_{34})}\frac{\Gamma(a_2)\Gamma(i_2+a_1)}{\Gamma(i_2+a_{12})}c_1^{i_1}c_2^{i_2}$
23.	$\left(1-c_1u_1^2\left(1+\frac{c_2(1-u_2)}{u_1u_2}\right)^2u_2^2\right)^d$	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{2i_1}\frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)}\binom{2i_1}{i_2}\frac{\Gamma(a_2)\Gamma(2i_1-i_2+a_1)}{\Gamma(2i_1-i_2+a_{12})}\frac{\Gamma(2i_1-i_2+a_3)\Gamma(i_2+a_4)}{\Gamma(2i_1+a_{34})}c_1^{i_1}c_2^{i_2}$
24.	$\left(\left(1-c_1u_1\right)\left(1-c_1c_3u_1\right)\left(1-c_2u_2\right)\right)^d$	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{i_1}\frac{\Gamma(i_2+d)}{\Gamma(d)\Gamma(i_2+1)}\frac{\Gamma(i_1-i_3+d)}{\Gamma(d)\Gamma(i_1-i_3+1)}\frac{\Gamma(i_3+d)}{\Gamma(d)\Gamma(i_3+1)}\frac{\Gamma(a_2)\Gamma(i_1+a_1)}{\Gamma(i_1+a_{12})}\frac{\Gamma(a_4)\Gamma(i_2+a_3)}{\Gamma(i_2+a_{34})}c_1^{i_1}c_2^{i_2}c_3^{i_3}$
25.	$\left(\left(1-c_1u_1\right)\left(1-c_1c_3u_1\right)\left(1-c_2u_2^2\right)\right)^d$	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{i_1}\frac{\Gamma(i_2+d)}{\Gamma(d)\Gamma(i_2+1)}\frac{\Gamma(i_1-i_3+d)}{\Gamma(d)\Gamma(i_1-i_3+1)}\frac{\Gamma(i_3+d)}{\Gamma(d)\Gamma(i_3+1)}\frac{\Gamma(a_2)\Gamma(i_1+a_1)}{\Gamma(i_1+a_{12})}\frac{\Gamma(a_4)\Gamma(2i_2+a_3)}{\Gamma(2i_2+a_{34})}c_1^{i_1}c_2^{i_2}c_3^{i_3}$
26.	$\left(1-c_1\left(1+c_2u_1\right)\left(1+c_3u_2\right)\right)^d$	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_1}\frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)}\binom{i_1}{i_2}\binom{i_1}{i_3}\frac{\Gamma(a_2)\Gamma(i_2+a_1)}{\Gamma(i_2+a_{12})}\frac{\Gamma(a_4)\Gamma(i_3+a_3)}{\Gamma(i_3+a_{34})}c_1^{i_1}c_2^{i_2}c_3^{i_3}$
27.	$\left(1-c_1\left(1+c_2u_1\right)\left(1+c_3u_2\right)^2\right)^d$	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{2i_1}\frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)}\binom{i_1}{i_2}\binom{2i_1}{i_3}\frac{\Gamma(a_2)\Gamma(i_2+a_1)}{\Gamma(i_2+a_{12})}\frac{\Gamma(a_4)\Gamma(i_3+a_3)}{\Gamma(i_3+a_{34})}c_1^{i_1}c_2^{i_2}c_3^{i_3}$
28.	$\left(1-c_1\left(1+c_2u_1\right)^2\left(1+c_3u_2\right)^2\right)^d$	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{2i_1}\sum_{i_3=0}^{2i_1}\frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)}\binom{2i_1}{i_2}\binom{2i_1}{i_3}\frac{\Gamma(a_2)\Gamma(i_2+a_1)}{\Gamma(i_2+a_{12})}\frac{\Gamma(a_4)\Gamma(i_3+a_3)}{\Gamma(i_3+a_{34})}c_1^{i_1}c_2^{i_2}c_3^{i_3}$
29.	$\left(\left(1-c_1u_2\right)\left(1-c_1c_2\left(1+c_3u_1\right)u_2\right)\right)^d$	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_2}\frac{\Gamma(i_1-i_2+d)}{\Gamma(d)\Gamma(i_1-i_2+1)}\frac{\Gamma(i_2+d)}{\Gamma(d)\Gamma(i_2+1)}\binom{i_2}{i_3}\frac{\Gamma(a_4)\Gamma(i_1+a_3)}{\Gamma(i_1+a_{34})}\frac{\Gamma(a_2)\Gamma(i_3+a_1)}{\Gamma(i_3+a_{12})}c_1^{i_1}c_2^{i_2}c_3^{i_3}$

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Table 4.4 continued

#	Integrand's Denominator (D) / Summation obtained from expanding the integration
30.	$\left((1-c_1c_2u_2)\left(1-c_1\left(1+\frac{c_3}{u_1}\right)u_1u_2\right)\right)^d$
	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_1-i_2}\frac{\Gamma(i_1-i_2+d)}{\Gamma(d)\Gamma(i_1-i_2+1)}\frac{\Gamma(i_2+d)}{\Gamma(d)\Gamma(i_2+1)}\binom{i_1-i_2}{i_3}\frac{\Gamma(a_4)\Gamma(i_1+a_3)}{\Gamma(i_1+a_{34})}\frac{\Gamma(a_2)\Gamma(i_1-i_2-i_3+a_1)}{\Gamma(i_1-i_2-i_3+a_{12})}c_1^{i_1}c_2^{i_2}c_3^{i_3}$
31.	$\left((1-c_1u_2)\left(1-c_1c_2\left(1+c_3u_1\right)^2u_2\right)\right)^d$
	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{2i_2}\frac{\Gamma(i_1-i_2+d)}{\Gamma(d)\Gamma(i_1-i_2+1)}\frac{\Gamma(i_2+d)}{\Gamma(d)\Gamma(i_2+1)}\binom{2i_2}{i_3}\frac{\Gamma(a_4)\Gamma(i_1+a_3)}{\Gamma(i_1+a_{34})}\frac{\Gamma(a_2)\Gamma(i_3+a_1)}{\Gamma(i_3+a_{12})}c_1^{i_1}c_2^{i_2}c_3^{i_3}$
32.	$\left(1-c_1\left(1+c_2u_1\left(1+c_3u_2\right)\right)^2\right)^d$
	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{2i_1}\sum_{i_3=0}^{i_2}\frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)}\binom{2i_1}{i_2}\binom{i_2}{i_3}\frac{\Gamma(a_2)\Gamma(i_2+a_1)}{\Gamma(i_2+a_{12})}\frac{\Gamma(a_4)\Gamma(i_3+a_3)}{\Gamma(i_3+a_{34})}c_1^{i_1}c_2^{i_2}c_3^{i_3}$
33.	$\left(1-c_1u_1^2\left(1+\frac{c_2(1+c_3u_2)}{u_1}\right)^2\right)^d$
	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{2i_1}\sum_{i_3=0}^{i_2}\frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)}\binom{2i_1}{i_2}\binom{i_2}{i_3}\frac{\Gamma(a_2)\Gamma(2i_1-i_2+a_1)}{\Gamma(2i_1-i_2+a_{12})}\frac{\Gamma(a_4)\Gamma(i_3+a_3)}{\Gamma(i_3+a_{34})}c_1^{i_1}c_2^{i_2}c_3^{i_3}$
34.	$\left(1-c_1u_1^2\left(1+\frac{c_2(1-u_1)\left(1+\frac{c_3}{u_2}\right)u_2}{u_1}\right)^2\right)^d$
	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{2i_1}\sum_{i_3=0}^{i_2}\frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)}\binom{2i_1}{i_2}\binom{i_2}{i_3}\frac{\Gamma(2i_1-i_2+a_1)\Gamma(i_2+a_2)}{\Gamma(2i_1+a_{12})}\frac{\Gamma(a_4)\Gamma(i_2-i_3+a_3)}{\Gamma(i_2-i_3+a_{34})}c_1^{i_1}c_2^{i_2}c_3^{i_3}$
35.	$\left((1-c_1u_1)(1-c_1c_3u_1)(1-c_2u_2)(1-c_2c_4u_2)\right)^d$
	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{i_1}\sum_{i_4=0}^{i_2}\frac{\Gamma(i_1-i_3+d)}{\Gamma(d)\Gamma(i_1-i_3+1)}\frac{\Gamma(i_3+d)}{\Gamma(d)\Gamma(i_3+1)}\frac{\Gamma(i_2-i_4+d)}{\Gamma(d)\Gamma(i_2-i_4+1)}\frac{\Gamma(i_4+d)}{\Gamma(d)\Gamma(i_4+1)}$ $\times\frac{\Gamma(a_2)\Gamma(i_1+a_1)}{\Gamma(i_1+a_{12})}\frac{\Gamma(a_4)\Gamma(i_2+a_3)}{\Gamma(i_2+a_{34})}c_1^{i_1}c_2^{i_2}c_3^{i_3}c_4^{i_4}$
36.	$\left(1-c_1\left(1+c_2\left(1+c_3u_1\right)\left(1+c_4u_2\right)\right)^2\right)^d$
	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{2i_1}\sum_{i_3=0}^{i_2}\sum_{i_4=0}^{i_2}\frac{\Gamma(i_1+d)}{\Gamma(d)\Gamma(i_1+1)}\binom{2i_1}{i_2}\binom{i_2}{i_3}\binom{i_2}{i_4}\frac{\Gamma(a_2)\Gamma(i_3+a_1)}{\Gamma(i_3+a_{12})}\frac{\Gamma(a_4)\Gamma(i_4+a_3)}{\Gamma(i_4+a_{34})}c_1^{i_1}c_2^{i_2}c_3^{i_3}c_4^{i_4}$
37.	$\left((1-c_1c_2u_2)\left(1-c_1\left(1+c_4u_1\right)\left(1+\frac{c_3}{1+c_4u_1}\right)u_2\right)\right)^d$
	$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_1-i_2}\sum_{i_4=0}^{i_1-i_2-i_3}\frac{\Gamma(i_1-i_2+d)}{\Gamma(d)\Gamma(i_1-i_2+1)}\frac{\Gamma(i_2+d)}{\Gamma(d)\Gamma(i_2+1)}\binom{i_1-i_2}{i_3}\binom{i_1-i_2-i_3}{i_4}$ $\times\frac{\Gamma(a_4)\Gamma(i_1+a_3)}{\Gamma(i_1+a_{34})}\frac{\Gamma(a_2)\Gamma(i_4+a_1)}{\Gamma(i_4+a_{12})}c_1^{i_1}c_2^{i_2}c_3^{i_3}c_4^{i_4}$

The denominators presented in table 4.4 represent all possible integrations (with two parameter variables, corresponding to the case of one-loop triangle diagrams) that may be reduced seamlessly using the available algorithms in the Z-Sum machinery, although not all will be obtained from a loop diagram. In the next section we cross check this information with all one-loop triangle diagrams that occur in any QFT in order to find how often the seamless Z-Sum reduction method is applicable.

4.2 Triangles in Quantum Field Theory

With the results of section 4.1 at hand we proceed to the simplest yet non-trivial loop calculations, one-loop triangles, and evaluate which diagrams lead to a seamless reduction. We consider both physical and unphysical calculations, in order to also include triangles which are part of larger multiloop diagrams. Since we are not interested in a single example but in all cases, first we need to figure out a way to obtain every possible triangle. For the Standard Model, that means considering every possible combination of three vertices, connecting them in every possible way with matching particles, while also considering that outgoing particles might be on or off-shell (at least one leg was considered off-shell).

Using all vertices present in the Standard Model, as shown in table 4.5, we obtain a very large number of possible combinations out of which 72 distinct cases, as labeled by the external momenta and internal masses of each diagram.

Being a bit overly ambitious we could also attempt to consider every triangle calculation present in any QFT. In this case, instead of focusing on the types of vertices present, we simply assign symbolic values for the six degrees of freedom (three external momenta, three internal masses) and consider all cases where these arguments might be zero or equal to one another. By doing so we obtain the maximal set of argument choices, which amounts to a total of 101, although it is very likely that no single theory could possibly involve all of them.

Table 4.6 presents all triangles with distinct parameters for both the SM and any QFT, with argument sets in red square brackets being present in the Standard Model.

The results presented in table 4.6 show that only a small number of triangles (18) can be reduced to Z-Sums using Taylor expansions and the reduction algorithms currently known. All other diagrams lead to well defined expansions that cannot be systematically reduced to Z-Sums. Since we do believe it should be possible to express these in terms of multiple polylogarithms, we see two different approaches in proceeding with this line of research. One would be to try to extend the Z-Sum machinery in order to obtain the necessary steps for an entirely systematic reduction, which is the topic of the next section, where we discuss

Table 4.5: All fundamental vertices in the Standard Model with particle content. Vertices obtained by full exchange of particles and antiparticles are implied if not explicitly included. Particles are considered as outgoing for simplicity (symmetry).

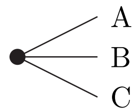
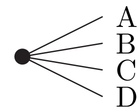
	Three Leg Vertices 	Four Leg Vertices 
QED	A: γ B: charged fermion C: $\bar{\mathrm{B}}$	
Weak Neutral	A: Z B: fermions C: $\bar{\mathrm{B}}$	
Weak Charged	A: W^+ (B,C): (upper quark, lower quark), (e,ν_e), (μ,ν_μ) or (τ,ν_τ)	
Weak Bosons Only	A: Z or γ B: W^+ C: W^-	A: W^+ B: W^- (C,D): (γ,γ), (γ,Z) or (Z,Z)
QCD	(A,B,C): (g,q,$\bar{\mathrm{q}}$) or (g,g,g)	(A,B,C,D): (g,g,g,g)
Higgs	A: h B: Z, h, W^- or fermions C: $\bar{\mathrm{B}}$	(A,B): (h,h) (C,D): (h,h), (Z,Z) or (W^+,W^-)
Ghosts	(A,B,C): (g,c,$\bar{\mathrm{c}}$)	

Table 4.6: One-loop triangle diagrams in QFTs ordered by degrees of freedom. Bracket refer to the invariants $(p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2)$, the external momenta and internal masses. Square brackets (red) represent cases present in the Standard Model. The third subcolumn (labeled \$) shows diagrams with seamless reduction, with the number indicating the corresponding polynomial in table 4.4, with 0 indicating no expansion necessary.

#	$(p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2)$	\$	#	$(p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2)$	\$	#	$(p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2)$	\$
1	$[0, 0, c_1, 0, 0, 0]$	0	2	$(0, 0, c_1, 0, 0, c_2)$	14	3	$[0, 0, c_1, 0, c_2, c_2]$	
4	$[0, 0, c_1, c_2, 0, 0]$	14	5	$(0, 0, c_1, c_2, 0, c_2)$	14	6	$[0, 0, c_1, c_2, c_2, c_2]$	
7	$[0, c_1, c_2, 0, 0, 0]$	15	8	$[0, c_1, c_2, 0, 0, c_1]$	26	9	$(0, c_1, c_2, 0, 0, c_2)$	26
10	$[0, c_1, c_2, 0, c_1, 0]$	15	11	$(0, c_1, c_2, 0, c_1, c_1)$		12	$[0, c_1, c_2, 0, c_2, c_2]$	
13	$[0, c_1, c_2, c_1, 0, c_1]$	14	14	$[0, c_1, c_2, c_1, c_1, c_1]$		15	$[c_1, c_1, c_2, 0, 0, 0]$	
16	$(c_1, c_1, c_2, 0, 0, c_1)$		17	$[c_1, c_1, c_2, 0, c_1, c_1]$	9	18	$[c_1, c_1, c_2, c_1, 0, 0]$	
19	$[c_1, c_1, c_2, c_1, 0, c_1]$		20	$[c_1, c_1, c_2, c_1, c_1, c_1]$		21	$[0, 0, c_1, 0, c_2, c_3]$	
22	$(0, 0, c_1, c_2, 0, c_3)$	14	23	$[0, 0, c_1, c_2, c_2, c_3]$		24	$[0, 0, c_1, c_2, c_3, c_3]$	
25	$[0, c_1, c_2, 0, 0, c_3]$	26	26	$(0, c_1, c_2, 0, c_1, c_3)$		27	$[0, c_1, c_2, 0, c_2, c_3]$	
28	$[0, c_1, c_2, 0, c_3, 0]$	15	29	$(0, c_1, c_2, 0, c_3, c_1)$		30	$[0, c_1, c_2, 0, c_3, c_2]$	
31	$[0, c_1, c_2, 0, c_3, c_3]$		32	$[0, c_1, c_2, c_1, 0, c_3]$	26	33	$[0, c_1, c_2, c_1, c_1, c_3]$	
34	$[0, c_1, c_2, c_1, c_3, c_1]$		35	$(0, c_1, c_2, c_1, c_3, c_3)$		36	$(0, c_1, c_2, c_2, 0, c_3)$	26
37	$(0, c_1, c_2, c_2, c_2, c_3)$		38	$(0, c_1, c_2, c_2, c_3, c_3)$		39	$[0, c_1, c_2, c_3, 0, c_3]$	14
40	$(0, c_1, c_2, c_3, c_1, c_3)$		41	$[0, c_1, c_2, c_3, c_3, c_3]$		42	$(c_1, c_1, c_2, 0, 0, c_3)$	
43	$(c_1, c_1, c_2, 0, c_1, c_3)$		44	$[c_1, c_1, c_2, 0, c_3, c_3]$		45	$(c_1, c_1, c_2, c_1, 0, c_3)$	
46	$[c_1, c_1, c_2, c_1, c_1, c_3]$		47	$[c_1, c_1, c_2, c_1, c_3, c_3]$		48	$[c_1, c_1, c_2, c_3, 0, 0]$	
49	$(c_1, c_1, c_2, c_3, 0, c_1)$		50	$(c_1, c_1, c_2, c_3, 0, c_3)$		51	$[c_1, c_1, c_2, c_3, c_1, c_1]$	
52	$(c_1, c_1, c_2, c_3, c_1, c_3)$		53	$[c_1, c_1, c_2, c_3, c_3, c_3]$		54	$[c_1, c_2, c_3, 0, 0, 0]$	
55	$[c_1, c_2, c_3, 0, 0, c_1]$		56	$[c_1, c_2, c_3, 0, 0, c_2]$		57	$[c_1, c_2, c_3, 0, c_1, c_1]$	
58	$(c_1, c_2, c_3, 0, c_1, c_2)$		59	$(c_1, c_2, c_3, 0, c_1, c_3)$		60	$[c_1, c_2, c_3, 0, c_2, c_1]$	9
61	$(c_1, c_2, c_3, 0, c_2, c_3)$		62	$[c_1, c_2, c_3, 0, c_3, c_3]$		63	$[c_1, c_2, c_3, c_1, c_1, c_1]$	
64	$[c_1, c_2, c_3, c_1, c_1, c_2]$		65	$(c_1, c_2, c_3, c_1, c_1, c_3)$		66	$[c_1, c_2, c_3, c_1, c_2, c_1]$	
67	$(0, 0, c_1, c_2, c_3, c_4)$		68	$[0, c_1, c_2, 0, c_3, c_4]$		69	$[0, c_1, c_2, c_1, c_3, c_4]$	
70	$(0, c_1, c_2, c_2, c_3, c_4)$		71	$[0, c_1, c_2, c_3, 0, c_4]$	26	72	$(0, c_1, c_2, c_3, c_1, c_4)$	
73	$[0, c_1, c_2, c_3, c_3, c_4]$		74	$[0, c_1, c_2, c_3, c_4, c_3]$		75	$(c_1, c_1, c_2, 0, c_3, c_4)$	
76	$[c_1, c_1, c_2, c_1, c_3, c_4]$		77	$(c_1, c_1, c_2, c_3, 0, c_4)$		78	$[c_1, c_1, c_2, c_3, c_1, c_4]$	
79	$[c_1, c_1, c_2, c_3, c_3, c_4]$		80	$[c_1, c_1, c_2, c_3, c_4, c_4]$		81	$[c_1, c_2, c_3, 0, 0, c_4]$	
82	$(c_1, c_2, c_3, 0, c_1, c_4)$		83	$[c_1, c_2, c_3, 0, c_2, c_4]$		84	$[c_1, c_2, c_3, 0, c_3, c_4]$	
85	$[c_1, c_2, c_3, 0, c_4, c_4]$		86	$[c_1, c_2, c_3, c_1, c_1, c_4]$		87	$[c_1, c_2, c_3, c_1, c_2, c_4]$	
88	$[c_1, c_2, c_3, c_1, c_3, c_4]$		89	$[c_1, c_2, c_3, c_1, c_4, c_1]$		90	$[c_1, c_2, c_3, c_1, c_4, c_2]$	
91	$[c_1, c_2, c_3, c_1, c_4, c_4]$		92	$[c_1, c_2, c_3, c_3, c_1, c_4]$		93	$[c_1, c_2, c_3, c_3, c_4, c_4]$	
94	$[c_1, c_2, c_3, c_4, c_4, c_4]$		95	$[0, c_1, c_2, c_3, c_4, c_5]$		96	$[c_1, c_1, c_2, c_3, c_4, c_5]$	
97	$[c_1, c_2, c_3, 0, c_4, c_5]$		98	$[c_1, c_2, c_3, c_1, c_4, c_5]$		99	$[c_1, c_2, c_3, c_3, c_4, c_5]$	
100	$[c_1, c_2, c_3, c_4, c_4, c_5]$		101	$[c_1, c_2, c_3, c_4, c_5, c_6]$				

all necessary yet missing steps. The other option would be to forgo the use of Taylor series and use Mellin-Barnes expansion instead. This will be left as a future development.

4.3 Missing Steps for an Entirely Systematic Reduction

We have seen in previous calculations that a reduction to Z-Sums is not always possible using solely the algorithms discussed in section 3.5. The problems arise from two key facts: the appearance of a factor of two multiplying the summation variable for quadratic calculations, and the occurrence of unbounded offsets. The first problem will be discussed on subsection 4.3.1, while the second will be covered in the following three subsections, where we will present the difficulties encountered to extend the algorithms previously developed with bounded offsets to the more general case of unbounded ones. The problems presented here are the only missing steps necessary for a full reduction to Z-Sums of any loop-like integration with a polynomial at most quadratic in the denominator, but as of now no solutions are known.

4.3.1 Quadratic Variations of Type C

In some situations quadratic polynomials will lead to gamma functions involving summation variables multiplied by a factor of two. For such cases it would be desirable to have a procedure for performing the reduction:

$$\sum_{i=0}^{n} \binom{n}{i} \frac{x^i}{(2i+b)^m} Z(2i,\ldots) \to Z(\ldots) \tag{4.5}$$

For summations without binomial coefficients this reduction was performed using roots of unity (section C.1). Unfortunately in this case such an approach is not applicable as it introduces unremovable half integers in the binomial, as in:

$$\sum_{i=0}^{n} \binom{n}{i} \frac{x^i}{(2i+a)^m} Z(2i,\ldots) = \frac{1}{2}\sum_{i=0}^{2n} \binom{n}{i/2} \frac{(\sqrt{x})^i}{(i+a)^m} Z(i,\ldots) + \frac{1}{2}\sum_{i=0}^{2n} \binom{n}{i/2} \frac{(-\sqrt{x})^i}{(i+a)^m} Z(i,\ldots) \tag{4.6}$$

While some sums of the form shown on the left of equation 4.5 may be reduced to Z-Sums by non systematic approaches, it is unknown at this point whether a systematic reduction is even possible.

4.3.2 Unbounded Shifting of Z-Sum Upper Limit

We have seen that some integrations lead to sums involving gamma functions with several summation variables in the argument. After performing an expansion in powers of ε, this will create Z-Sums that will require synchronization. For example, let's imagine we need to treat a sum of the type:

$$\sum_{i=1}^{n_1} \frac{x^i}{i+a} Z(i-1,\ldots) Z(i+n_2,\ldots) \tag{4.7}$$

After synchronization and multiplication we obtain sums of the form:

$$\sum_{i=1}^{n_1} \frac{x^i}{i+a} Z(i-1,\ldots) \sum_{j_1=0}^{n_2} \frac{1}{i+j_1} \sum_{j_2=0}^{j_1-1} \frac{1}{i+j_2} \cdots \sum_{j_\alpha=0}^{j_{\alpha-1}-1} \frac{1}{i+j_\alpha} \;, \tag{4.8}$$

where the first step introduces the j sums. In order to perform the summation on i we need to use partial fractioning. This will lead to several terms including:

$$\sum_{j_1=0}^{n_2} \frac{1}{j_1-a} \sum_{j_2=0}^{j_1-1} \frac{1}{j_1-j_2} \sum_{j_3=0}^{j_2-1} \frac{1}{j_1-j_3} \cdots \sum_{j_\alpha=0}^{j_{\alpha-1}-1} \frac{1}{j_1-j_\alpha} \sum_{i=1}^{n_1} \frac{x^i}{i+j_1} Z(i-1,\ldots) \tag{4.9}$$

Equation 4.9 matches the Z-Sum reduction algorithms if n_2 is a bounded number. On the other hand, if it depends on summation variables (from hidden outer sums) we face two problems. First we have to deal with the inner sum involving an unbounded offset in the denominator, which is the topic of the next two subsections. Second, the outer sums on j do not match the Z-Sum machinery because of factors like $1/(j_1 - j_k)$. A solution for this problem has not yet been found.

4.3.3 Unbounded Offset of Type A

Assuming we were able somehow to deal with the problem involving j sums, we are left to treat summations of the type:

$$\sum_{i=1}^{n_1} \frac{x^i}{(i+n_2)^m} Z(i-1;\ldots) \;, \tag{4.10}$$

where n_2 involves outer hidden summation variables. If the depth of the inner sum is zero, equations 3.95 and 3.96 can be applied even with unbounded n_2. The problem occurs when the inner sum has depth larger than zero. Equation 3.98, repeated here:

$$\sum_{i=1}^{n} \frac{x^i}{(i+b)^m} Z(i-1; m_1, \ldots) = \frac{x^n}{(n+b)^m} Z(n-1; m_1, \ldots) +$$
$$+\frac{1}{x} \sum_{i=1}^{n} \frac{x^i}{(i+b-1)^m} Z(i-1; m_1, \ldots) - \sum_{i=1}^{n-1} \frac{x^i}{(i+b)^m} \frac{x_1^i}{i^{m_1}} Z(i-1; m_2, \ldots) \quad ,$$

is recursive and cannot reduce an unbounded offset by a finite number of applications. Attempts at obtaining an explicit full form identity from repeated applications of the recursive algorithm fail because the final expression is of similar form as the initial one. A full reduction is yet to be found.

4.3.4 Unbounded Offset of Type C

Similarly to the previous case, we would like to be able to perform Type C reduction on summations with an unbounded offset, as in:

$$\sum_{i=1}^{n_1} \binom{n_1}{i} \frac{x^i}{(i+n_2)^m} S(i; \ldots) \quad . \tag{4.11}$$

Equation 3.110, repeated here:

$$\sum_{i=1}^{n} \binom{n}{i} \frac{x^i}{(i+b)^m} S(i; \ldots) = \frac{1}{x} \frac{1}{n+1} \sum_{i=1}^{n+1} \binom{n+1}{i} \frac{x^i}{(i+b-1)^m} i S(i-1; \ldots) \quad ,$$

is also recursive and cannot remove an unbounded offset by a finite number of applications. It is possible to write an explicit full form identity by repeated applications, however the expression obtained is not helpful. Below we illustrate the problem. We start by shifting the inner sum $S(i-1; \ldots)$ upper limit to obtain:

$$\sum_{i=1}^{n} \binom{n}{i} \frac{x_0^i}{(i+b_1')^{m_0}} S(i; m_1, \ldots, m_k; x_1, \ldots, x_k) = \frac{1}{x_0} \frac{1}{n+1} \sum_{i=1}^{n+1} \binom{n+1}{i} \frac{x_0^i}{(i+b-1)^{m_0}} i$$
$$\times \left(S(i; m_1, \ldots) - S(i; m_2, \ldots) \frac{x_1^i}{i^{m_1}} + S(i; m_3, \ldots) \frac{(x_1 x_2)^i}{i^{m_1+m_2}} - \ldots \pm \frac{(x_1 \ldots x_k)^i}{i^{m_1+\ldots+m_k}} \right) \quad . \tag{4.12}$$

After partial fractioning, we obtain two types of sums: one with zero offset which may be readily reduced using the standard Type C algorithm, and another of the form:

$$\frac{1}{x_0} \frac{1}{n+1} \sum_{i=1}^{n+1} \binom{n+1}{i} \frac{x^i}{(i+b-1)^m} i S(i; \ldots) \quad . \tag{4.13}$$

Repeated applications lead to the reduction:

$$\sum_{i=1}^{n}\binom{n}{i}\frac{x_0^i}{(i+b_1')^{m_0}}S(i;m_1,\ldots,m_k;x_1,\ldots,x_k)\to(\text{Zero offset sums}) \tag{4.14}$$

$$+\quad \frac{\Gamma(n+1)}{\Gamma(n+b+1)}\frac{1}{x^b}\sum_{i=1}^{n+b}\binom{n+b}{i}\frac{x^i}{i^m}\frac{\Gamma(i+1)}{\Gamma(i-b+1)}S(i;\ldots)\quad .$$

While the first type of sums have zero offset, the second includes the factor $\frac{\Gamma(i+1)}{\Gamma(i-b+1)}$, which cannot be reduced to a Z-Sum for unbounded b, making the expression circular. An effective procedure for removal of unbounded offset of type C is yet to be found.

CHAPTER 5

APPLICATION

In the last chapter we surveyed the applicability of the Taylor expansion method and found that not all calculations could be performed using the Z-Sum machinery as it is, specially if several external legs are off-shell. Since some of these diagrams are used as building blocks for multiloop calculations this demonstrates the limitations of the procedure, with similar difficulties to be expected for box diagrams. While this is a concern if the ultimate goal is to develop an automatic computational package capable of performing any loop calculation, it does not necessarily put this approach in a worse position than the alternatives, given that such a general survey has not yet been done for the other methods and so their limitations are still unknown.

In this chapter, therefore, we would like to focus on the successes of the approach by applying it to a problem of physical interest. One interesting application is the calculation of heavy flavor corrections to the renormalized scattering amplitude for deep inelastic scattering (DIS) [3, 38]. In such reactions, the internal structure of a proton, which cannot be calculated from first principles, is probed by a highly energetic particle, for example an electron in the reaction $e^- + N \to e^- + X$, where X represents a generic final state containing strong interacting bound states of quarks and gluons.

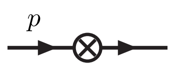
The scattering amplitude can be represented on a basis of matrix elements of universal operators whose coefficients are process dependent and can be calculated perturbatively in QCD. The effects of massive quarks show up mainly in the matrix elements of the operators and in the scale evolution of the coefficients. The calculation of heavy quark effects in the operators matrix elements has been performed both approximately and exactly at $O(\alpha_s^2)$, while partial contributions have been obtained at $O(\alpha_s^3)$. In both cases, the effects are sizable and definitely to be included in the theoretical predictions to be safely compared with the current experimental results. In particular, in [17, 15, 16, 18] the calculation of the most important set of contributions, from operators with external flavor line, have been calculated analytically at $O(\alpha_s^2)$, introducing new techniques based on Mellin-Barnes representations of loop integrals. As a test of the applicability and potential of the method proposed in this thesis, we here reproduced these results.

In practice this means that we will have to calculate two-loop heavy quark corrections to the gluon self-energy and to the matrix elements of certain operators. The operators that we consider are given as vertices with specific Feynman rules [15], shown in fig. 5.1. The self-energy diagrams that we need to calculate are represented in fig. 5.2, while the ones that are involved in the $O(\alpha_s^2)$ corrections to the matrix elements operators in question have been separated in two groups, those that do not involve a bubble loop as a building block and those that do, as shown in figs. 5.3 and 5.4. We will focus our attention in the first group since these lead to the most complicated calculations. Regarding the second group, the presence of a bubble loop generally leads to a simpler calculation and so we will not be performing them explicitly, although we did verify that the method may be applied in all cases.

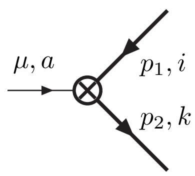
5.1 Self Energy Diagrams Without Operator Insertions

In this section we will calculate the integrations originating from diagrams shown in fig. 5.2. For clarity, we list here the steps we will be following:

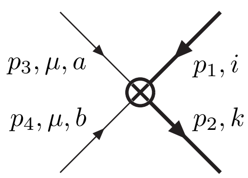
- perform first momentum integration and express result in terms of an artificial propagator in second integration;
- perform second momentum integration;
- expand parameter integrand using Taylor series and perform integration;
- expand gamma functions in powers of ε and apply Z-Sum reduction algorithms;



$$\not{\Delta}\,(\Delta\cdot p)^{N-1} \qquad N \geq 1$$



$$g t^a_{ki} \Delta^\mu \,\not{\Delta} \sum_{j=0}^{N-2} (\Delta\cdot p_1)^j \, (\Delta\cdot p_2)^{N-j-2} \qquad N \geq 2$$



$$\begin{aligned} & g^2 \Delta^\mu \Delta^\nu \,\not{\Delta} \sum_{j_2=0}^{N-2} \sum_{j_1=0}^{j_2} \\ & \times \Big[\left(t^a t^b\right)_{ki} (\Delta\cdot p_1)^{N-j_2-2} \, (\Delta\cdot p_1 + \Delta\cdot p_4)^{j_2-j_1-1} \, (\Delta\cdot p_2)^{j_1} \\ & + \left(t^b t^a\right)_{ki} (\Delta\cdot p_1)^{N-j_2-2} \, (\Delta\cdot p_1 + \Delta\cdot p_3)^{j_2-j_1-1} \, (\Delta\cdot p_2)^{j_1} \Big] \end{aligned} \qquad N \geq 3$$

Figure 5.1: Feynman rules for operator insertions.

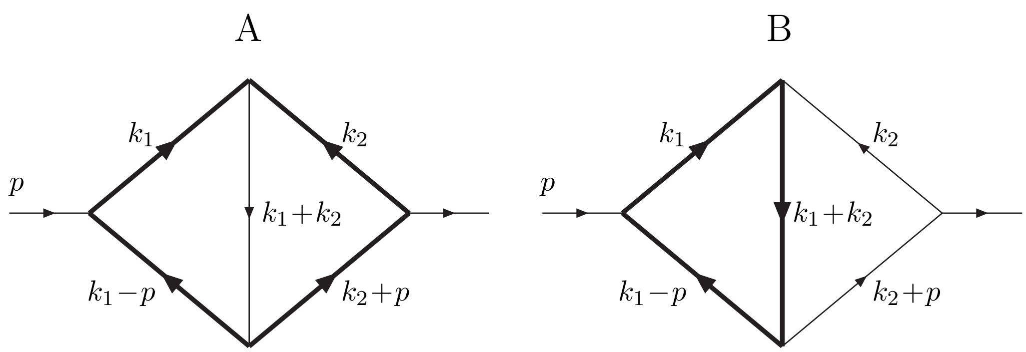


Figure 5.2: Two loop diagrams with massive internal particles (represented by thick lines) and massless external one. Arrows represent momentum direction. These diagrams represent the integrals arising in self energy corrections for a gluon, with a massive internal quark loop.

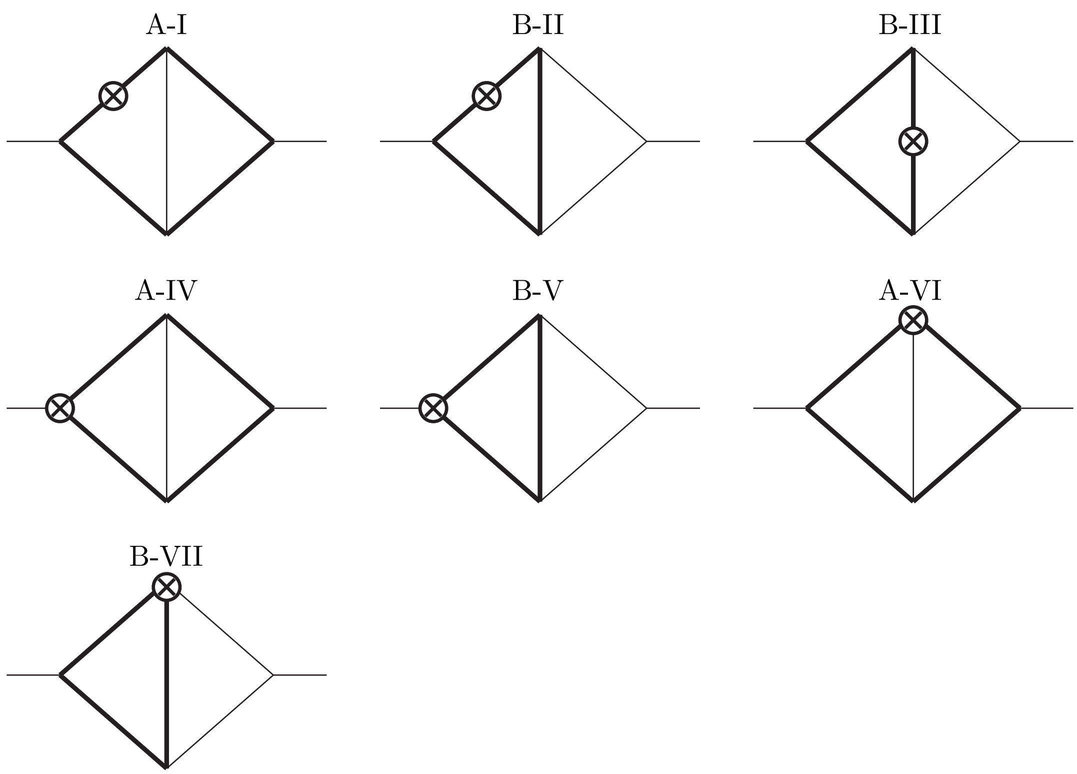


Figure 5.3: Comprehensive list of two triangle loop diagrams with operator insertion stemming from the calculation the renormalized heavy quark effects in the operator matrix elements of DIS at $O(\alpha_s^2)$.

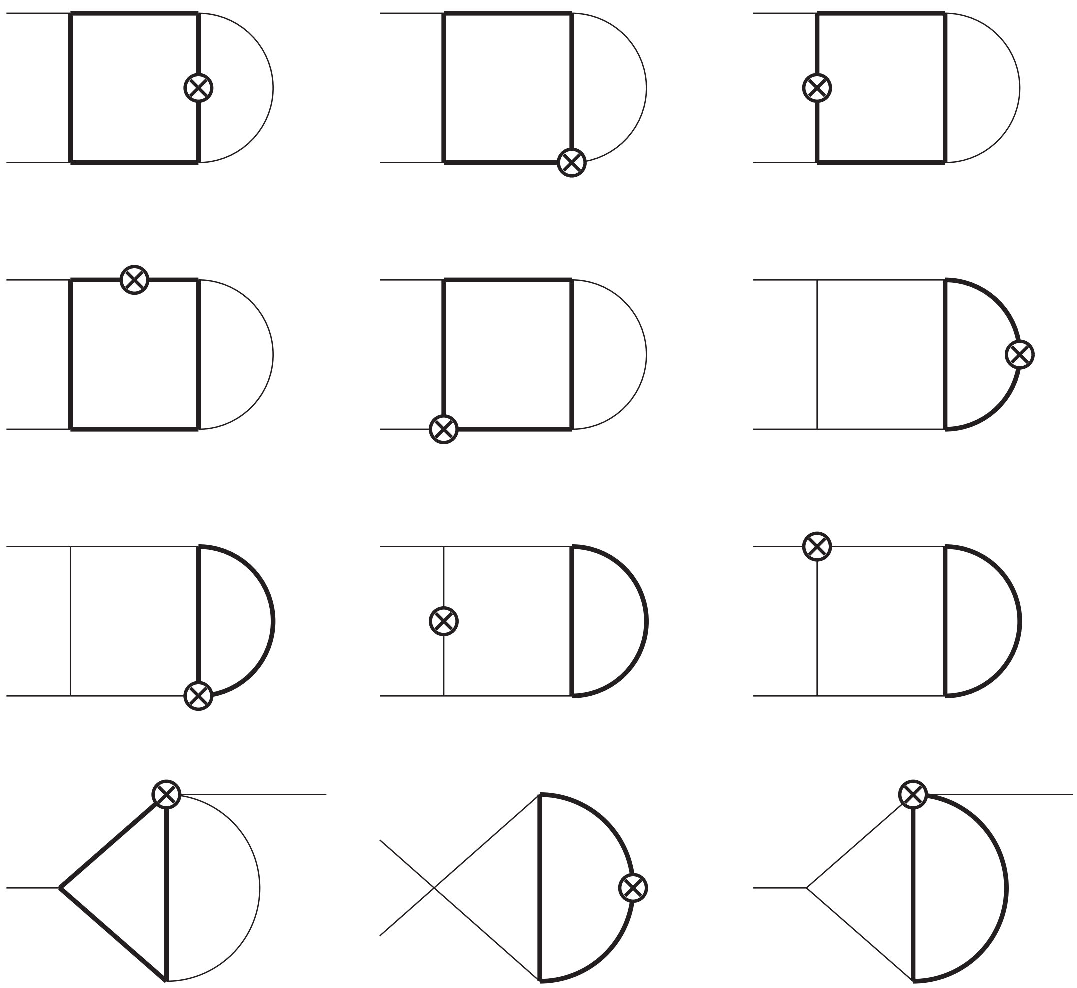


Figure 5.4: Remaining two loop diagrams necessary for the calculation of the renormalized heavy quark effects in the operator matrix elements of DIS at $O(\alpha_s^2)$. Because of the presence of a bubble loop, these calculations are generally simpler than those presented in fig. 5.3.

- obtain result in terms of polylogarithms.

5.1.1 Diagram A

We start with an explicit calculation of the integration for diagram A (fig. 5.2). The initial expression is given by:

$$\int \frac{d^D k_2}{(2\pi)^D} \frac{1}{\left(k_2^2 - m^2\right)^{\nu_4} \Big((k_2+p)^2 - m^2\Big)^{\nu_5}} \qquad (5.1)$$
$$\times \quad \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{\left(k_1^2 - m^2\right)^{\nu_1} \Big((k_1+k_2)^2\Big)^{\nu_2} \Big((k_1-p)^2 - m^2\Big)^{\nu_3}} \quad .$$

We perform the k_1 integration and re-express the result in terms of an artificial propagator:

$$\frac{i(-1)^{\nu_{123}}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\big(\nu_{123}-\frac{D}{2}\big)}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)} \int_0^1 \frac{dx\ \delta_{123}\ x_1^{\nu_1-1}\ x_2^{\nu_2-1}\ x_3^{\nu_3-1}}{(-x_2(1-x_2))^{\nu_{123}-\frac{D}{2}}}$$
$$\times \quad \int \frac{d^D k_2}{(2\pi)^D} \frac{1}{\left(k_2^2 - m^2\right)^{\nu_4} \Big((k_2+p)^2 - m^2\Big)^{\nu_5} \left(\Big(k_2 + \frac{x_3\ p}{1-x_2}\Big)^2 - \frac{m^2}{x_2}\right)^{\nu_{123}-\frac{D}{2}}} \quad , (5.2)$$

where $\delta_{123} = \delta(1 - x_1 - x_2 - x_3)$ and dx represents the integration on all x_ℓ ($\ell = 1,\ 2,\ 3$). At this point the k_2 integration may be performed to obtain:

$$\frac{(-1)^{\nu_{12345}+1}}{(4\pi)^D\,(m^2)^{\nu_{12345}-D}} \frac{\Gamma(\nu_{12345}-D)}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)\,\Gamma(\nu_4)\,\Gamma(\nu_5)} \qquad (5.3)$$
$$\times \quad \int_0^1 \frac{dx\ \delta_{123}\ \delta_{456}\ x_1^{\nu_1-1}\ x_2^{\nu_{245}-\frac{D}{2}-1}\ (1-x_2)^{-\nu_{123}+\frac{D}{2}}\ x_3^{\nu_3-1}\ x_4^{\nu_4-1}\ x_5^{\nu_5-1}\ x_6^{\nu_{123}-\frac{D}{2}-1}}{(x_2\,(1-x_6)+x_6)^{\nu_{12345}-D}} \quad .$$

The integration deltas are removed by applying a change of parameter variables:

$$\frac{(-1)^{\nu_{12345}+1}}{(4\pi)^D\,(m^2)^{\nu_{12345}-D}} \frac{\Gamma(\nu_{12345}-D)}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)\,\Gamma(\nu_4)\,\Gamma(\nu_5)} \int_0^1 du\ u_1^{-\nu_2+\frac{D}{2}-1}\ (1-u_1)^{\nu_{245}-\frac{D}{2}-1}$$
$$\times \quad \frac{u_2^{\nu_1-1}\ (1-u_2)^{\nu_3-1}\ u_3^{-\nu_4-1}\ (1-u_3)^{\nu_5-1}\ u_4^{-\nu_{45}-1}\ (1-u_4)^{\nu_{123}-\frac{D}{2}-1}}{(1-u_1\ u_4)^{\nu_{12345}-D}} \quad . \qquad (5.4)$$

With the inclusion of a regulator α the denominator may be expanded leading to:

$$\frac{(-1)^{\nu_{12345}+1}}{(4\pi)^D\,(m^2)^{\nu_{12345}-D}} \frac{\Gamma\big(\nu_{245}-\frac{D}{2}\big)\,\Gamma\big(\nu_{123}-\frac{D}{2}\big)}{\Gamma(\nu_2)\,\Gamma(\nu_{13})\,\Gamma(\nu_{45})} \qquad (5.5)$$
$$\times \quad \sum_{i=0}^{\infty} \alpha^i\ \frac{\Gamma(i+\nu_{12345}-D)}{\Gamma(i+1)} \frac{\Gamma\big(i-\nu_2+\frac{D}{2}\big)}{\Gamma\big(i+\nu_{12345}-\frac{D}{2}\big)} \quad .$$

Equation 5.5 may be expressed in terms of Z-Sums and ultimately polylogarithms even for generic values of ν_ℓ and D by using identities 3.47 and 3.48. However for simplicity we will finish this calculation using the specific values of $\nu_\ell = 1$ and $D = 4 - 2\varepsilon$. Our expression becomes:

$$\frac{1}{(4\pi)^4 m^2}\left(\frac{4\pi}{m^2}\right)^{2\varepsilon}\Gamma(1+\varepsilon)^2\sum_{i=0}^{\infty}\alpha^i\,\frac{\Gamma(i+1+2\varepsilon)}{\Gamma(i+1)}\frac{\Gamma(i+1-\varepsilon)}{\Gamma(i+3+\varepsilon)}\quad . \tag{5.6}$$

It is interesting to notice that this expression is simpler than the one obtained when using Mellin-Barnes splitting [17], as it involves a single infinite summation, with both leading to the same result.

Since equation 5.6 is finite in ε, we apply the limit $\varepsilon \to 0$, leading to:

$$\begin{aligned}
&\frac{1}{(4\pi)^4 m^2}\sum_{i=0}^{\infty}\alpha^i\,\frac{\Gamma(i+1)}{\Gamma(i+3)}\\
=&\frac{1}{(4\pi)^4 m^2}\sum_{i=0}^{\infty}\alpha^i\,\left(\frac{1}{i+1}-\frac{1}{i+2}\right)\\
=&\frac{1}{(4\pi)^4 m^2}\left(\frac{1}{\alpha}-\log(1-\alpha)\left(\frac{1}{\alpha}-\frac{1}{\alpha^2}\right)\right)\quad .
\end{aligned} \tag{5.7}$$

The final step is to remove the regulator α. In the limit $\alpha \to 1$ the last terms vanish and we obtain:

$$\frac{1}{(4\pi)^4 m^2}\quad . \tag{5.8}$$

The simplicity of the result is due to the fact that this is a two point function with massless on-shell external particle.

5.1.2 Diagram B

The calculation for diagram B (fig. 5.2) follows the same procedure and so we omit the intermediate equations. The initial expression is given by:

$$\begin{aligned}
&\int\frac{d^Dk_2}{(2\pi)^D}\frac{1}{\left(k_2^2-m^2\right)^{\nu_4}\left((k_2+p)^2-m^2\right)^{\nu_5}}\\
\times&\int\frac{d^Dk_1}{(2\pi)^D}\frac{1}{\left(k_1^2\right)^{\nu_1}\left((k_1+k_2)^2-m^2\right)^{\nu_2}\left((k_1-p)^2\right)^{\nu_3}}\quad .
\end{aligned} \tag{5.9}$$

We proceed by performing the momentum integrations, expanding the denominator after inclusion of a regulator α, and finally performing the parameter integrations. We obtain:

$$\begin{aligned}
&\frac{(-1)^{\nu_{12345}+1}}{(4\pi)^D\,(m^2)^{\nu_{12345}-D}}\,\frac{\Gamma\big(\nu_{1345}-\frac{D}{2}\big)\,\Gamma\big(\nu_{123}-\frac{D}{2}\big)}{\Gamma(\nu_2)\,\Gamma(\nu_{13})\,\Gamma(\nu_{45})} \\
\times\;&\sum_{i=0}^{\infty}\alpha^i\;\frac{\Gamma(i+\nu_{12345}-D)}{\Gamma(i+1)}\,\frac{\Gamma\big(i-\nu_{13}+\frac{D}{2}\big)}{\Gamma\big(i+\nu_{12345}-\frac{D}{2}\big)} \quad .
\end{aligned} \tag{5.10}$$

After setting $\nu_\ell = 1$ and $D = 4-2\varepsilon$ the expression becomes:

$$\frac{\Gamma(2+\varepsilon)\,\Gamma(1+\varepsilon)}{(4\pi)^{4-2\varepsilon}\,(m^2)^{1+2\varepsilon}}\sum_{i=0}^{\infty}\alpha^i\;\frac{\Gamma(i+1+2\varepsilon)}{\Gamma(i+1)}\,\frac{\Gamma(i-\varepsilon)}{\Gamma(i+3+\varepsilon)} \quad . \tag{5.11}$$

Unlike diagram A, this expression is not finite so we need to expand the gamma functions in powers of ε. This can be done using the method discussed in previous chapters, after which we obtain:

$$\frac{\Gamma(2+\varepsilon)\,\Gamma(1+\varepsilon)}{(4\pi)^{4-2\varepsilon}\,(m^2)^{1+2\varepsilon}}\left(\frac{-1}{2\varepsilon}+\frac{3}{2}-\frac{1}{2\alpha}-\frac{1}{2\alpha}\left(1-\frac{2}{\alpha}+\frac{1}{\alpha^2}\right)\log\left(1-\alpha\right)\right) \quad . \tag{5.12}$$

When we take the limit $\alpha \to 0$ the last term vanishes and the final result becomes:

$$\frac{\Gamma(2+\varepsilon)\,\Gamma(1+\varepsilon)}{(4\pi)^{4-2\varepsilon}\,(m^2)^{1+2\varepsilon}}\left(-\frac{1}{2\varepsilon}+1\right) \quad . \tag{5.13}$$

Now we move to more complicated diagrams involving insertion of operators.

5.2 Diagrams with Operator Insertions

In this section we will reproduce results presented in [17] by calculating diagrams involving operator insertions, shown in fig. 5.3. These diagrams appear in the renormalization of the operator matrix elements for DIS. We perform the calculation explicitly for two examples, and also present results for the remaining ones.

5.2.1 Momentum Integration with Insertion Operators

Before we start with specific calculations let's examine how the new factors in the numerator originating from the modified Feynman rules of fig. 5.1 affects a general triangle integration. Consider a prototype integration of the form:

$$\int\frac{d^Dk}{(2\pi)^D}\,\frac{(\Delta\cdot k)^s}{\left(k^2-m_1^2\right)^{\nu_1}}\,\frac{1}{\left((k-a_2)^2-m_2^2\right)^{\nu_2}}\,\frac{1}{\left((k-a_3)^2-m_3^2\right)^{\nu_3}} \quad . \tag{5.14}$$

The initial steps in the normal procedure, that is, introduction of Feynman parameters and completion of the square in the denominator, may be applied as before. We obtain:

$$\frac{\Gamma(\nu_{123})}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)} \int_0^1 dx\ \delta_{123}\ x_1^{\nu_1-1}\ x_2^{\nu_2-1}\ x_3^{\nu_3-1} \int \frac{d^D k'}{(2\pi)^D} \frac{(\Delta\cdot(k'+x_2a_2+x_3a_3))^s}{\left(k'^2-\Delta\right)^{\nu_{123}}} \quad . \tag{5.15}$$

Before we can proceed, we need to expand the factor in the numerator using binomials:

$$\begin{aligned} &\frac{\Gamma(\nu_{123})}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)} \sum_{j_1=0}^{s} \binom{s}{j_1} \sum_{j_2=0}^{j_1} \binom{j_1}{j_2} (\Delta\cdot a_2)^{j_1-j_2} (\Delta\cdot a_3)^{j_2} \\ \times\ & \int_0^1 dx\ \delta_{123}\ x_1^{\nu_1-1}\ x_2^{j_1-j_2+\nu_2-1}\ x_3^{j_2+\nu_3-1} \int \frac{d^D k'}{(2\pi)^D} \frac{(\Delta\cdot k')^{s-j_1}}{\left(k'^2-\Delta\right)^{\nu_{123}}} \quad . \end{aligned} \tag{5.16}$$

Because $\Delta^2 = 0$, terms involving non-zero powers of $(\Delta\cdot k')^{s-j_1}$ inside the integration vanish, and so our expression becomes

$$\begin{aligned} &\frac{\Gamma(\nu_{123})}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)} \sum_{j_2=0}^{s} \binom{s}{j_2} (\Delta\cdot a_2)^{s-j_2} (\Delta\cdot a_3)^{j_2} \\ \times\ & \int_0^1 dx\ \delta_{123}\ x_1^{\nu_1-1}\ x_2^{s-j_2+\nu_2-1}\ x_3^{j_2+\nu_3-1} \int \frac{d^D k'}{(2\pi)^D} \frac{1}{\left(k'^2-\Delta\right)^{\nu_{123}}} \quad . \end{aligned} \tag{5.17}$$

At this point the regular procedure may be resumed. The end result is simply the normal expression for a triangle loop summed over powers of $\Delta\cdot a$, as in

$$I^*_{cc} = \sum_{j=0}^{s} \binom{s}{j} (\Delta\cdot a_2)^{s-j} (\Delta\cdot a_3)^j\ I_{cc} \quad , \tag{5.18}$$

where I^*_{cc} and I_{cc} represent the triangle integration with and without insertion operators, respectively.

5.2.2 Diagram B-II

The integral for diagram B-II (fig. 5.3) is given by:

$$\begin{aligned} &\int \frac{d^D k_2}{(2\pi)^D} \frac{(\Delta\cdot k_2)^{N-1}}{(k_2^2-m^2)^{\nu_4}\left((k_2+p)^2-m^2\right)^{\nu_5}} \\ \times\ & \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{(k_1^2)^{\nu_1}\left((k_1+k_2)^2-m^2\right)^{\nu_2}\left((k_1-p)^2\right)^{\nu_3}} \quad . \end{aligned} \tag{5.19}$$

The k_1 integration does not involve operator insertions and so it is performed as usual leading to:

$$\frac{i(-1)^{\nu_{123}}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\big(\nu_{123}-\frac{D}{2}\big)}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)} \int_0^1 \frac{dx\ \delta_{123}\ x_1^{\nu_1-1}\ x_2^{\nu_2-1}\ x_3^{\nu_3-1}}{(-x_2(1-x_2))^{\nu_{123}-\frac{D}{2}}} \\ \times \int \frac{d^Dk_2}{(2\pi)^D} \frac{(\Delta\cdot k_2)^{N-1}}{\left(k_2^2-m^2\right)^{\nu_4}\left((k_2+p)^2-m^2\right)^{\nu_5}\left(\left(k_2+\frac{x_3\ p}{1-x_2}\right)^2-\frac{m^2}{x_2}\right)^{\nu_{123}-\frac{D}{2}}}\quad . \tag{5.20}$$

Following the steps discussed in the previous subsection, we perform the k_2 integration involving the insertion operator and obtain:

$$\frac{(-1)^{\nu_{12345}+1}\,(-\Delta\cdot p)^{N-1}}{(4\pi)^D\,(m^2)^{\nu_{12345}-D}} \frac{\Gamma(\nu_{12345}-D)}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)\,\Gamma(\nu_4)\,\Gamma(\nu_5)} \sum_{j=0}^{N-1}\binom{N-1}{j} \\ \times \int \frac{dx\ \delta_{123}\ \delta_{456}\ x_1^{\nu_1-1} x_2^{-\nu_{13}+\frac{D}{2}-1}\ (1-x_2)^{-j+\nu_{45}-\frac{D}{2}}\ x_3^{j+\nu_3-1} x_4^{\nu_4-1} x_5^{N-j+\nu_5-2} x_6^{j+\nu_{123}-\frac{D}{2}-1}}{((1-x_6)\,(1-x_2)+x_6)^{\nu_{12345}-D}}. \tag{5.21}$$

Applying a change of integration variables and introducing a regulator α we get:

$$\frac{(-1)^{\nu_{12345}+1}\,(-\Delta\cdot p)^{N-1}}{(4\pi)^D\,(m^2)^{\nu_{12345}-D}} \frac{\Gamma(\nu_{12345}-D)}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)\,\Gamma(\nu_4)\,\Gamma(\nu_5)} \sum_{j=0}^{N-1}\binom{N-1}{j} \\ \times \int du\ u_1^{-\nu_{13}+\frac{D}{2}-1}\ (1-u_1)^{\nu_{1345}-\frac{D}{2}-1}\ u_2^{\nu_1-1}\ (1-u_2)^{j+\nu_3-1} \\ \times \frac{u_3^{\nu_4-1}\ (1-u_3)^{N-j+\nu_5-2}\ u_4^{N-j+\nu_{45}-2}\ (1-u_4)^{j+\nu_{123}-\frac{D}{2}-1}}{(1-\alpha\ u_1\ u_4)^{\nu_{12345}-D}}\quad . \tag{5.22}$$

After expanding the denominator and performing the parameter integrations we obtain:

$$\frac{(-1)^{\nu_{12345}+1}\,(-\Delta\cdot p)^{N-1}}{(4\pi)^D\,(m^2)^{\nu_{12345}-D}} \frac{\Gamma\big(\nu_{1345}-\frac{D}{2}\big)}{\Gamma(\nu_2)\,\Gamma(\nu_3)\,\Gamma(\nu_5)} \\ \times \sum_{j=0}^{N-1}\binom{N-1}{j} \frac{\Gamma(j+\nu_3)}{\Gamma(j+\nu_{13})} \frac{\Gamma(N-j+\nu_5-1)}{\Gamma(N-j+\nu_{45}-1)} \\ \times \sum_{i=0}^{\infty}\alpha^i \frac{\Gamma(i+\nu_{12345}-D)}{\Gamma(i+1)} \frac{\Gamma\big(i-\nu_{13}+\frac{D}{2}\big)}{\Gamma(i+\nu_{45})} \frac{\Gamma(i+N-j+\nu_{45}-1)\,\Gamma\big(j+\nu_{123}-\frac{D}{2}\big)}{\Gamma\big(i+N+\nu_{12345}-\frac{D}{2}-1\big)}\quad . \tag{5.23}$$

Using equations 3.47 and 3.48 it is possible to systematically express the above equation in terms of Z-Sum and ultimately polylogarithms for generic values of N, ν_ℓ and D. For simplicity, we will calculate it for $\nu_\ell = 1$ and $D = 4-2\varepsilon$. Apart from the prefactor

$\frac{(-\Delta p)^{N-1}}{(4\pi)^{4-2\varepsilon}(m^2)^{1+2\varepsilon}}$ we get:

$$\Gamma(2+\varepsilon)\sum_{j=0}^{N-1}\binom{N-1}{j}\frac{\Gamma(j+1)\,\Gamma(j+1+\varepsilon)\,\Gamma(N-j)}{\Gamma(j+2)\,\Gamma(N-j+1)}$$
$$\times\quad\sum_{i=0}^{\infty}\alpha^i\frac{\Gamma(i-\varepsilon)\,\Gamma(i+1-2\varepsilon)\,\Gamma(i-j+N+1)}{\Gamma(i+1)\,\Gamma(i+2)\,\Gamma(i+N+2+\varepsilon)}\quad. \tag{5.24}$$

At this point the regular Z-Sum machinery may be applied. This calculation has been performed for N equal to 2 through 5 in order to reproduce previous results obtained by other methods, and are presented in table 5.1.

5.2.3 Diagram A-VI

Diagram A-VI involves the insertion of a vertex operator, and so it is a bit more complicated than the previous example. We start the calculation with:

$$\sum_{j_1=0}^{N-2}\int\frac{d^Dk_2}{(2\pi)^D}\frac{(-\Delta\cdot k_2)^{N-j_1-1}}{\left(k_2^2-m^2\right)^{\nu_4}\left((k_2+p)^2-m^2\right)^{\nu_5}} \tag{5.25}$$
$$\times\quad\int\frac{d^Dk_1}{(2\pi)^D}\frac{(\Delta\cdot k_1)^{j_1}}{\left(k_1^2-m^2\right)^{\nu_1}\left((k_1+k_2)^2\right)^{\nu_2}\left((k_1-p)^2-m^2\right)^{\nu_3}}\quad.$$

In this case both integrations include a factor in the numerator. After the k_1 integration we obtain:

$$\frac{i(-1)^{\nu_{123}}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma\left(\nu_{123}-\frac{D}{2}\right)}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)}\sum_{j_1=0}^{N-2}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}(-1)^{N-j_1+j_2-2}\,(\Delta\cdot p)^{j_1-j_2}$$
$$\times\quad\int_0^1\frac{dx\;\delta_{123}\;x_1^{\nu_1-1}\;x_2^{\nu_2-1}\;x_3^{\nu_3-1}}{\left(-x_2\,(1-x_2)\right)^{\nu_{123}-\frac{D}{2}}} \tag{5.26}$$
$$\times\quad\int\frac{d^Dk_2}{(2\pi)^D}\frac{(\Delta\cdot k_2)^{N-j_1+j_2-2}}{\left(k_2^2-m^2\right)^{\nu_4}\left((k_2+p)^2-m^2\right)^{\nu_5}\left(\left(k_2+\frac{x_3p}{1-x_2}\right)^2-\frac{m^2}{x_2}\right)^{\nu_{123}-\frac{D}{2}}}\quad.$$

We proceed with the k_2 integration and change parameter variables:

$$\begin{aligned}
&\frac{(-1)^{\nu_{12345}+1}}{(4\pi)^D}\frac{(\Delta\cdot p)^{N-2}}{(m^2)^{\nu_{12345}-D}}\frac{\Gamma(\nu_{12345}-D)}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)\,\Gamma(\nu_4)\,\Gamma(\nu_5)}\\
\times\;&\sum_{j_1=0}^{N-2}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{N-j_1+j_2-2}\binom{N-j_1+j_2-2}{j_3}\\
\times\;&\int_0^1 du\; u_1^{j_1-j_2-\nu_2+\frac{D}{2}-1}\;(1-u_1)^{j_2+\nu_{245}-\frac{D}{2}-1}\;u_2^{\nu_1-1}\;(1-u_2)^{j_1-j_2+j_3+\nu_3-1}\\
\times\;&\frac{u_3^{\nu_4-1}\;(1-u_3)^{N-j_1+j_2-j_3+\nu_5-3}\;u_4^{N-j_1+j_2-j_3+\nu_{45}-3}\;(1-u_4)^{j_3+\nu_{123}-\frac{D}{2}-1}}{(1-\alpha\;u_1\;u_4)^{\nu_{12345}-D}}\quad.
\end{aligned}\tag{5.27}$$

After denominator expansion and parameter integration we get:

$$\begin{aligned}
&\frac{(-1)^{\nu_{12345}+1}}{(4\pi)^D}\frac{(\Delta\cdot p)^{N-2}}{(m^2)^{\nu_{12345}-D}}\frac{1}{\Gamma(\nu_2)\,\Gamma(\nu_3)\,\Gamma(\nu_5)}\sum_{j_1=0}^{N-2}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\\
\times\;&\sum_{j_3=0}^{N-j_1+j_2-2}\binom{N-j_1+j_2-2}{j_3}\frac{\Gamma(j_1-j_2+j_3+\nu_3)}{\Gamma(j_1-j_2+j_3+\nu_{13})}\frac{\Gamma(N-j_1+j_2-j_3+\nu_5-2)}{\Gamma(N-j_1+j_2-j_3+\nu_{45}-2)}\\
\times\;&\sum_{i=0}^{\infty}\alpha^i\frac{\Gamma(i+\nu_{12345}-D)}{\Gamma(i+1)}\frac{\Gamma\left(i+j_1-j_2-\nu_2+\frac{D}{2}\right)\Gamma\left(j_2+\nu_{245}-\frac{D}{2}\right)}{\Gamma(i+j_1+\nu_{45})}\\
\times\;&\frac{\Gamma(i+N-j_1+j_2-j_3+\nu_{45}-2)\,\Gamma\left(j_3+\nu_{123}-\frac{D}{2}\right)}{\Gamma(i+N-j_1+j_2+\nu_{12345}-2)}\quad.
\end{aligned}\tag{5.28}$$

For the purpose of this example we set $\nu_\ell = 1$ and $D = 4$. Apart from a prefactor we get:

$$\begin{aligned}
&\sum_{j_1=0}^{N-2}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{N-j_1+j_2-2}\binom{n-j_1+j_2-2}{j_3}\frac{\Gamma(j_2+1)\,\Gamma(j_3+1)\,\Gamma(j_1-j_2+j_3+1)}{\Gamma(j_1-j_2+j_3+2)}\\
\times\;&\frac{\Gamma(N-j_1+j_2-j_3-1)}{\Gamma(N-j_1+j_2-j_3)}\sum_{i=0}^{\infty}\alpha^i\frac{\Gamma(i+j_1-j_2+1)\,\Gamma(i+N-j_1+j_2-j_3)}{\Gamma(i+j_1+2)\,\Gamma(i+N-j_1+j_2+1)}\quad.
\end{aligned}\tag{5.29}$$

5.2.4 Remaining Diagrams

Having explicitly discussed two calculations as examples, we end this chapter by presenting results for the remaining operator insertion diagrams required in the calculation of the process. Calculations for all diagrams in fig. 5.3 have been reproduced and are presented in table 5.1 for values of N from 2 through 5. Note that our normalization choice has not been included in order to match the results obtained by [17].

Table 5.1: Results for insertion diagrams.

N	2	3	4	5
A-Ia	$\frac{1}{2}$	$\frac{67}{216}$	$\frac{31}{144}$	$\frac{2161}{13500}$
A-Ib	$-\frac{13}{144}$	$-\frac{19}{432}$	$-\frac{17}{675}$	$-\frac{431}{27000}$
B-II	$-\frac{1}{4\varepsilon}+\frac{1}{4}+\frac{\gamma_E}{2}$	$-\frac{11}{72\varepsilon}+\frac{23}{144}+\frac{11\gamma_E}{36}$	$-\frac{5}{48\varepsilon}+\frac{11}{96}+\frac{5\gamma_E}{24}$	$-\frac{137}{1800\varepsilon}+\frac{949}{10800}+\frac{137\gamma_E}{900}$
A-III	0	$-\frac{1}{24\varepsilon}+\frac{1}{48}+\frac{\gamma_E}{12}$	0	$-\frac{1}{90\varepsilon}+\frac{1}{270}+\frac{\gamma_E}{45}$
A-IV	1	0	$\frac{31}{72}$	0
B-V	$-\frac{1}{2\varepsilon}+\frac{1}{2}+\gamma_E$	0	$-\frac{5}{24\varepsilon}+\frac{11}{48}+\frac{5\gamma_E}{12}$	0
A-VI	1	1	$\frac{65}{72}$	$\frac{29}{36}$
B-VII	$-\frac{1}{2\varepsilon}+\frac{1}{2}+\gamma_E$	$-\frac{1}{4\varepsilon}+\frac{1}{4}+\frac{\gamma_E}{2}$	$-\frac{5}{24\varepsilon}+\frac{29}{144}+\frac{5\gamma_E}{12}$	$-\frac{5}{36\varepsilon}+\frac{7}{48}+\frac{5\gamma_E}{18}$

While we will not be presenting results for diagrams involving bubbles as building blocks (fig. 5.4) because of their simplicity, we note that the method can be successfully applied in all cases.

CHAPTER 6

CONCLUSIONS

We started this project with the intention of understanding the applicability of the Z-Sum reduction algorithms to multi-loop diagram integrations involving massive degrees of freedom. Unlike most high energy calculations, our work did not aim at performing a full calculation to obtain a physically measurable quantity for a specific reaction, but rather focused solely on the integration techniques, trying to obtain a general procedure applicable to a variety of processes.

In order to apply the existing Z-Sum reduction algorithms to loop integrals, the first step involves performing an expansion followed by the integrations, leading to an expression in terms of concatenated sums. As discussed in chapter 2, we found there are three different approaches for this expansion:

A: Standard multi-loop approach followed by Taylor series expansion:

1. Introduce Feynman parameters and perform momentum integration of first loop.
2. Express result of first step in terms of one artificial propagator in second loop.
3. Introduce Feynman parameters and perform momentum integration of second loop.
4. Expand parameter integrand using Taylor series and perform parameter integrations.

B: Standard multi-loop approach followed by Mellin-Barnes expansion:

1. Introduce Feynman parameters and perform momentum integration of first loop.
2. Express result of first step in terms of one artificial propagator in second loop.

3. Introduce Feynman parameters and perform momentum integration of second loop.
4. Re-express parameter integrand using the Mellin-Barnes representation and perform parameter integrations.
5. Perform complex integrations leading to concatenated sums.

C: Mellin-Barnes splitting multi-loop approach:

1. Introduce Feynman parameters and perform momentum integration of first loop.
2. Using Mellin-Barnes representation, express result of first step in terms of several propagators in the second loop. These propagators may be identical or not to already existing ones.
3. Introduce Feynman parameters and perform momentum integration of second loop.
4. Perform complex integrations leading to concatenated sums.

All of these approaches offer some advantages. Approach A is the simplest in obtaining a general and systematic expression valid for any diagram, since it does not involve complex integrations.

Approach B shares many of the same steps as A, and so expansions can be easily compared. We found that whenever a specific expansion may be legally performed using Taylor series (Approach A), the related Mellin-Barnes expansion (Approach B) is also applicable and leads to an expression of the same exact form. The reverse, however, is not always true. There are cases when the Mellin-Barnes is applicable while the equivalent approach using Taylor expansion leads to an illegal (non convergent) expansion. The reason for this difference is the moment at which the condition of convergence must be imposed on both expansions. While for Taylor series it applies the moment the expansion is performed (before the parameter integration), for Mellin-Barnes it is only enforced when completing the complex integration contour (after the parameter integration), with the second case being less restrictive. In this sense, we may think of Approach B as a generalization of A.

Approach C has the advantage of not creating artificial propagators in the second loop integration, meaning they do not involve artificial masses or momenta dependent on Feynman parameters. This leads to simpler parameter integrations and consequently “cleaner” concatenated sums. The disadvantage is that since the method introduces complex integrations early in the procedure it is more difficult to obtain a general expression applicable to any diagram.

All these methods are interesting and deserve a thorough study and development of a general and systematic final expression in terms of concatenated sums. After such equation is found, it should be easy to understand how often the Z-Sum algorithms may be successfully applied leading to a result in term of multiple polylogarithms. Since it would not be possible to evaluate all of these in a timely manner, the first procedure was chosen because of its simplicity.

After deciding to pursue Approach A, we focused on vertex diagrams since they are neither trivial nor too complex, representing a good testing ground. We studied different ways of performing the Taylor expansion and how different choices at each step affect the final result, that is, the form of the concatenated sums. As discussed in chapter 3, we found there is a large number of decisions to be made, as shown in column 1 of table 6.1. Different choices made at each step may lead to summations with different forms, and while

Table 6.1: Possible choices when performing Taylor expansion of parameter integrand.

Possible Choice	Choice Made
Parametrization of integration	Brute force
Factorization of denominator	Maximum factorization, without creating fractional powers of integration variables
Number of terms in denominator factors	Minimum number possible
Ordering of denominator factors	Brute force
Specific sequence of expansions	Single denominator expansion followed by as many numerator expansions as necessary

two (legally expanded) series obtained from the same diagram are equivalent they are not always easily related. In order to compare and manipulate these expressions we developed a series of systematic procedures as listed in appendix C.

Further investigation showed that not all possible summations obtained were desirable, which lead to some restrictions, as shown in column 2 of table 6.1. For example, we found that expressing the integrand's denominator with more terms than the minimum necessary generally leads to unnecessarily complicated summations. Similarly, it is almost always better to factorize the denominator as much as possible. On the other hand, all choices of parametrization potentially lead to an interesting summation, and so we considered every possible option, with the same applying to the ordering of the denominator terms. The brute force approach is acceptable in these cases since the total number of variations is not very large. Finally, in order to obtain a simple general expression we chose to always perform the expansions in a specific way, starting with one denominator expansion followed by as many numerator expansions as necessary.

Following this recipe we were able to obtain a general expression involving concatenated sums applicable to any triangle loop diagram (Eq. 3.19), giving a better insight into the types of summations we would have to deal with. Studying this equation we found that the main difference between diagrams (and what would define whether the Z-Sum approach is applicable to a specific calculation) was in the argument of gamma functions, which potentially consisted of an infinitesimal, a constant and summation variables.

Infinitesimal. The infinitesimal, if present, is simply dealt with by performing an expansion which introduces Z and S-Sum.

Constant term. Previously the constant term could only be treated when a numerical value was applied. By obtaining a general expression for partial fractioning (Eq. 3.43) and applying it repeatedly, we were able to derive equations 3.47 and 3.48, which automatizes partial fractioning of gamma pairs. This step allowed to obtain Z-Sums even for generic (non numerical) values of the constant term (assuming the summation variables have the right structure). Related to the previous identities, we also obtained equations 3.49 and 3.50, which involve the mixing of arguments between gamma functions. This allows for obtaining results for an unknown integration from two other known sums.

Summation variables. When studying the summation variables content it became clear that, depending on its complexity, the existing Z-Sum algorithms could potentially be insufficient to reduce the full expression to multiple polylogarithms. In order to find out if this would happen in integrations that arise from actual Feynman diagrams, we considered every possible triangle diagram that could occur in any QFT, and in particular the Standard Model, by finding every possible combination of values for the six invariants (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2), while requiring at least one external leg to be off-shell, and found a total of 101 different triangles. Note that if we allow all external legs to be on-shell this number would be considerably larger.

In order to find out how many of these would fit the Z-Sum machinery we had two possible methods:

Forward Approach. In the Forward approach we simply apply all invariant combinations to the general expression, while considering every parametrization and denominator ordering, as described above, and verify whether any of the final summations match the Z-Sum algorithms.

Backwards Approach. In the backwards approach, we use all summations that match the Z-Sum algorithms, and follow the steps backwards into the integration that would lead to such a summation when Taylor expanded. We then compare these results with the integration for every triangle.

These two options are similar but give slightly different information. The Backwards Approach will lead to all possible integrations that may be systematically reduced using Taylor series and the Z-Sum algorithms, but it does not give any information on integrations that cannot, unlike the Forward Approach.

In chapter 4, when we used the Backwards Approach and verified the results against the 101 possible triangles with at least one off-shell leg, we found that some diagrams could be systematically reduced to multiple polylogarithms using present Z-Sum reduction algorithms while others could not. This result is important for both negative and positive reasons. First, it shows that further development must be done on the Z-Sum machinery before the method may be implemented as a computational algorithm applicable to any loop calculation. Second, it gives us a list of which building blocks may be used in multi-loop diagrams while still obtaining a systematically reducible expression.

Using the Forward approach, we were able to obtain more information on triangles in which the method could not be successfully applied. We found that while all diagrams may be expanded and integrated using Taylor expansion, the concatenated sums obtained do not match the existing Z-Sum algorithms, and so cannot be systematically reduced to multiple polylogarithms. However, it is generally believed that it should be possible to express loop integrations in terms of such functions, and in fact some of the diagrams that failed in this approach have been solved using other methods [39, 40]. By using the expressions described in appendix C, we were able to compare these results on the summation level hoping to find a systematic way of relating them, but found that the steps required change for every order of every diagram, and we could not find a generalization. In section 4.3 we list all missing steps required for a fully systematic reduction applicable to any loop integration.

Finally, having discovered in which situations the method can be successfully used, in chapter 5 we concluded this project by applying it to a problem of physical interest, namely the calculation of heavy flavor corrections to the renormalized scattering amplitude for deep inelastic scattering (DIS), which involved the calculation of two-loop heavy quark self-energy integrals both with and without operator insertions. By doing so, we were able to reproduce results presented in [17].

These calculations show that there are in fact cases of multi-loop diagrams involving massive particles where the method may be successfully applied. While the study of loop integrations is clearly far from being over, we believe this work sets a useful foundation for further investigation into applications of Z-Sum algorithms for calculations of loop integrals.

APPENDIX A

NOTATION AND CONVENTIONS

Throughout this work we defined some conventions for functions which we list here for for ease of access.

Boolean Step Function. Whenever we use the step function θ it will have a boolean argument as in:

$$\theta(x) = \begin{cases} 1 & \text{if } x = true \\ 0 & \text{if } x = false \end{cases} \tag{A.1}$$

Bounded Numbers and Summation Variables. Bounded numbers are defined as involving integers and variables from summations with a finite number of terms only. Furthermore, when we mention "summation variable" without further explanation it usually means a variable from a sum with an infinite number of terms.

Floor and Ceiling Function. The floor function $\lfloor x \rfloor$ evaluates as the largest integer smaller or equal to x. Similarly, the ceiling function $\lceil x \rceil$ equals the smallest integer larger or equal to x. Namely:

$$\begin{aligned} \lfloor x \rfloor &= n, \qquad n \in \mathbb{I}, \qquad 0 \leq x - n < 1 \\ \lceil x \rceil &= n, \qquad n \in \mathbb{I}, \qquad 0 \leq n - x < 1 \end{aligned} \tag{A.2}$$

SCALAR AND TENSOR INTEGRALS. In this work we will be using the following notation for tensor and scalar loop integrations:

$$\begin{aligned} A_0 &= \int \frac{d^D k}{(2\pi)^D} \frac{1}{P_1} , \\ B_{0,\mu,\mu\nu,\ldots} &= \int \frac{d^D k}{(2\pi)^D} \frac{1, k_\mu, k_\mu k_\nu, \ldots}{P_1 P_2} , \\ C_{0,\mu,\mu\nu,\ldots} &= \int \frac{d^D k}{(2\pi)^D} \frac{1, k_\mu, k_\mu k_\nu, \ldots}{P_1 P_2 P_3} , \\ D_{0,\mu,\mu\nu,\ldots} &= \int \frac{d^D k}{(2\pi)^D} \frac{1, k_\mu, k_\mu k_\nu, \ldots}{P_1 P_2 P_3 P_4} , \end{aligned} \tag{A.3}$$

where the indexes refer to the tensor structure of the integral (with 0 for scalars) and can be further generalized to include more propagators. The propagator P_i are given by:

$$P_i = ((k - a_i)^2 - m_i^2 + i\lambda) \quad , \tag{A.4}$$

where a_i are vectors related to the momenta entering the loop. While we could shift the integration variable to absorb one of the a_i in every integration we prefer to leave it in a more symmetric form. The propagators in equation A.4 include a infinitesimal complex term $i\lambda$ added to avoid poles whenever $(k - a_i)^2 = m_i^2$. We might sometimes omit this term but its presence should be implied. Integrals of type B_0, C_0, and D_0 are referred as bubbles, triangles and boxes, and their topologies are shown in fig. 2.2.

APPENDIX B

SPECIAL FUNCTIONS

In this appendix we list definitions and a few properties of some functions of interest. More information can be found in [34].

B.1 Gamma Function

The gamma function is defined by:

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t) dt \tag{B.1}$$

for $Re(z) > 0$. For positive integers it generalizes the factorial function with

$$\Gamma(n) = (n-1)! \, . \tag{B.2}$$

Some useful identities are given by:

$$\Gamma(z+1) = z\Gamma(z), \tag{B.3}$$

$$\Gamma(-z) = -\frac{\pi \csc(\pi z)}{\Gamma(z+1)} \ , \tag{B.4}$$

$$\Gamma(-n+a) = (-1)^n \frac{\Gamma(a)\Gamma(1-a)}{\Gamma(n+1-a)} \qquad \text{for integer n} \tag{B.5}$$

and

$$\Gamma(nz+b) = n^{nz+b-\frac{1}{2}} (2\pi)^{\frac{1-n}{2}} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{b+k}{n}\right), n \in \mathbb{N}^+, \tag{B.6}$$

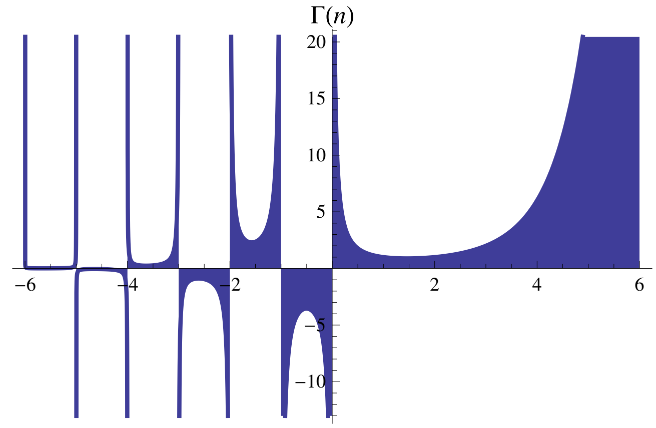


Figure B.1: Plot of $\Gamma(n)$ for real n from -6 to 6.

which generalizes

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) \ . \tag{B.7}$$

It is an analytical function of z over the whole complex plane except at zero and negative integers, where it has simple poles with residue:

$$res_z(\Gamma(z))(-k) = \frac{(-1)^k}{k!}, k \in \mathbb{N} \ . \tag{B.8}$$

Expansions around these poles can be found from the expansion around n=1 given by:

$$\Gamma(1+\epsilon) = \exp\left(-\gamma_E \epsilon + \sum_{i=2}^{\infty} \epsilon^i \frac{(-1)^i}{i} \sum_{j=1}^{\infty} \frac{1}{j^i}\right) , \tag{B.9}$$

where γ_E is Euler's constant

$$\gamma_E = \lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{1}{j} - \ln n\right) \approx 0.577, \tag{B.10}$$

or directly with the use of Z and S-Sums:

$$\begin{aligned} \Gamma(n+\epsilon) &= \theta(n>0)\ \Gamma(1+\epsilon)\ \Gamma(n) \sum_{i=0}^{n-1} \epsilon^i\ Z_i(n-1) \\ &+ \theta(n \le 0) \frac{\Gamma(1+\epsilon)\ (-1)^n}{\epsilon\ \Gamma(1-n)} \sum_{i=0}^{\infty} \epsilon^i\ S_i(-n) \end{aligned} \tag{B.11}$$

and

$$
\begin{aligned}
\frac{1}{\Gamma(n+\epsilon)} &= \theta(n>0)\ \frac{1}{\Gamma(1+\epsilon)\ \Gamma(n)} \sum_{i=0}^{\infty} (-\epsilon)^i\ S_i(n-1) \\
&+ \theta(n \le 0)\ \frac{\epsilon\ \Gamma(1-n)\ (-1)^n}{\Gamma(1+\epsilon)} \sum_{i=0}^{-n} (-\epsilon)^i\ Z_i(-n)
\end{aligned} \tag{B.12}
$$

The gamma function does not have branch points or branch cuts.

B.2 Beta Function

The beta function is defined by:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\ . \tag{B.13}$$

It can also be defined in integral form for $Re(x) > 0$ and $Re(y) > 0$ by:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{B.14}$$

or

$$B(x,y) = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}}dt\ . \tag{B.15}$$

Some useful identities include:

$$B(x,y) = \sum_{j=0}^{\infty} \binom{x-1}{j} \frac{(-1)^j}{j+y} \tag{B.16}$$

with y not a negative integer and $x > 0$, and

$$B(n+1,i+1)^{-1} = (n+i+1)\sum_{j=0}^{n} i^j Z_j(n) = \frac{n+i+1}{\Gamma(n+1)} \sum_{j=0}^{n} \kappa_{n-k}(1,\ldots,n)\ i^j \tag{B.17}$$

with $n \in \mathbb{N}$, i not a negative integer and κ is the Kappa function (section B.3).

The beta function $B(a,b)$ is an analytical function of a and b except for negative integers including zero, where it displays simple poles, and has no branch cuts or branch points.

B.3 Kappa Function

The kappa function $\kappa_s(t)$ is defined to be the sum of the products of every subset of t with s elements, for example:

$$\begin{aligned} \kappa_0(\ldots) &= 1 \\ \kappa_1(a,b,c) &= a+b+c \\ \kappa_2(a,b,c) &= (ab+ac+bc) \\ \kappa_3(a,b,c) &= abc \end{aligned} \tag{B.18}$$

It generalizes the binomial coefficient:

$$\binom{i}{j} = \kappa_j(\underbrace{1,\ldots,1}_{i}) \ , \tag{B.19}$$

and can be related to Z-Sums as given by:

$$\kappa_{n-j-1}(1,\ldots,n-1) = \Gamma(n) \ Z_j(n-1) \ . \tag{B.20}$$

B.4 Polylogarithms

Polylogarithms [33] are defined by:

$$Li_n(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^n} \tag{B.21}$$

for $|z| < 1$ and in the rest of the complex plane by analytical continuation. Alternative definitions are given by:

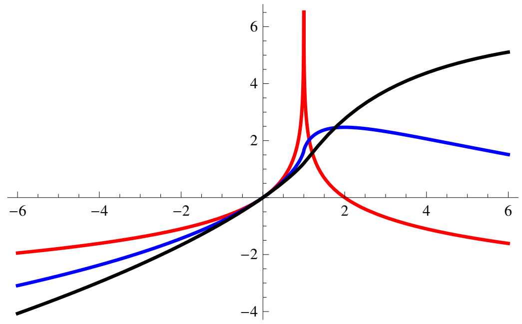
$$Li_n(z) = \frac{z}{\Gamma(n)} \int_1^{\infty} \frac{dt \ \log^{n-1}(t)}{t(t-z)} \text{ for } Re(n) > 0, \tag{B.22}$$

and also

$$Li_n(z) = \frac{1}{\Gamma(n-1)} \int_0^z \frac{dt \ \log^{n-1}(\frac{z}{t})}{1-t} \ . \tag{B.23}$$

A few useful identities are given by:

$$\begin{aligned} Li_n(z^2) &= 2^{n-1}\left(Li_n(z) + Li_n(-z)\right) \ , \\ Li_n(z) &= \int_0^z \frac{dt \ Li_{n-1}(t)}{t} \ . \end{aligned} \tag{B.24}$$



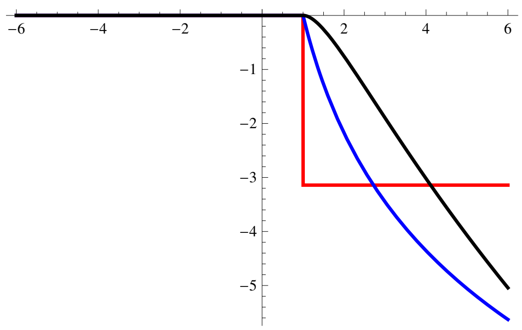


Figure B.2: Plot of the real part (left) and imaginary part (right) of $Li_1(x)$ (Red), $Li_2(x)$ (Blue) and $Li_3(x)$ (Black) for real x from -6 to 6.

In the limit $|z| \to \infty$ it behaves as:

$$Li_n(z) \propto -\frac{\log^n(-z)}{n!} \text{ for } n \in N^+ \tag{B.25}$$

For a fixed n, it does not show poles or essential singularities. It does have branch points at $z = 1$ and in the limit $z \to \infty$, and a branch cut in the real axis from 1 to ∞, where it is continuous from below:

$$\begin{aligned} \lim_{\varepsilon \to +0} Li_n(x - i\varepsilon) &= Li_n(x) \\ \lim_{\varepsilon \to -0} Li_n(x + i\varepsilon) &= Li_n(x) + \frac{2i\pi}{\Gamma(n)} \log^{n-1}(x) \ , \end{aligned} \tag{B.26}$$

for $x > 1$.

For fixed n we have:

$$\begin{aligned} Li_{-1}(z) &= \frac{z}{(z-1)^2} \ , \\ Li_0(z) &= \frac{z}{(1-z)} \ , \\ Li_1(z) &= -\log(1-z) \ . \end{aligned} \tag{B.27}$$

For $n = 2$ it is called dilogarithm and for $n = 3$ trilogarithm. The dilogarithm has a large number of identities, some of which are given by:

$$\begin{aligned} Li_2(z) &= -Li_2(1-z) - \log(1-z)\log(z) + \frac{\pi^2}{6} \ , \\ Li_2(z) &= -Li_2\left(\frac{1}{z}\right) - \frac{1}{2}\log^2(-z) - i\pi\left(\sqrt{\frac{z-1}{z}}\sqrt{\frac{z}{z-1}} - 1\right)\log(z) - \frac{\pi^2}{6} \ . \end{aligned} \tag{B.28}$$

Logarithms and dilogarithms are important as physical one-loop integrations can be expressed just in terms of these functions.

B.5 Inverse Trigonometric and Hyperbolic Functions

It is not uncommon to find integration results expressed in terms of the inverse of trigonometric and hyperbolic functions. Here we list the definitions in terms of polylogarithms:

Inverse Cosine.

$$ArcCos(z) = \frac{\pi}{2} + i \log\left(iz + \sqrt{1-z^2}\right) \tag{B.29}$$

Inverse Sine.

$$ArcSin(z) = -i \log\left(iz + \sqrt{1-z^2}\right) \tag{B.30}$$

Inverse Tangent.

$$ArcTan(z) = \frac{i}{2}\left(\log(1-iz) - \log(1+iz)\right) \tag{B.31}$$

Inverse Cotangent.

$$ArcCot(z) = \frac{i}{2}\left(\log\left(1 - \frac{i}{z}\right) - \log\left(1 + \frac{i}{z}\right)\right) \ , \ z \neq 0 \tag{B.32}$$

Inverse Hyperbolic Cosine.

$$ArcCosh(z) = \log(z + \sqrt{z+1}\sqrt{z-1}) \tag{B.33}$$

Inverse Hyperbolic Sine.

$$ArcSinh(z) = \log(z + \sqrt{z^2+1}) \tag{B.34}$$

Inverse Hyperbolic Tangent.

$$ArcTanh(z) = \frac{1}{2}\left(\log(1+z) - \log(1-z)\right) \tag{B.35}$$

Inverse Hyperbolic Cotangent.

$$ArcCoth(z) = \frac{1}{2}\left(\log\left(1 + \frac{1}{z}\right) - \log\left(1 - \frac{1}{z}\right)\right) \ , \ z \neq 0 \tag{B.36}$$

APPENDIX C

SUMMATION SPLITTING AND REORDERING

When dealing with concatenated sums, it is useful and sometimes even necessary to shift the summation variable and its limits or reorder sums, for example to compare two different functions by expressing them in the same base of monomials. In this appendix we list some useful identities.

C.1 Basic Identities with a Single Sum

We start with very basic ones involving one sum only:

$$\sum_{i=0}^{n} f(i) = \sum_{i=0}^{n} f(n-i) \tag{C.1}$$

$$\sum_{j=0}^{n+i} f(j) = \sum_{j=0}^{i} f(j) + \sum_{j=0}^{n} f(j+i) - f(i) \tag{C.2}$$

$$\sum_{j=0}^{n-i} f(j) = \sum_{j=0}^{n} f(j) + \sum_{j=0}^{i} f(j+n-i) - f(n-i) \tag{C.3}$$

In equation C.3 we added and subtracted terms originally outside the range of the original summation, which might lead to badly defined terms (for example involving $\Gamma(-1)$). These will eventually cancel out but a regularizer should be used so that intermediate steps are well defined.

It is possible to express a sum in terms of other summations with multiplied upper limit.

A trivial example comes from splitting into odd and even sums:

$$\sum_{i=0}^{n} f(i) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} f(2i) \; + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} f(2i+1) \;, \tag{C.4}$$

with the inverted identity given by:

$$\sum_{i=1}^{n} f(i) = \frac{1}{2} \left(\sum_{i=1}^{2n} f(i/2) + \sum_{i=1}^{2n} (-1)^i \; f(i/2) \right) \;, \tag{C.5}$$

where odd terms will simply cancel out.

These equations can be generalized to:

$$\sum_{i=0}^{n} f(i) = \sum_{p=0}^{q-1} \sum_{i=0}^{\lfloor \frac{n-p}{q} \rfloor} f(qi+p) \tag{C.6}$$

and

$$\sum_{i=1}^{n} f(i) = \frac{1}{q} \sum_{p=0}^{q-1} \sum_{i=1}^{qn} (r_q^p)^i f(i/q) \;, \tag{C.7}$$

where the coefficient r_q^p is called root of unity and is defined by:

$$r_q^p = \exp\left(\frac{2\pi i p}{q}\right) \tag{C.8}$$

with the properties:

$$\left(r_q^p\right)^{j+q} = \left(r_q^p\right)^j \tag{C.9}$$

and

$$\sum_{p=0}^{q-1} \left(r_q^p\right)^m = \begin{cases} q & \text{if } m \bmod q = 0, \\ 0 & \text{if } m \bmod q \neq 0. \end{cases} \tag{C.10}$$

C.2 Splitting and Shifting Concatenated Sums

C.2.1 Splitting Identities

Z-Sums multiplication is a result of sum splitting:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} f(i,j) = \sum_{i=1}^{n} \sum_{j=1}^{i-1} f(i,j) + \sum_{j=1}^{n} \sum_{i=1}^{j-1} f(i,j) + \sum_{i=1}^{n} f(i,i) \;. \tag{C.11}$$

A minor limit modification (useful for sums originating from denominator Taylor series) gives:

$$\sum_{i=0}^{n}\sum_{j=0}^{n} f(i,j) = \sum_{i=0}^{n}\sum_{j=0}^{i} f(i,j) + \sum_{j=0}^{n}\sum_{i=0}^{j} f(i,j) - \sum_{i=0}^{n} f(i,i)\ . \tag{C.12}$$

Concatenated sums involving floor function in the limit may be split into odd and even parts according to:

$$\sum_{i=0}^{a}\ \sum_{j=0}^{\lfloor\frac{i+b}{2}\rfloor+c} f(i,j) = \sum_{i=0}^{\lfloor\frac{a}{2}\rfloor}\ \sum_{j=0}^{i+\lfloor\frac{b}{2}\rfloor+c} f(2i,j) + \sum_{i=0}^{\lfloor\frac{a-1}{2}\rfloor}\ \sum_{j=0}^{i+\lfloor\frac{b+1}{2}\rfloor+c} f(2i+1,j) \tag{C.13}$$

where a, b and c are integer numbers.

C.2.2 Shifting Identities

Other identities obtained from sum reordering are:

$$\sum_{i=0}^{n}\sum_{j=0}^{i} f(i,j) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} f(i+j,j) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} f(i+j,i) \tag{C.14}$$

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i} f(i,j) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} f(j,i) = \sum_{i=0}^{n}\sum_{j=0}^{i} f(i-j,j) \tag{C.15}$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i} f(i,j) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f(i+j,j) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f(i+j,i) \tag{C.16}$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f(i,j) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f(j,i) = \sum_{i=0}^{\infty}\sum_{j=0}^{i} f(i-j,j)\ . \tag{C.17}$$

Equation C.15 is simply equation C.14 rewritten with $f(x,y) \to f(x-y,y)$, while the last two are obtained from the first ones after taking the limit $n \to \infty$.

Similarly to equations C.16 and C.17, further shifting leads to:

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i} f(i,j) = \sum_{i=0}^{\infty}\sum_{j=0}^{\lfloor\frac{i}{2}\rfloor} f(i-j,j) = \sum_{i=0}^{\infty}\sum_{j=0}^{\lfloor\frac{i}{m}\rfloor} f(i-(m-1)j,j) \tag{C.18}$$

and

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i} f(i,j) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i} f\left(i+\lfloor\frac{j}{2}\rfloor,j\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{m\,i} f\left(i-(j-1)-\lfloor\frac{(j-1)}{m}\rfloor,j\right)\ . \tag{C.19}$$

All infinite summations being reordered must be absolutely convergent for these identities to be valid.

C.3 Reordering of Concatenated Sums

Now we present the algorithm for reordering two sums where the inner sum limits depend linearly on the outer sum's variable. Since this identity does not change the arguments within the sum we omit $f(i,j)$. Expressions involving more than two sums can be reordered iteratively two sums at a time.

The initial sum is given by:

$$\sum_{i=a}^{b} \sum_{j=ci+d}^{ei+f} \tag{C.20}$$

where a, b, c, d, e, f are real numbers. We can express the limits of this summation as:

$$\begin{aligned} a \leq \quad & i \quad \leq b \\ ci + d \leq \quad & j \quad \leq ei + f \end{aligned} \tag{C.21}$$

where i and j are integers. We would like to invert the condition C.21 as blocks of the form:

$$\begin{aligned} f_1(a,b,c,d,e,f) \leq \quad & j \quad \leq f_2(a,b,c,d,e,f) \\ f_3(j,a,b,c,d,e,f) \leq \quad & i \quad \leq f_4(j,a,b,c,d,e,f) \; . \end{aligned} \tag{C.22}$$

which will be rewritten as

$$\sum_{j=f_1(a,b,c,d,e,f)}^{f_2(a,b,c,d,e,f)} \; \sum_{i=f_3(j,a,b,c,d,e,f)}^{f_4(j,a,b,c,d,e,f)} \; . \tag{C.23}$$

In case we obtain a block of the form C.22 involving $<$ or $>$ instead of $\leq$ or $\geq$ we make the substitutions $\alpha < i \to \lfloor \alpha + 1 \rfloor \leq i$ and $i < \alpha \to i \leq \lceil \alpha - 1 \rceil$, which is correct whether α is an integer or not. In all other cases whenever the summation limits are not integers the ceiling function for the lower limit and the floor function for the upper limit are implied.

We separate the solution in eight cases depending on the values of c and e. All cases will have constant blocks present for all values of a, b, c, d, e and f and also conditional blocks with a boolean step function.

Case 1. $e = 0 \wedge c > 0$

$$\sum_{i=a}^{b} \sum_{j=ci+d}^{f} = \sum_{j=\lfloor cb+d+1 \rfloor}^{f} \sum_{i=a}^{b} + \theta(cb + d \leq f) \sum_{j=ca+d}^{cb+d} \sum_{i=a}^{\frac{j-d}{c}} + \theta(cb + d > f) \sum_{j=ca+d}^{f} \sum_{i=a}^{\frac{j-d}{c}} \tag{C.24}$$

Case 2. $e = 0 \wedge c < 0$

$$\sum_{i=a}^{b} \sum_{j=ci+d}^{f} = \sum_{j=ca+d}^{f} \sum_{i=a}^{b} + \theta(ca+d \leq f) \sum_{j=cb+d}^{\lceil ca+d-1 \rceil} \sum_{i=\frac{j-d}{c}}^{b} + \theta(ca+d > f) \sum_{j=cb+d}^{f} \sum_{i=\frac{j-d}{c}}^{b} \tag{C.25}$$

Case 3. $c = 0 \wedge e > 0$

$$\sum_{i=a}^{b} \sum_{j=d}^{ei+f} = \sum_{j=d}^{\lceil ea+f-1 \rceil} \sum_{i=a}^{b} + \theta(ea+f \leq d) \sum_{j=d}^{eb+f} \sum_{i=\frac{j-f}{e}}^{b} + \theta(ea+f > d) \sum_{j=ea+f}^{eb+f} \sum_{i=\frac{j-f}{e}}^{b} \tag{C.26}$$

Case 4. $c = 0 \wedge e < 0$

$$\sum_{i=a}^{b} \sum_{j=d}^{ei+f} = \sum_{j=d}^{\lceil eb+f-1 \rceil} \sum_{i=a}^{b} + \theta(eb+f \leq d) \sum_{j=d}^{ea+f} \sum_{i=a}^{\frac{j-f}{e}} + \theta(eb+f > d) \sum_{j=eb+f}^{ea+f} \sum_{i=a}^{\frac{j-f}{e}} \tag{C.27}$$

Case 5. $c > 0 \wedge e > 0$

$$\begin{aligned} \sum_{i=a}^{b} \sum_{j=ci+d}^{ei+f} \quad = \quad & \sum_{j=\lfloor ea+f+1 \rfloor}^{cb+d} \sum_{i=\frac{j-f}{e}}^{\frac{j-d}{c}} + \sum_{j=\lfloor cb+d+1 \rfloor}^{ea+f} \sum_{i=a}^{b} \\ + \quad & \theta(cb+d \leq ea+f) \left(\sum_{j=ca+d}^{cb+d} \sum_{i=a}^{\frac{j-d}{c}} + \sum_{j=\lfloor ea+f+1 \rfloor}^{eb+f} \sum_{i=\frac{j-f}{e}}^{b} \right) \\ + \quad & \theta(cb+d > ea+f) \left(\sum_{j=ca+d}^{ea+f} \sum_{i=a}^{\frac{j-d}{c}} + \sum_{j=\lfloor cb+d+1 \rfloor}^{eb+f} \sum_{i=\frac{j-f}{e}}^{b} \right) \end{aligned} \tag{C.28}$$

Case 6. $c > 0 \wedge e < 0$

$$\begin{aligned} \sum_{i=a}^{b} \sum_{j=ci+d}^{ei+f} \quad = \quad & \sum_{j=eb+f}^{\frac{cf-ed}{c-e}} \sum_{i=a}^{\frac{j-d}{c}} + \sum_{j=\lfloor \frac{cf-ed}{c-e}+1 \rfloor}^{cb+d} \sum_{i=a}^{\frac{j-f}{e}} + \sum_{j=\lfloor cb+d+1 \rfloor}^{\frac{cf-ed}{c-e}} \sum_{i=a}^{b} + \sum_{j=\lfloor \frac{cf-ed}{c-e}+1 \rfloor}^{\lceil eb+f-1 \rceil} \sum_{i=a}^{b} \\ + \quad & \theta(cb+d \geq eb+f) \left(\sum_{j=ca+d}^{\lceil eb+f-1 \rceil} \sum_{i=a}^{\frac{j-d}{c}} + \sum_{j=\lfloor cb+d+1 \rfloor}^{ea+f} \sum_{i=a}^{\frac{j-f}{e}} \right) \\ + \quad & \theta(cb+d < eb+f) \left(\sum_{j=ca+d}^{cb+d} \sum_{i=a}^{\frac{j-d}{c}} + \sum_{j=eb+f}^{ea+f} \sum_{i=a}^{\frac{j-f}{e}} \right) \end{aligned} \tag{C.29}$$

Case 7. $c < 0 \wedge e > 0$

$$\begin{aligned}\sum_{i=a}^{b}\sum_{j=ci+d}^{ei+f} \quad = \quad & \sum_{j=\frac{cf-ed}{c-e}}^{ea+f}\sum_{i=a}^{b} + \sum_{j=ca+d}^{\lceil\frac{cf-ed}{c-e}-1\rceil}\sum_{i=a}^{b} + \sum_{j=\frac{cf-ed}{c-e}}^{\lceil ca+d-1\rceil}\sum_{i=\frac{j-f}{e}}^{b} + \sum_{j=\lfloor ea+f+1\rfloor}^{\lceil\frac{cf-ed}{c-e}-1\rceil}\sum_{i=\frac{j-d}{c}}^{b} \\ + \quad & \theta(ca+d \leq ea+f)\left(\sum_{j=\lfloor ea+f+1\rfloor}^{eb+f}\sum_{i=\frac{j-f}{e}}^{b} + \sum_{j=cb+d}^{\lceil ca+d-1\rceil}\sum_{i=\frac{j-d}{c}}^{b}\right) \\ + \quad & \theta(ca+d > ea+f)\left(\sum_{j=ca+d}^{eb+f}\sum_{i=\frac{j-f}{e}}^{b} + \sum_{j=cb+d}^{ea+f}\sum_{i=\frac{j-d}{c}}^{b}\right)\end{aligned} \tag{C.30}$$

Case 8. $c < 0 \wedge e < 0$

$$\begin{aligned}\sum_{i=a}^{b}\sum_{j=ci+d}^{ei+f} \quad = \quad & \sum_{j=eb+f}^{\lceil ca+d-1\rceil}\sum_{i=\frac{j-d}{c}}^{\frac{j-f}{e}} + \sum_{j=ca+d}^{\lceil eb+f-1\rceil}\sum_{i=a}^{b} \\ + \quad & \theta(ca+d \geq eb+f)\left(\sum_{j=ca+d}^{ea+f}\sum_{i=a}^{\frac{j-f}{e}} + \sum_{j=cb+d}^{\lceil eb+f-1\rceil}\sum_{i=\frac{j-d}{c}}^{b}\right) \\ + \quad & \theta(ca+d < eb+f)\left(\sum_{j=eb+f}^{ea+f}\sum_{i=a}^{\frac{j-f}{e}} + \sum_{j=cb+d}^{\lceil ca+d-1\rceil}\sum_{i=\frac{j-d}{c}}^{b}\right)\end{aligned} \tag{C.31}$$

BIBLIOGRAPHY

[1] P. A. Rottmann and L. Reina. In preparation. 2011.

[2] K. Nakamura et al. (Particle Data Group). Review of particle physics. *J. Phys*, G 37(075021), 2010.

[3] Michael E. Peskin and Dan V. Schroeder. *An Introduction to quantum field theory.* Frontiers in Physics. HarperCollins Publishers, 1995.

[4] Steven Weinberg. *The Quantum theory of fields*, volume 1 - 3. Cambridge University Press, 2005.

[5] Steven Weinberg. A model of leptons. *Phys. Rev. Lett.*, 19(21):1264–1266, 1967.

[6] Sheldon Lee Glashow. Partial symmetries of weak interactions. *Nucl. Phys.*, 22:579–588, 1961.

[7] Abdus Salam. *Proceedings of the 8th Nobel Symposium.* Almqvist and Wiksell, 1968.

[8] Guido Altarelli. Particle Physics in the LHC Era and beyond. 2010, 1002.4957.

[9] Stefan Weinzierl. Algebraic algorithms in perturbative calculations. 2003, hep-th/0305260.

[10] Sven Moch, Peter Uwer, and Stefan Weinzierl. Nested sums, expansion of transcendental functions and multi-scale multi-loop integrals. *J. Math. Phys.*, 43:3363–3386, 2002, hep-ph/0110083.

[11] Sven Moch, Peter Uwer, and Stefan Weinzierl. QCD two-loop amplitudes for $e^+e^- \to$ 3jets: The fermionic contribution. *Nucl. Phys. Proc. Suppl.*, 116:8–12, 2003, hep-ph/0211156.

[12] Sven Moch, Peter Uwer, and Stefan Weinzierl. Scattering amplitudes for $e^+e^- \to$ 3jets at next-to-next- to-leading order QCD. *Nucl. Phys. Proc. Suppl.*, 121:37–41, 2003, hep-ph/0210009.

[13] Isabella Bierenbaum. *The Massless Two-loop Two-point Function and Zeta Functions in Counterterms of Feynman Diagrams.* PhD thesis, Universität Mainz, February 2005.

[14] Isabella Bierenbaum and Stefan Weinzierl. The massless two-loop two-point function. *Eur. Phys. J.*, C32:67–78, 2003, hep-ph/0308311.

[15] I. Bierenbaum, J. Blumlein, and S. Klein. Calculation of massive 2-loop operator matrix elements with outer gluon lines. *Phys. Lett.*, B648:195–200, 2007, hep-ph/0702265.

[16] I. Bierenbaum, J. Blumlein, and S. Klein. Two-Loop Massive Operator Matrix Elements for Polarized and Unpolarized Deep-Inelastic Scattering. 2007, 0706.2738.

[17] Isabella Bierenbaum, Johannes Blumlein, and Sebastian Klein. Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals. *Nucl. Phys. Proc. Suppl.*, 160:85–90, 2006, hep-ph/0607300.

[18] Isabella Bierenbaum, Johannes Blumlein, Sebastian Klein, and Carsten Schneider. Two–Loop Massive Operator Matrix Elements for Unpolarized Heavy Flavor Production to $O(\epsilon)$. *Nucl. Phys.*, B803:1–41, 2008, 0803.0273.

[19] J. Ablinger et al. Heavy Flavor DIS Wilson coefficients in the asymptotic regime. *Nucl. Phys. Proc. Suppl.*, 205-206:242–249, 2010, 1007.0375.

[20] Stefan Weinzierl. Hopf algebra structures in particle physics. *Eur. Phys. J.*, C33:s871–s875, 2004, hep-th/0310124.

[21] Stefan Weinzierl. The art of computing loop integrals. 2006, hep-ph/0604068.

[22] Ansgar Denner and S. Dittmaier. Reduction of one-loop tensor 5-point integrals. *Nucl. Phys.*, B658:175–202, 2003, hep-ph/0212259.

[23] Ansgar Denner and S. Dittmaier. Reduction schemes for one-loop tensor integrals. *Nucl. Phys.*, B734:62–115, 2006, hep-ph/0509141.

[24] J. Fleischer and T. Riemann. A complete algebraic reduction of one-loop tensor Feynman integrals. 2010, 1009.4436.

[25] Vladimir A. Smirnov. *Evaluating Feynman Integrals.* Springer, 2004.

[26] M. Czakon. Automatized analytic continuation of Mellin-Barnes integrals. *Comput. Phys. Commun.*, 175:559–571, 2006, hep-ph/0511200.

[27] Stefan Weinzierl. Expansion around half-integer values, binomial sums and inverse binomial sums. *J. Math. Phys.*, 45:2656–2673, 2004, hep-ph/0402131.

[28] Leonhard Euler. Meditationes circa singulare serierum genus. *Novi Comm. Acad. Sci. Petropol.*, 20:140–186, 1775.

[29] Don Zagier. Values of zeta functions and their applications. In *First European Congress of Mathematics*, volume 2, pages 497–512, 1992.

[30] Alexander Goncharov. Multiple polylogarithms, cyclotomy and modular complexes. *Mathematical Research Letters*, 5(3):497–516, 1998.

[31] E. Remiddi and J. A. M. Vermaseren. Harmonic polylogarithms. *Int. J. Mod. Phys.*, A15:725–754, 2000, hep-ph/9905237.

[32] N. Nielsen. Der Eulersche Dilogarithmus und seine Verallgemeinerungen. *Nova Acta Leopoldina (Halle)*, (90):123, 1909.

[33] Leonard Lewin. *Polylogarithms and Associated Functions*. Elsevier Science Ltd, 1981.

[34] The Wolfram functions site. `http://functions.wolfram.com/`.

[35] G. J. van Oldenborgh and J. A. M. Vermaseren. The formula manipulation program Form. Prepared for International Workshop on Software Engineering, Artificial Intelligence and Expert Systems for High-energy and Nuclear Physics, Lyon, France, 19-24 Mar 1990.

[36] J. A. M. Vermaseren. New features of FORM. 2000, math-ph/0010025.

[37] S. Moch and P. Uwer. XSummer: Transcendental functions and symbolic summation in Form. *Comput. Phys. Commun.*, 174:759–770, 2006, math-ph/0508008.

[38] M. Buza, Y. Matiounine, J. Smith, R. Migneron, and W. L. van Neerven. Heavy quark coefficient functions at asymptotic values $Q\ 2 \gg m\ 2$. *Nucl. Phys.*, B472:611–658, 1996, hep-ph/9601302.

[39] R. Keith Ellis and Giulia Zanderighi. Scalar one-loop integrals for QCD. *JHEP*, 02:002, 2008, 0712.1851.

[40] Carlo Oleari. Next-to-leading-order corrections to the production of heavy-flavour jets in e+ e- collisions. 1997, hep-ph/9802431.

[41] A. Denner. Techniques for the calculation of electroweak radiative corrections at the one-loop level and results for w-physics at lep 200. *Fortschritte der Physik/Progress of Physics*, 41(4):307–420, 1993.

[42] L. Reina, S. Dawson, and D. Wackeroth. QCD corrections to associated t anti-t h production at the Tevatron. *Phys. Rev.*, D65:053017, 2002, hep-ph/0109066.

[43] W. Beenakker et al. NLO QCD corrections to t anti-t H production in hadron collisions. ((U)). *Nucl. Phys.*, B653:151–203, 2003, hep-ph/0211352.

[44] S. Dawson, L. H. Orr, L. Reina, and D. Wackeroth. Associated top quark Higgs boson production at the LHC. *Phys. Rev.*, D67:071503, 2003, hep-ph/0211438.

[45] S. Dawson, C. Jackson, L. H. Orr, L. Reina, and D. Wackeroth. Associated Higgs production with top quarks at the Large Hadron Collider: NLO QCD corrections. *Phys. Rev.*, D68:034022, 2003, hep-ph/0305087.

[46] Stefan Dittmaier, 1 Michael Kramer, and Michael Spira. Higgs radiation off bottom quarks at the Tevatron and the LHC. *Phys. Rev.*, D70:074010, 2004, hep-ph/0309204.

[47] S. Dawson, C. B. Jackson, L. Reina, and D. Wackeroth. Higgs boson production with one bottom quark jet at hadron colliders. *Phys. Rev. Lett.*, 94:031802, 2005, hep-ph/0408077.

[48] S. Dawson, C. B. Jackson, L. Reina, and D. Wackeroth. Exclusive Higgs boson production with bottom quarks at hadron colliders. *Phys. Rev.*, D69:074027, 2004, hep-ph/0311067.

[49] F. Febres Cordero, L. Reina, and D. Wackeroth. NLO QCD corrections to W boson production with a massive b-quark jet pair at the Tevatron p anti-p collider. *Phys. Rev.*, D74:034007, 2006, hep-ph/0606102.

[50] F. Febres Cordero, Laura Reina, and Doreen Wackeroth. NLO QCD corrections to W-b-bbar and Z-b-bbar production. *PoS*, RADCOR2007:012, 2007, 0801.2374.

[51] Christian Bogner and Stefan Weinzierl. Feynman graph polynomials. 2010, 1002.3458.

[52] Stefan Weinzierl. Feynman integrals and multiple polylogarithms. 2007, 0705.0900.

[53] J. Alcaraz. Precision Electroweak Measurements and Constraints on the Standard Model. 2009, 0911.2604.

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- Second Annual Dirac Lectures - Cosmology: From Inflation to the Cosmic Microwave Background, Florida State University, March 2008.
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- XII Jorge Andre Swieca Summer School - Particles and Fields, Brazilian Society of Physics, Campos do Jordão, Brazil, January 2003.
- University Extension School - Topics in Theoretical Physics, Institute of Theoretical Physics (IFT), São Paulo, Brazil, July 2002.
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