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Universal Mappings and the Metric Geometry of Functional Data

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UNIVERSAL MAPPINGS

AND THE METRIC GEOMETRY OF FUNCTIONAL DATA

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This thesis is dedicated to my mother, Ehteram Zarei, my late father, Mahmoud Anbouhi, and my siblings. Without their support and encouragement, I would never have been able to complete my graduate studies.

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ABSTRACT

We study functional data defined on metric spaces or metric measure spaces, modeled as a 1-Lipschitz map between Polish spaces with the domain possibly equipped with a probability measure. We refer to such an object as a functional space.

If the domain of the 1-Lipschitz map is a metric space, we refer to the functional space as a functional metric space. We construct a unique, up to isometry, Urysohn (i.e., universal and homogeneous) functional metric space for the class of functional metric spaces. We prove a characterization for the Urysohn functional space and show how it relates to the notion of Gromov-Hausdorff distance between functional metric spaces.

A functional space with a metric measure domain is called functional metric measure space (fmm-space). We introduce a functional analogue of the Gromov's Box distance, Gromov-Prokhorov distance, and Gromov-Wasserstein distance and investigate their properties. Adapting the notion of distance matrices to functional data, we formulate discrete models and obtain an empirical estimation result that provides theoretical assurance that these discrete models can be used reliably in functional data analysis. We also prove a functional analogue of Gromov's Reconstruction Theorem.

CHAPTER 1

INTRODUCTION

Modeling structural data as geometric objects has been a subject of extensive study using tools from areas such as metric geometry and optimal transport. [28], [27], [2], [35], [8], [1]. For instance, one can express a point cloud (a set of data points in space representing a 3D shape) as a compact metric space and study the class of compact metric spaces with the *Gromov-Hausdorff* distance to compare the degree of similarity between point clouds. In [28], the authors present a theoretical and computational geometric framework for comparing manifolds given by point clouds. In many applications, data points are assigned different weights; these weights can describe, for example, the degree to which a given measurement is relevant [27]. To account for this information, one can model the data as a metric space with a probability measure on it (metric measure spaces or mm-spaces). The following figure compares a metric space with an mm-space.

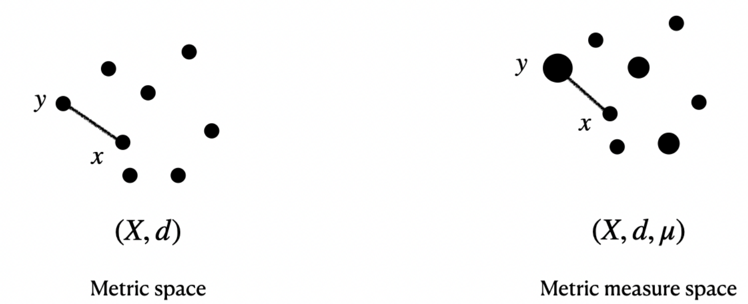


Figure 1.1: Left: A metric space. Right: The same metric space with a probability measure on it.

Gromov [15] studied the class of mm-spaces in detail. He introduced a notion of convergence for mm-spaces based on pairwise distances between random points in the space. This notion is expressible in terms of several distance functions, namely *the Gromov's box* distance, *the Gromov-Prohorov* distance, and *the Gromov–Wasserstein* distance. Informally, two mm-spaces are close to each other in the Gromov-Prohorov (and the Gromov's box) distance if they are almost everywhere similar, but this may disregard subspaces of "small" probability. On the other hand, the Gromov–Wasserstein distance, adopting ideas from optimal transport theory, measures the minimum *cost* required to transform one distribution into another.

Given a collection of metric spaces $\mathcal{C}$, a metric space U is called universal for $\mathcal{C}$ if it includes an isometric image of every element of $\mathcal{C}$. U is said to be Urysohn if, in addition, it is *homogeneous*. That means any isometry between finite subsets of U extends to an isometry of U. It is well-known that the class of Polish spaces (i.e., complete and separable metric spaces) admits a Polish Urysohn space. Universal spaces provide a framework to study the theoretical aspects of the class for which they are universal. For instance, the collection of compact subsets of the Urysohn space with the Hausdorff distance, modulo the action of the isometry group of the Urysohn space, is isometric to the space of the isometry classes of compact metric spaces with the Gromov-Hausdorff distance [15]. In [38], it is shown that the space of compact ultrametric spaces with the *Gromov-Hausdorff ultrametric* distance is Urysohn. As another example, in [13], authors identify Polish spaces with closed subsets of the Urysohn universal space to study the classification of Polish spaces.

1.1 Contribution

This work studies the class of data that are in form of functions (functional data) defined on the geometric domains. For instance, f could represent the Functional Magnetic Resonance Imaging (fMRI); or the labels defined on the nodes of a graph, where the underlying metric structure is given by the shortest path distance. As another example, f could be the phenotypic attributes associated with the nodes of a phylogenetic tree with the metric given by cophenetic distance.

To that end, we study the class of 1-Lipschitz maps defined between Polish spaces with the domain sometimes equipped with a probability measure. If the domain of the 1-Lipschitz map is a metric space, we refer to such an object as *functional metric space*. Formally, a functional metric space over a base space B is a triple $\mathcal{X} = (X, B, \pi_)$ where X and B are Polish spaces and $\pi_X : X \to B$ is a 1-Lipschitz map. We construct a Urysohn functional space $\mathcal{U}_B$ for the class of functional spaces over a fixed base B. We also show that the following properties characterize $\mathcal{U}_B$

- Universality: for every functional space $\mathcal{X}$ there is an isometric embedding $\phi : X \to B$ such that $\pi_X = \pi_U \circ \phi$.
- Homogeneity: every isometry between finite subsets of $\mathcal{U}_B$ extends to an isometry of $\mathcal{U}_B$.

Following Gromov, we prove that the collection of compact subspaces of $\mathcal{U}_B$ with the Hausdorff distance, modulo the action of the isometries $\mathcal{U}_B$ is isometric to the space of the isometry classes of compact functional spaces with Gromov-Hausdorff distance.

If the domain of the 1-Lipschitz map is a metric measure space (mm-space), we refer to it as *functional metric measure space* (fmm-space). Precisely, an fmm-space is a quadruple $\mathcal{X} =$

(X, B, π_X, μ_X) where (X, B, π_X) is a functional metric space and μ_X is a probability measure on X. The following figure compares a functional metric space with an fmm-space.

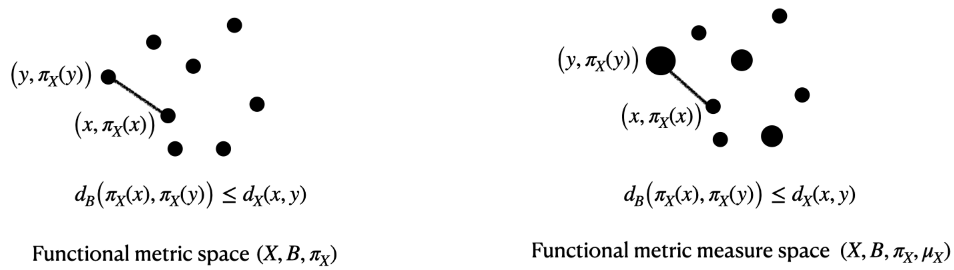


Figure 1.2: Left: A functional metric space viewed as a labeled metric space. Right: The same functional metric space with a probability measure on X.

We develop a functional analogues of the Gromov box distance $\square_a$, Gromov-Prokhorov distance $d_{GP,a}$, and Gromov-Wasserstein distance $d_{\mathrm{GW},p}$, that have been studied extensively for *mm*-spaces [15],[26], [37],[23]. In Propositions 3.2.4 and 3.2.8, we establish some basic relationships between these metrics; namely,

$$\square_a(\mathcal{X}, \mathcal{X}') = 2d_{GP,2a}(\mathcal{X}, \mathcal{X}') \quad \text{and} \quad d_{\mathrm{GW},p}(\mathcal{X}, \mathcal{Y}) \leq \frac{1}{2}\left(\square_a(\mathcal{X}, \mathcal{Y}) + a^{1/p} M^{(p+1)/p}\right), \tag{1.1}$$

Moreover, we show that the Gromov-Wasserstein distance $d_{\mathrm{GW},\infty}$ is realized by uniformly distributed sequences, where a sequence (x_i) in X is uniformly distributed if the empirical measures $\sum_{i=1}^n \delta_{x_i}/n$ converge weakly to μ. More formally,

Theorem 3.2.9. *Let $\mathcal{X}$ and $\mathcal{Y}$ be fmm-spaces, and U_X and U_Y denote the set of uniformly distributed sequences in X and Y, respectively.*

$$d_{\mathrm{GW},\infty}(\mathcal{X}, \mathcal{Y}) = \inf_{(x_n)\in U_X, (y_n)\in U_Y} \max\left\{\frac{1}{2}\sup_{i,j} m_{X,Y}(x_i, y_i, x_j, y_j),\ \sup_i d_{X,Y}(x_i, y_i)\right\}.$$

Furthermore, the infimum in the right-hand side is realized.

We also prove a functional version of Gromov's Reconstruction Theorem [15] which states that the functional matrix distribution $\mathcal{D}_\mathcal{X}$ gives a faithful representation of $\mathcal{X}$.

Theorem 4.0.5 (Fmm-Reconstruction Theorem)**.** *Let $\mathcal{X} = (X, B, \pi, \mu)$ and $\mathcal{X}' = (X', B', \pi', \mu')$ be mm-fields. Then,*

$$\mathcal{X} \simeq \mathcal{X}' \iff \mathcal{D}_\mathcal{X} = \mathcal{D}_{\mathcal{X}'}.$$

In addition, we study the Wasserstein distance and Prokhorov distance between augmented matrix distributions $\mathcal{D}_{\mathcal{X}}$ and $\mathcal{D}'_{\mathcal{X}}$ as a measure of dissimilarity between $\mathcal{X}$ and $\mathcal{X}'$. Moreover, we relate these notions to the Gromov-Prokhorov distance between fmm-spaces:

Theorem 4.0.6. *Let $\mathcal{X} = (X, B, \pi, \mu)$ and $\mathcal{X}' = (X', B', \pi', \mu')$ be functional metric measure spaces. For any $a \geq 0$, we have*

$$d_{GP,2a}(\mathcal{X}, \mathcal{X}') \leq d_{P,a}(\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}'}) \leq d_{GP,0}(\mathcal{X}, \mathcal{X}').$$

Furthermore, if $a\, d_{P,a}(\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}'}) < 1$, then

$$d_{P,a}(\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}'}) = d_{P,0}(\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}'}) = d_{GP,0}(\mathcal{X}, \mathcal{X}'). \tag{1.2}$$

Our main result supporting empirical estimation of the Gromov-Wasserstein distance between fmm-spaces is the following convergence theorem that has an empirical estimation result as a corollary.

Theorem 4.0.12. *Let $\mathcal{X}$ and $\mathcal{Y}$ be bounded functional metric measure spaces. Then for any $p \geq 1$ we have*

$$\lim_{N}(d_{\mathrm{W},p}(\mathcal{D}^N_{\mathcal{X}}, \mathcal{D}^N_{\mathcal{Y}})) = d_{\mathrm{W},p}(\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}'}) = d_{\mathrm{GW},\infty}(\mathcal{X}, \mathcal{X}') \tag{1.3}$$

1.2 Historical Remarks

The interest in the universal metric spaces goes back to 1910 when Fréchet (see [11], [12]) showed that l^∞, the spaces of real-valued bounded sequences with the supremum norm, is universal for the class of (completed) separable metric spaces $\mathcal{S}$. Around the same time Banach and Mazur showed that Banach space $C[0,1]$, the space of continuous functions with the supremum norm, is also universal for $\mathcal{S}$ ([3], 1933). Note that neither $C[0,1]$, nor l^∞ are separable. In 1924, Urysohn constructed a universal separable space U for the class $\mathcal{S}$ [33]. What distinguishes U is that any isometry between finite subsets of U extends to an isometry of the U (*homogeneity*). Urysohn proved that universality and homogeneity characterize his space. In other words, metric spaces with these properties are all isometric. He also studied homogeneity further and showed there are infinitely many isometries between subsets of U that cannot be extended to the space U. In ([29], 1953) Mrowka gave an example of an isometry between countable subsets of U that is not extendable to U. Huhunaišvili ([17], 1955) proved isometries between compact subsets of U can be extended to U (compact homogeneity). Melleray ([24], 2007) showed the strongest possible form of homogeneity in U is compact homogeneity.

It turns out a metric space is Urysohn if and only if it has the following *one-point extension* property: given a finite subset $X \subset U$ and any one-point metric extension of it $(X \cup \{x^*\}, d^*)$, there is $u \in U$ such that $d^*(x, x^*) = d_U(x, u)$ for all $x \in X$. Most constructions of the Urysohn space are based on finding a metric space with the one-point extension property. A well-known construction is due to Katětov ([19], 1986). He began with an arbitrary separable metric space X_0 and built a metric space X_1 containing an isometric copy of X_0 and all possible one-point metric extensions of X_0. Iterating the process, he defined X_n inductively and set $X_\infty := \cup_{n\in\mathbb{N}} X_n$. It is easy to see that X_∞, and therefore its completion, has the one-point extension property and is Urysohn. Hušek in [18] compared three approaches to the construction of the Urysohn space, namely Urysohn's, Hausdorff's, and Katětov's approaches. He also extended Hausdorff's construction to higher cardinals. In [34], Uspenskij used Katětov's method to show that the group of isometries of U is a universal Polish group.

The Urysohn space can be considered as a generalization of the universal graph (Radó 's graph). Vershik in [36] showed that the Urysohn space, similar to the universal graph, is generic. He constructed a universal distance matrix (a countable metric space with the *approximate-one point extension*) and proved that the universal matrix is (a) the generic distance matrix in the topological sense and (b) the random distance matrix in the probabilistic sense. Further similarities between Urysohn space and the universal graph were studied in [7].

Using methods of descriptive set theory and model theory, it has been shown that the Urysohn space is the completion of the inductive limit (Fraïssé limit) of the class of its finite subspaces (Fraïssé class), see for example [39] and [21]. In [9], Doucha used Fraïssé methods to furnish the Urysohn space with additional universal and homogeneous structures.

Gromov [15] studied the space of compact metric spaces with the Gromov–Hausdorff distance. He discussed that this distance can be interpreted as the Hausdorff in the Urysohn space U, modulo the action of isometries in U. He also introduced the notion of the box distance to study the class of mm-spaces. The Gromov–Prokhorov distance $d_{GP,a}$ $(a \geq 0)$, motivated by the Gromov-Hausdorff distance, is introduced by Villani in [37] and Greven *et al.* in [14], as a natural extension of Prokhorov distance to the class of mm-spaces. Löhr showed that the Gromov-Prokhorov distance is bi-Lipschitz equivalent to the box distance ($\square_a = 2\, d_{GP,2a}$ for $a > 0$, [23]). Mémoli [26], and Sturm [32] introduced their own version of the Gromov-Wasserstein distance ($d_{\mathrm{GW},p}$ for $p \geq 1$) between mm-spaces based on the ideas from optimal mass transport. Mémoli focused on applications in object matching and comparison, and Sturm introduced a notion of lower curvature bounds (formulated in the framework of mm-spaces) and shows they are stable under the Gromov–Wasserstein distance convergence ([32], Theorem 4.20).

In myriad real world scenarios, structured data is associated with both structure and feature information. To encode both [35] introduced the notion of *Fused* Gromov–Wasserstein distance between structured objects. Formally, a structured object over the metric space (feature space) (Ω, d_Ω) is a triple $(X \times \Omega, d_X, \mu)$ where (X, d_X) is a metric space and μ is a probability measure on $X \times \Omega$.

In the context of the mathematical population model (genealogy trees), any population has an underlying metric structure (the genealogical distance) and comes with a sampling measure on the population set. Moreover, each individual carries a genetic type with an impact on the dynamics of the population. In [8], the notions of *marked metric measure space* (mmm-space) and the corresponding marked-Gromov-weak topology were introduced to involve the genealogical relations of individuals. Here, a mmm-space is a triple $(X, d_X, \mu_{X\times B})$ where (X, d_X) is a Polish space and $\mu_{X\times B}$ is a probability measure on $X \times B$ for a fixed Polish space B of all types.

1.3 Organization of the Manuscript

Chapter 2 begins with a brief discussion on the Urysohn space and its genericity. We then give the construction of the Urysohn functional space for the class of functional spaces and prove that the Urysohn functional space is the generic functional space in the topological sense. Moreover, we introduce the Gromov-Hausdorff distance between compact functional spaces and relate this notion to the Urysohn functional space. Chapter 3 introduces the class of functional metric measure spaces (fmm-space). We extend the Box distance, the Gromov-Prokhorov distance, and the Gromov-Wasserstein distance to the functional setting and study several ways of comparing fmm-spaces. Chapter 4 introduces the functional version of matrix distributions and studies the Wasserstein and Prokhorov distance between them. We end the chapter by providing some examples.

CHAPTER 2

FUNCTIONAL SPACES

In this section, we introduce some notation and review basic concepts and results from metric geometry (cf. [25] and [16]) that are used in this manuscript. We also introduce extensions of these basic concepts to the realm of functional spaces.

2.1 Preliminaries

A metric space (X, d) is a set X together with a function $d : X \times X \to \mathbb{R}$, called distance function, if the following are satisfied for all $x, y, z \in X$:

- (i): $d(x, y) = 0 \iff x = y$
- (ii): $d(x, y) = d(y, x)$
- (iii): $d(x, y) \leq d(x, y) + d(y, x)$

If we allow zero distance between different points, we call X *pseudometric* space. Given a pseudometric space, we obtain a metric space by identifying zero-distance points. If the distance function d can take infinity, X is called an *extended metric space*. For a given extended metric space, (X, d), define the following relation R: $(x, y) \in R \iff d(x, y) \neq \infty$. Then, R is an equivalence relation, and each equivalence class with d becomes a metric space.

Let (X, d) be a pseudometric space. The open ball centered at $x \in X$ with radius $r > 0$ is denoted by $B_r(x)$ and defined by $B_r(x) := \{y \in X : \ d_X(x, y) < r\}$. The closed ball centered at $x \in X$ with radius $r > 0$ is $\overline{B_r(x)} := \{y \in X : \ d_X(x, y) \leq r\}$.

Definition 2.1.1. Let A be a subset of a metric space X.

1. An *ϵ-neighborhood of* A is denoted by A^ϵ and defined by $A^\epsilon := \cup_{a \in A} B_\epsilon(a)$.
2. A is called an ϵ-net for X, if X is an ϵ-neighborhood of A, that is $X = A^\epsilon$.
3. The *diameter* of A is denoted by $\text{diam}(A)$ and defined by $\text{diam}(A) := \sup\{d_X(a, b) : \ a, b \in A\}$. A is *bounded* if $\text{diam}(A) < \infty$ and is *unbounded* if $diam(A) = \infty$.
4. A is called a G_δ subset if it is a countable intersection of open subsets in X.

Definition 2.1.2. A map $f : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called

1. *isometric embedding* if for all x and x' in X we have $d_X(x, x') = d_Y(f(x), f(x'))$. In addition, if f is surjective, we call it an *isometry* and in this case X and Y are called isometric (denoted by $X \simeq Y$).

2. *Lipschitz* if there is a real number k such that for all $x, x' \in X$, $d_Y(f(x), f(x')) \leq k\, d_X(x, x')$. Moreover, f is called k-Lipschitz if k is the smallest constant that satisfies the above inequality.

Definition 2.1.3 (Whitney-McShane Extension)**.** Let (X, d_X) be a metric space and $\emptyset \neq A \subseteq X$. Given a 1-Lipschitz function $f : A \to \mathbb{R}$, the *Whitney-McShane* extension of f is the function $F : X \to \mathbb{R}$ given by

$$F(x) := \inf_{a \in A} f(a) + d(a, x)\,.$$

Remark 2.1.4. The Whitney-McShane extension is the maximal 1-Lipschitz extension of f, which implies that if $\emptyset \neq A \subseteq B \subseteq X$, taking two consecutive Whitney-McShane extensions, from A to B and then B to X, is the same as taking the Whitney-McShane extension from A to X.

Definition 2.1.5. A metric space (X, d_X) is called

- *Separable* if it has a countable dense subset.
- *Complete* if every Cauchy sequence in X converges in X.
- *Polish* if it is complete and separable.

The following definition introduces a distance between subsets of a metric space.

Definition 2.1.6. (Hausdorff Distance) Let A and B be subsets of a metric space (X, d_X). The Hausdorff distance between A and B, denoted by $d_H^X(A, B)$ is defined by

$$d_H^X(A, B) := \max\{\sup_{a \in A} d_X(a, B),\ \sup_{b \in B} d_X(b, A)\}$$

where $d_X(a, B) := \inf_{b \in B} d_X(a, b)$. Equivalently, $d_H^X(A, B) = \inf\{\epsilon > 0 :\ A \subset B^\epsilon \quad \text{and} \quad B \subset A^\epsilon\}$. The Hausdorff distance defines an extended pseudometric on the power set of X and an extended metric space on $M(X)$, the set of closed subsets of X ([6], Proposition 7.3.3).

Proposition 2.1.7. *([6], Proposition 7.3.8.) If (X, d) is compact, then $(M(X), d_H)$ is a compact metric space.*

The Gromov-Hausdorff distance, a generalization of the Hausdorff distance, is a measure of dissimilarity between compact metric spaces.

Definition 2.1.8. (Gromov-Hausdorff Distance) The Gromov-Hausdorff distance between two compact metric spaces (X, d_X) and (Y, d_Y), denoted by $d_{\mathrm{GH}}(X, Y)$, is the infimum of the numbers

$$d_H^Z(\phi(X), \psi(Y))$$

for all compact space Z and isometric embeddings $\phi : X \to Z$ and $\psi : Y \to Z$.

Theorem 2.1.9. *([6], Section 7.3) The Gromov-Hausdorff distance defines a complete metric on the set of isometry classes of compact metric spaces.*

Definition 2.1.10. (Universal Spaces) Let $\mathcal{M}$ be a class of metric spaces. A metric space U is *universal* for $\mathcal{M}$, if it contains an isometric copy of each metric space in $\mathcal{M}$. More precisely, if for every $X \in \mathcal{M}$ there is an isometric embedding $\phi : X \to U$.

Example 2.1.11. (Fréchet Embedding) The space of bounded sequences $l^\infty (= L^\infty(\mathbb{N}))$, with sup-norm is universal for the class of separable metric spaces.

Proof. Suppose (X, d) is a separable metric space and $A = \{x_0, x_1, \dots\}$ is a dense subset of X. Define $\phi : X \to l^\infty$ by $x \mapsto (f_i^x)_{i \in \mathbb{N}}$, where $f_i^x = d(x, x_i) - d(x_i, x_0)$. For any $i \in \mathbb{N}$ we have

$$|f_i^x - f_i^y| \leq |d(x, x_i) - d(y, x_i)| \leq d(x, y)$$

by taking supremum over all $i \in \mathbb{N}$, we have

$$\|(f_i^x)_{i \in \mathbb{N}} - (f_i^y)_{i \in \mathbb{N}}\|_\infty \leq d(x, y).$$

Equality follows when x_i approaches x. □

Example 2.1.12. (Kuratowski Embedding) Every metric space (X, d) embeds isometrically in $C_b(X)$, the the Banach space of all bounded continuous real-valued functions on X with the sup-norm $\|f\|_\infty := \sup_{x \in X} |f(x)|$.

Proof. Fix $x_0 \in X$ and define $\phi : X \to C_b(X)$ by $x \mapsto f_x,\ f_x(a) := d(x, a) - d(a, x_0)$. For every $a \in X$ We have:

$$|f_x(a) - f_y(a)| \leq |d(x, a) - d(y, a)| \leq d(x, y)$$

by taking supremum over all $a \in X$, we have

$$\|f_x - f_y\|_\infty \leq d(x, y)$$

with equality at $a = x$. □

Note that if X is bounded, there is no need to fix a point in advance, and $x \to f_x := d(x, -)$, defines an isometric embedding from X to $C_b(X)$. Both Fréchet and Kuratowski embedding depend on the choice of x_0. Moreover, in both cases, the destination space is not necessarily Polish.

Theorem 2.1.13. *(Urysohn) Let $\mathcal{M}$ be the class of Polish spaces. There exists a Polish space $U \in \mathcal{M}$ such that U is universal for $\mathcal{M}$.*

Urysohn's space U is Polish, and embedding into U does not depend on the choice of a base point. Furthermore, any isometry between two finite subsets of U extends to an isometry on U. The latter property is known as *homogeneity.* Universality and homogeneity characterize the Urysohn space. Precisely, all universal and homogeneous Polish space are isometric. The Urysohn space U also enjoys the following one-point extension property:
Given a finite subspace $F := \{x_i : i \in \{1, 2, .., n\}\} \subset U$ and any one-point metric extension of F,

$$(F \sqcup \{x^*\}, d^*): \quad d^*|_{F\times F} = d_U|_{F\times F}$$

there exists a point $u \in U$ such that

$$d^*(x_i, x^*) = d_U(x_i, u)$$

for every $1 \leq i \leq n$.

It turns out any Polish space with the one-point extension property is isometric to Urysohn space:

Theorem 2.1.14. *(Urysohn) A Polish space has the one-point extension property if and only if it is both universal and homogeneous.*

To construct such space, Urysohn built a countable metric space $U_{\mathbb{Q}}$ with a rational-valued distance function that has the one-point extension property for the class of countable metric spaces with a rational-valued distance function. Then, he proved that the metric completion has the one-point extension property for the class of Polish spaces and hence, is Urysohn.

Definition 2.1.15. A space (X, d) has the approximate one-point extension property if for any finite set $F \subset X$ and any one-point extension $(F \sqcup \{x^*\}, d^*)$, the new point x^* can be approximated it by elements in X, that is

$$\forall \epsilon > 0 \ \exists z \in X \quad : \quad \forall x \in F \quad |d^*(x, x^*) - d(x, z)| \leq \epsilon$$

Theorem 2.1.16. *(Urysohn) If a Polish space* (U, d_U) *has the approximate one-point extension property, it has the one-point extension property.*

One can consider the class of metric spaces with distance function taking values in $A \subset \mathbb{R}$, and obtain a metric space U_A with the one-point extension property for this class using the same method. For $A = \{0, 1, 2\}$, U_A has a graph structure as follows: join $x, y \in U_A$ by an edge if and only if $d_U(x, y) = 1$. A straightforward argument shows that this countable graph is isometric to the *Erdös-Rényi* graph (the universal graph). That is, any finite or countable graph embeds isometrically (as an induced subgraph) in U_A (see [7]). Consequently, it is natural to think of the Urysohn space as a generalization of the random graph. In fact, in [36] A. Vershik shows that similar to the universal graph, the Urysohn space is a generic Polish space.

2.1.1 Construction of the Urysohn Space: Katětov's Approach (1986)

As cited in [18], M. Katětov announced his construction of the Urysohn space at the Prague Topological Symposium in 1986. He started with an arbitrary Polish space (X, d_X) and constructed a separable space $E(X)$ containing an isometric copy of X and all possible one-point extensions of it. More explicitly, $E(X)$ is defined to be the collection of all functions $g : X \to \mathbb{R}$ such that

$$\forall x, y \in X: \qquad |g(x) - g(y)| \leq d(x, y) \leq g(x) + g(y), \tag{2.1}$$

and there is a finite set $S_g \subset X$ such that

$$\forall x \in X: \qquad g(x) = \inf\{g(a) + d(x, a) : a \in S_g\}. \tag{2.2}$$

S_g is called the support of g. We equip $E(X)$ with the following sup-metric:

$$d_E(f, g) := sup\{|f(x) - g(x)| : x \in X\}$$

Lemma 2.1.17. *X embeds isometrically into $E(X)$ via Kuratowski embedding $a \to f_a := d_X(a, -))$. Thus, we can assume $X \subset E(X)$ and for each $a \in X$ and $f \in E(X)$ we have $d_E(f_a, f) = f(a)$.*

Proof. By 2.1.12, $a \to f_a := d_X(a, -))$ is an isometry. For any f in $E(X)$ we have

$$d_E(f_a, f) = \sup|f_a(x) - f(x)| = \sup_{x \in X}|d_X(a, x) - f(x)| \leq f(a)$$

with the equality at $x = a$.

□

Remark 2.1.18. (Geometric interpretation of $E(X)$)

(i) Each point f in $E(X)$ corresponds to a one-point metric extension $X \sqcup \{x^*\}$ of X. Precisely, for any one-point extension of X, $(X \sqcup \{x^*\}, d^*)$, $f(x) := d^*(x^*, x)$ belongs to $E(X)$ and conversely, given $f \in E(X)$, $d^*(x, x^*) := f(x)$ extends d_X to a metric on $X \sqcup \{x^*\}$.

(ii) $d_E(f, g)$ may be interpreted as the smallest distance between x^* and y^* on $X \sqcup \{x^*, y^*\}$ (a two-point metric extension of X).

(iii) Let $A = \{x_1, x_2, \ldots, x_n\} \subset X$, $f \in E(A)$, and F be the Whitney-McShane extension of f to X. Then, $F \in E(X)$ and $d_E(F, d_X(x_i, -) = f(x_i)$. In fact, the Whitney-McShane extension from A to X induces an isometric embedding of $E(A)$ to $E(X)$ and as a result, for any pair of $f, g \in E(X)$ with a common support $S \subset X$ we have:

$$d_E(f, g) = \sup\{|f(x) - g(x)| : \quad x \in S\}.$$

Katětov starts with an arbitrary Polish space X. Iterating the process,

$$X_0 := X \text{ and } X_n := E(X_{n-1}), \ \forall n \in \mathbb{N}$$

he builds a metric space,

$$X^\infty := \sqcup_{n \geq 0} X_n$$

X^∞ has the one-point extension property by construction. Therefore, theorem 2.1.16, implies that its completion $\overline{X_\infty}$ is Urysohn.

Note that functions that take rational values in their supports are dense in $E(X)$, and without assuming finiteness of the support, $E(X)$ is not necessarily separable. For more details, see [25], where the author discusses necessary and sufficient conditions for separability of $E(X)$ in a general case.

2.1.2 Construction of the Urysohn Space: Vershik's Approach

This construction appears in [36]. Vershik describes a correspondence between the set of distance matrices and the collection of all Polish spaces with a fixed countable dense subset. He uses this model (the set of distance matrices) to prove the existence of a "universal distance matrix". Such a distance matrix corresponds to the Urysohn space. Like Urysohn and Katětov methods, Vershik constructs a space with the approximate one-point extension property, which implies that its completion has the one-point extension property and consequently is Urysohn. Vershik method has the advantage of allowing us to introduce a topology and define a measure on the space of

Polish metric spaces with a fixed countable dense subset [36], as he eventually proves the Urysohn space is the generic Polish space.

Consider the set of all infinite distance matrices:

$$\mathcal{R} := \{ \ [r_{ij}]_{i,i=1}^{\infty} : \ r_{ij} \geq 0, \ r_{ij} = r_{ji}, \ r_{ii} = 0, \ r_{ij} \leq r_{ik} + r_{jk} \}$$

Note that off-diagonal entries can be zero. $\mathcal{R}$ is a closed subspace of the euclidean space $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ with respect to weak topology. Every element $r \in \mathcal{R}$ corresponds to a Polish space with a fixed countable dense subset in the following way: $r \in \mathcal{R}$ defines a pseudometric on natural numbers $\mathbb{N}$, which we denote by $(\mathbb{N}, r)$. Completion of the metric space obtained by identifying zero-distance points in $(\mathbb{N}, r)$, denoted by (X_r, d_r), is Polish. On the other hand, given a Polish space (X, d_X) with a fixed dense countable subset $\{x_i\}_{i=1}^{\infty}$, we have that $[d_X(x_i, x_j)]_{i,j=1}^{\infty} \in \mathcal{R}$.

Similarly, let $\mathcal{R}_n$ to be the set of $n \times n$ distance matrices. Note that $r \in \mathcal{R}_n$ defines a pseudometric space on $\{1, 2, \ldots, n\}$. Define projection maps $P_n^{n+1} : \mathcal{R}_{n+1} \to \mathcal{R}_n$, sending $[r_{ij}]_{i,j=1}^{n+1} \in \mathcal{R}_{n+1}$ to $[r]_{i,j=1}^{n} \in \mathcal{R}_n$. Also $P_n : \mathcal{R} \to \mathcal{R}_n$, mapping $[r_{ij}]_{i,j=1}^{\infty}$ to $[r]_{i,j=1}^{n}$. That is, P_n^{n+1} deletes the last row and column of r (or equivalently it deletes the point $n+1$ in $(\{1, 2, \ldots, n, n+1\}, r)$) and P_n sends r to the north west-corner (NW-corner) of r of order n (or equivalently, it keeps the first n points in $(\mathbb{N}, r)$ and deletes the rest).

Under these projection maps, $\mathcal{R}$ can be seen as the inverse limit of towers of $\mathcal{R}_n$:

$$(0 =) \ \mathcal{R}_1 \xleftarrow{P_1^2} (\mathbb{R}_+ =) \ \mathcal{R}_2 \xleftarrow{P_2^3} \mathcal{R}_3 \xleftarrow{P_3^4} \ldots \xleftarrow{P_{n-1}^n} \mathcal{R}_n \xleftarrow{P_n^{n+1}} \ldots$$

Suppose $r \in \mathcal{R}_n$ and let $q = (q_1, q_2, ..., q_n) \in \mathbb{R}^n$ such that

$$r^q := \left[\begin{array}{cccc|c} 0 & r_{12} & \ldots & r_{1n} & q_1 \\ r_{12} & 0 & \ldots & r_{2n} & q_2 \\ \vdots & \vdots & \ddots & \vdots & \\ r_{1n} & r_{2n} & \ldots & 0 & q_n \\ \hline q_1 & q_2 & \ldots & q_n & 0 \end{array}\right] \in \mathcal{R}_{n+1}$$

The distance matrix r^q may be interpreted as a one-point metric extension of $(\{1, 2, \ldots, n\}, r)$. Precisely, each q_i is the distance between $(n+1)$ and i in $(\{1, 2, \ldots n, n+1\}, r^q)$. The set of all admissible vectors for $r \in \mathcal{R}_n$ is denoted by $A(r)$ and we have:

$$A(r) = \{q \in \mathbb{R}^n : |q_i - q_j| \leq r_{ij} \leq q_i + q_j\}$$

Definition 2.1.19. (Universal Distance Matrix) $r \in \mathcal{R}$ is called universal if for each natural number n, $\{(r_{1j}, r_{2j}, ..., r_{nj}) : j \geq n+1\}$ is dense in $A(P_n(r))$.

Remark 2.1.20. As a direct consequence of the definition of universality, $r \in \mathcal{R}$ is universal if for each $n \in \mathbb{N}$, the set of distance sub-matrices of r is dense in $\mathcal{R}_n$. Note that the metric space (X_r, d_r), induced by a universal distance matrix, has the approximate one-point extension property, and its completion is Urysohn.

Theorem 2.1.21. *There exist a universal distance matrix.*

Vershik constructs the universal matrix by induction. By definition, the first row ($n = 1$) should be dense in $A([0]) \simeq \mathbb{R}_+$. Therefore, for the first row, choose any arbitrary dense subset $(r_{ij})_{j=1}^{\infty} \subset \mathbb{R}_+$. Now assume the first $k-1$ rows has been constructed (i.e., $\big((r_{ij})_{i=1}^{k-1}\big)_{j=k}^{\infty}$ is dense in $(A(r_{k-1}))$, where r_{k-1} is the NW-corner of r of size $k-1$). He supplements the columns with a dense countable set $(r_{kj})_{j=k+1}^{\infty} \subseteq \mathbb{R}_+$ in a way that $\big((r_{ij})_{i=}^{k}\big)_{j=k+1}^{\infty}$ is dense in $A(r_k)$, where r_k is the NW-corner of r of order k (such a sequence always exists). Thus the existence of universal matrices is proved.

Vershik also shows that the set M of universal distance matrices is a dense and G_δ-subset of $\mathcal{R}$. Thus, universality is a generic property in the topological sense (i.e, $\mathcal{R} \setminus M$ is nowhere dense).

Theorem 2.1.22 (Genericity of Urysohn space)**.** *The set of universal distance matrices, M, is dense and a G_δ-subset of $\mathcal{R}$.*

Proof. It is straight forward to show M is dense in $\mathcal{R}$ with weak topology. Fix an order for the n-dimensional rational vectors, $\mathbb{Q}^n = \{q^j = (q_1^j, q_2^j, \ldots, q_n^j) : j \in \mathbb{N}\}$. By definition of universality for distance matrices, we have that

$$M = \cap_{k\in\mathbb{N}} \cap_{n\in\mathbb{N}} \cap_{j\in\mathbb{N}} \cup_{m\in\mathbb{N}} \{r \in \mathcal{R} : q^j \notin A(p_n(r)) \text{ or } \max_{i\leq n} |q_i^j - r_{im}| < \frac{1}{k}\}$$

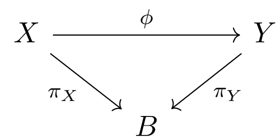
This proves M is a G_δ-subset of $\mathcal{R}$. □

2.2 Functional Spaces

This section introduces the class of *functional spaces* and the Gromov-Hausdorff distance between them. Throughout this section, unless specified otherwise, all metric spaces are considered Polish.

Definition 2.2.1. (Functional Spaces) A functional space over a base space B, is a triple (X, B, π_X), where X and B are Polish metric spaces and $\pi_X : X :\rightarrow B$ is a 1−Lipschitz map. Moreover, if X is a compact metric space, $\mathcal{X}$ is called a compact function space. Throughout this section all functional spaces are assumed to have the same base space which we denote by B. We also use a curly letter $\mathcal{X}$ to denote the functional space (X, B, π_X), where X is its underlying metric structure.

Definition 2.2.2. Let $\mathcal{X} = (X, B, \pi_X)$ and $\mathcal{Y} = (Y, B, \pi_Y)$ be two functional spaces. A mapping $\phi : \mathcal{X} \to \mathcal{Y}$ is said to be an isometry if $\phi : X \to Y$ is an isometry of metric spaces such that the following diagram commutes



$\mathcal{X}$ and $\mathcal{Y}$ are called isometric, denoted by $\mathcal{X} \simeq \mathcal{Y}$, if there is an isometry $\phi : \mathcal{X} \to \mathcal{Y}$ between them. Furthermore, if $\phi : X \to Y$ is an isometric embedding from the metric space X to the metric space Y, $\phi : \mathcal{X} \to \mathcal{Y}$ is called an isometric embedding from $\mathcal{X}$ to $\mathcal{Y}$.

Definition 2.2.3. Denote by $\mathcal{F}_B$ the class of functional spaces over the base space B. A functional space $\mathcal{Z} \in \mathcal{F}_B$ is called

- *Universal $\mathcal{F}_B$, if for every $\mathcal{X} \in \mathcal{F}_B$ there is an isometric embedding $\phi : \mathcal{X} \to \mathcal{Z}$.*
- Homogeneous if any isometry between finite subspaces of $\mathcal{Z}$ extends to an isometry of $\mathcal{X}$.

Definition 2.2.4. (The Urysohn Universal Functional Space over the base B) A functional space $\mathcal{U} = (U, B, \pi_U)$ is called Urysohn if it has the following one-point extension property:
given a finite subspace $F := \{x_i : i \in \{1, 2, .., n\}\} \subset U$ and any one-point (function) extension of F that is

$$(\ F \sqcup \{x^*\}, d^*, \pi^*\ ) : \quad d^*|_{F\times F} = d_U,\ \pi^*|_{F\times F} = \pi_U,$$

there exists a point $u \in U$ such that

$$d^*(x_i, x^*) = d_U(x_i, u) \quad \text{and} \quad \pi^*(x^*) = \pi_U(u).$$

We call u the realization of x^* in U.

The following theorem is the functional version of Theorem 2.1.14.

Theorem 2.2.5. *A functional space is Urysohn if and only if it is both universal and homogeneous.*

Proof. Universality: let (U, B, π_U) be a Urysohn functional space, (X, B, π) be a functional space and $D = \{x_1,\ x_2,\ x_3,\ ...\}$ be a dense subset of X. We show that every isometric embedding

$$\phi_n : (\{x_1,\ x_2,\ x_3,\ ...,\ x_n, B, \pi_X)\} \to (U, B, \pi_U)$$

extends to an isometric embedding

$$\phi_{n+1} : (\{x_1,\ x_2,\ x_3,\ ...,\ x_n,\ x_{n+1}\}, B, \pi_X) \to (U, B, \pi_U),$$

and therefore, iterating the process, we have an isometric embedding $\phi_\infty : (D, B, \pi_X) \to (U, B, \pi_U)$. Since X and B are Polish, ϕ_∞ extends to an isometric embedding $\Phi : (X, B, \pi_X) \to (U, B, \pi_U)$. Without loss of generality, assume $\{x_1, x_2, x_3, ..., x_n\} \subseteq U$ (the image of $x_i \in X$ under ϕ_n, is shown by the same letter $x_i \in U$). Consider $\{x_1,\ x_2,\ x_3,\ ...\ x_n,\ x_{n+1}\} \subset X$ as a one-point extension of $\{x_1,\ x_2,\ x_3,\ ...,\ x_n\} \subset U$. Then,

$$\exists u \in U : \quad d_X(x_i, x_{n+1}) = d_U(x_i, u) \quad \text{and} \quad \pi_X(x_{n+1}) = \pi_U(u)$$

Now define

$$\phi_{n+1} : (\{x_1,\ x_2,\ x_3,\ ...\ x_n,\ x_{n+1}\}, B, \pi_X) \to (U, B, \pi_U)$$

where $\phi_{n+1}(x_i) := \phi_n(x_i)$ for all $i \leq n$, and $\phi_{n+1}(x_{n+1}) := u$.

Homogeneity: let $C = \{c_1,\ c_2,\ ...,\ c_n\}$, $D = \{d_1,\ d_2,\ ...,\ d_n\}$ be two isometric subspace of U, and assume $\phi : (C, B, \pi_U) \to (D, B, \pi_U)$ is an isometry. We use a *back-and-forth argument* to extend ϕ to a countable dense subset of U: suppose $\{u_1,\ u_2,\ u_3, ...\}$ is a dense subset of U with no intersection with C and D. Consider $C_1 := C \sqcup \{u_1\}$ as a possible one-point extension of C. Then, there exists $v_1 \in U$ such that $D_1 := D \sqcup \{v_1\}$ is isometric to C_1. Let $\phi_1 : (C_1, B, \pi_U) \to (D_1, B, \pi_U)$ denote the extension of ϕ to C_1. Define $D_2 := D_1 \sqcup \{u_2\}$. D_2 is a possible one-point extension of D_1. Therefore, there is $v_1 \in U$ such that $C_2 := C_1 \sqcup \{v_2\}$ is isometric to D_2. Let $\phi_2 : (C_2, B, \pi_U) \to (D_2, B, \pi_U)$ denote the extension of ϕ_1 to C_2. Iterating the process infinitely many times, we get two countable isometric dense subsets C_∞ and D_∞ of U, and an isometry $\phi_\infty : (C_\infty, B, \pi_U) \to (D_\infty, B, \pi_U)$. Since U and B are Polish, ϕ_∞ extends to an isometry of (U, B, π_U).

Now suppose U is both universal and homogeneous. Let $A := \{a_1, a_2, .., a_n\} \subset U$ and $(A^* = \{a_1,\ a_2,\ ...,\ a_n, x^*\}, B, \pi^*)$ be a one-point extension of (A, B, π_B). We are looking for a point $y \in U$ such that

$$d^*(a_i, x^*) = d_U(a_i, y) \quad \text{and} \quad \pi^*(x^*) = \pi_U(y)$$

Since (U, B, π_U) is universal, it contains an isometric copy of (A^*, B, π^*). Let $\{b_1,\ b_2,\ ...,\ b_n, z\}$ be the isometric copy of A^* in (U, B, π_U) under isometry map ϕ. That is, $\phi : (A^*, B, \pi^*) \to (U, B, \pi_U)$ sending a_i to b_i, and x^* to z. By homogeneity of U, the isometry $\phi|_A : \{a_1, a_2, \ldots, a_n\} \to \{b_1, b_2, \ldots, b_n\}$ extends to an isometry $\Phi : (U, B, \pi_U) \to (U, B, \pi_U)$. Let $y := \Phi^{-1}(z)$ we have

$$d^*(a_i, x^*) = d_U(b_i, z) = d_U(a_i, y) \quad \text{and} \quad \pi^*(x^*) = \pi_U(z) = \pi_U(y).$$

□

The following proposition shows that there are universal and non-homogeneous functional spaces.

Proposition 2.2.6. *Let U be the Urysohn metric space, B be a Polish space, and $U \times B$ be the product space with the max metric. Moreover, let $\pi : X \times B \to B$ be the projection map. If B has more than one element, then $(U \times B, B, \pi)$ is universal and not homogeneous.*

Proof. We first show the universality of $(U \times B, B, \pi)$, and then we prove it does not have the one-point extension property and, therefore, is not homogeneous.

Let (X, B, π_X) be a functional space and let $\phi : X \to U$ be an isometric embedding (in the sense of metric spaces). Note that such an embedding always exists as (U, d_U) is universal. Then, the map

$$\Phi : X \to U \times B,\ \Phi(x) = (\phi(x), \pi_X(x))$$

is an isometric embedding (in the sense of functional spaces). Indeed,

$$d_{U\times B}(\Phi(x), \Phi(y)) = d_U(\phi(x), \phi(y)) = d_X(x, y) \quad \text{and} \quad \pi_X(x) = \pi(\phi(x))$$

Assume $(U \times B, B, \pi)$ has the one-point extension property. Let b_0, b_1 be distinct points in B, $\delta := d_B(b_0, b_1)$, and u_0 an arbitrary point in U. Since U is a Urysohn space, there exists $u_1 \in U$ such that $d_U(u_0, u_1) < \delta$. Let $A := \{(u_0, b_0), (u_1, b_1)\} \subseteq U \times B$, $A^* := A \sqcup \{a^*\}$ a one-point extension of A satisfying

$$d_{A^*}((u_0, b_0), a^*) = 2\delta \quad \text{and} \quad d_{A^*}((u_1, b_1), a^*) = 3\delta\,, \tag{2.3}$$

and $\phi\colon A^* \to B$ the 1-Lipschitz extension of $\pi|_A$ given by $\phi(a^*) = b_0$. Since π is a Urysohn map, there is $u \in U$ such that

$$d_{U\times B}((u, b_0), (u_0, b_0)) = 2\delta \quad \text{and} \quad d_{U\times B}((u, b_0), (u_1, b_1)) = 3\delta\,. \tag{2.4}$$

Since $U \times B$ is endowed with the max metric, this implies that $d_U(u, u_0) = 2\delta$ and $d_U(u, u_1) = 3\delta$, which is a contradiction since $d_U(u_0, u_1) < \delta$. □

Definition 2.2.7. A functional space (X, B, π) has the approximate one-point extension property if for any $A := \{x_1, x_2, x_3, ..., x_n\} \subset X$ and any one-point extension $A \sqcup \{x^*\}$, x^* can be approximated by elements of X:

$$\forall\epsilon > 0\ \exists u \in X\ : |d^*(x_i, x^*) - d(x_i, u)| \le \epsilon \quad \text{and} \quad d_B(\pi^*(x^*), \pi(u)) \le \epsilon \ \ (i \le n)$$

The following theorem is the functional version of 2.1.16.

Theorem 2.2.8. *If a functional space (X, B, π) has the approximate one-point extension property, it has the one-point extension property.*

Proof. Suppose $A = \{x_1, x_2, \dots, x_m\} \subset X$ and $(A \sqcup \{x^*\}, B, \pi^*)$ is a one-point extension of A. We show that x^* is realized in X, by buildings a sequence $(z_n)_{n=1}^{\infty}$ in X inductively such that for all $i \leq m$ and all $n \in \mathbb{N}$:

1. $|d^*(x_i, x^*) - d(x_i, z_n)| \leq 2^{-n}$
2. $|\pi^*(x^*) - \pi(z_n)| \leq 2^{-n}$
3. $d(z_n, z_{n+1}) \leq 2^{1-n}$

Using the approximate one-point extension property of X, we define z_1. Assume we have defined $z_1, z_2, \dots, z_n$. Let $Y := A \sqcup \{z_n, y\}$ be a one-point extension of $A \sqcup \{z_n\}$, where

$$d_Y(x_i, y) := d^*(x_i, x^*), \quad d_Y(z_n, y) := \sup_{i \leq m}\{|d^*(x_i, x^*) - d(x_i, z_n)|\} \quad \text{and} \quad \pi_Y(y) := \pi(z_n)$$

By the assumption, there exists $z \in X$ such that

$$|d_Y(x_i, y) - d(x_i, z)| \leq 2^{-(n+1)} \quad \text{and} \quad |d_Y(z_n, y) - d(z_n, z)| \leq 2^{-(n+1)}$$

Consequently,

$$d(z, z_n) \leq d_Y(z_n, y) + 2^{-(n+1)} \leq 2^{1-n}$$

We also have that

$$|\pi_Y(y) - \pi(z)| = |\pi(z_n) - \pi(z)| \leq d(z_n, z) \leq 2^{-(n+1)}$$

This proves the claim. □

Using this theorem we construct the Urysohn functional space. It is enough to construct a functional space V with the one-point extension property, then the completion of V is Urysohn. Here we adapt Katětov's method. Given an arbitrary functional space (X, B, π_X), define $E(\pi : X \to B)$ as follows:

$$E(\pi : X \to B) := \{\ (f, b) \in E(X) \times \mathbb{R} :\ d_B(b, \pi_X(x)) \leq f(x),\ \forall x \in X\}$$

We equip $E(\pi : X \to B)$ with the following max-distance

$$d_E\big(\ (f, b), (f', b')\ \big) := max\left(\ \sup_{x \in X}|f(x) - f'(x)|\ ,\ d_B(b, b')\ \right), \quad \forall\ (f, b), (f', b') \in E(\pi : X \to B),$$

and the projection map:

$$\pi_B : E(\pi : X \to B) \to B$$

Note that the projection map π_B is a 1-Lipschitz function. From now on, by $E(\pi : X \to B)$ we mean the functional space $(E(\pi : X \to B), B, \pi_B)$.

Lemma 2.2.9. *(X, B, π) embeds isometrically into $E(\pi : X \to B)$ via $y \to (d_X(y, x), \pi(y))$.*

Proof. For all $y, y' \in X$ we have

$$d_E\left((d_X(y,x), \pi(y)), (d(y',x), \pi(y')\right) = \sup_{x \in X} |d(y,x) - d(y',x)| \leq d_X(y,y')$$

By setting $x = y$, we obtain the equality:

$$\sup_{x \in X} |d(y,x) - d(y',x)| = d_X(y,y')$$

We also have that the following diagram commutes:

$$\begin{array}{ccc} X & \hookrightarrow & E(\pi\colon X \to B) \\ & \overset{\pi}{\searrow} & \downarrow{\scriptstyle \pi_B} \\ & & B \end{array}$$

□

Remark 2.2.10. (Geometric interpretation of $E(\pi : X \to B)$)

(i) $E(\pi : X \to B)$ contains (X, B, π) and all one-point extension of it. Each point (f, b) in $E(\pi : X \to B)$ corresponds to a one-point extension of X, $X \sqcup \{u\}$, in the following sense: for all $x \in X$, $f(x)$ defines the distance between x and u ($d(x, u) = f(x)$), and by setting $\pi(x) = \pi_X(x)$ and $\pi(u) = b$, π extends π_X to a 1-Lipschitz function on $X \sqcup \{u\}$.

(ii) If X and B are separable, then so is $E(\pi : X \to B)$.

(iii) Let $\pi\colon X \to B$ be 1-Lipschitz and $A \subseteq X$. If $(f, b) \in E(\pi\colon A \to B)$ and $F\colon X \to \mathbb{R}$ is the Whitney-McShane extension of f to X, then $(F, b) \in E(\pi\colon X \to B)$. We call (F, b) the Whitney-McShane extension to X of (f, b).

Theorem 2.2.11. *The Urysohn functional space (over a fixed base B) exists.*

Proof. Start with $\mathcal{X}_0 = (B, B, id)$ and define $\mathcal{X}_{n+1} := E(\pi_n : X_n \to B)$ inductively. Then, the diagram

$$\begin{array}{ccccccc} \mathcal{X}_0 & \hookrightarrow & \mathcal{X}_1 & \hookrightarrow & \mathcal{X}_2 & \hookrightarrow & \cdots \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \\ B & \xrightarrow{id} & B & \xrightarrow{id} & B & \xrightarrow{id} & \cdots \end{array} \tag{2.5}$$

is commutative, where the mappings on the first row are isometric embeddings. By viewing the isometric embeddings $\mathcal{X}_n \hookrightarrow \mathcal{X}_{n+1}$ as inclusions, we may write the metric co-limit as $X_\infty = \cup_{n \geq 0} X_n$. We denote the limiting 1-Lipschitz map by $\pi_\infty : X_\infty \to B$. By construction, $(X_\infty, B, \pi_\infty)$ has the one-point extension property. Indeed, if F is a finite subset of X_∞, for large enough M, F is contained in X_M. The Whitney-Mcshane extension to X_M of any $(f, b) \in E(\pi : F \to B)$ is an element of $\mathcal{X}_{m+1} = E(\pi : X_m \to B)$, which shows that any one-point extension of F is realized in (X_∞, B, π). Let U be the metric completion of X_∞ and $\pi\colon U \to B$ be the 1-Lipschitz extension of π_∞. We prove that (U, B, π) has the approximate one-point extension property, and thus, by Theorem 2.2.8, is Urysohn. Let $A = \{u_1, u_2, \ldots u_m\} \subset U$, $(A^* := A \sqcup u^*, B, \pi^*)$ be a one-point extension of A. For any $\epsilon > 0$, we show that there is $x \in X_\infty$ such that

$$|d^*(u^*, u_i) - d_U(x, u_i)| \leq \epsilon.$$

Let $\epsilon > 0$. There exist a natural number N and points $x_1,\ x_2, \ldots x_m \in X_N$ such that $d_U(x_i, u_i) \leq \frac{\epsilon}{2}$ for all $i = 1, \ldots, n$. Let $g : A \to \mathbb{R}$, $g(u_i) = d^*(u, u_i)$ and $b = \pi^*(u^*)$. By definition, $(g, b) \in E(\pi : A \to B)$. Denote by $\hat{g}$ the Whitney-Mcshane extension of g to U, and let $\tilde{g}$ denote the Whitney-Mcshane extension of $\hat{g}|_{\{x_1, x_2, \ldots, x_n\}}$ to X_N. Then $x := (\tilde{g}, b) \in \mathcal{X}_{N+1} \subset (X_\infty, B, \pi_\infty)$. Note that $\hat{g}(x_i) = \tilde{g}(x_i) = d_U(x, u_i)$. Moreover, the setting $d^*(u^*, x_i) := \hat{g}(x_i)$ extends (A^*, B, π^*) to the following one-point extension of $(A \cup \{x_1, x_2, \ldots, x_m\})$:

$$\big((A \cup \{x_1, x_2, \ldots, x_m\} \sqcup u^*), B, \pi^*\big).$$

It follows that

$$|d^*(u^*, u_i) - d_U(x, u_i)| \leq |d^*(u^*, x_i) + d_U(x_i, u_i) - d_U(x, u_i)| = |d_U(x, x_i) - d_U(x, u_i) + d_U(x_i, u_i)| \leq \epsilon$$

□

Theorem 2.2.12. *(Uniqueness) The Urysohn functional space is unique up to isometry.*

Proof. Let (U, B, π_U) and (V, B, π_V) be two Urysohn functional spaces, and $C = \{c_1, c_2, \ldots\}$ and $D = \{d_1, d_2, \ldots\}$ be dense subsets of U and V, respectively. Starting with subsets $C_0 = \{c_1\}$ and $D_0 = \{d_1\}$ and by a back-and-forth argument as in the proof of homogeneity in the Theorem 2.2.5, we can construct two dense subsets C_∞ and D_∞ of U and V, respectively, and an isometry $\phi_\infty : C_\infty \to D_\infty$. Since U, V, and B are complete, ϕ_∞ extends to an isometry $\Phi : U \to V$ such that $\pi_U = \pi_V \circ \Phi$. □

By definition, the Urysohn functional space $\mathcal{U} = (U, B, \pi_U)$ has the one-point extension property for its finite subsets. The following theorem extends this property to compact subsets of $\mathcal{U}$.

Proposition 2.2.13. *Suppose $\mathcal{U} = (U, B, \pi_U)$ is the Urysohn functional space. Lets A be a compact subspace of U and $\mathcal{A}^* := (A \sqcup \{a^*\}, B, \pi^*)$ be a one-point functional extension of (A, B, π_U). Then there is $u \in U$ such that for all $a \in A$:*

$$d_U(u, a) = d^*(a^*, a) \quad \text{and} \quad \pi^*(a^*) = \pi_U(u) \tag{2.6}$$

Proof. We construct a sequence (u_n) of elements of U inductively such that

(i) for all n: $\pi^*(a^*) = \pi(u_n)$

(ii) for all $a \in A$: $|d^*(a^*, a) - d(u_n, a)| \leq 2^{-n}$

(iii) for all n: $d_U(u_n, u_{n+1}) \leq 2^{1-n}$

Note that the limit point $u := \lim_{n\to\infty} u_n$ is the realization of a^* in U and satisfies the equalities in 2.6.

Construction of (u_n): Let A_n be an increasing 2^{-n}-net subsets of A. Denote by B_1 the one-point extension $A_1 \sqcup \{a^*\}$. Due to the one-point extension property of $\mathcal{U}$, there is $u_1 \in U$ such that for all $a \in A_1$, $d^*(a^*, a) = d_U(u_1, a)$ and $\pi^*(a^*) = \pi_U(u_1)$. Suppose, u_n is already constructed and let $B_{n+1} := A_{n+1} \sqcup \{a^*, u_n\}$, be a one-point function extension of $A_{n+1} \sqcup \{u_n\}$, where $d^*(a^*, u_n) := \sup_{a\in A_{n+1}} |d^*(a^*, a) - d_U(u_n, a)|$. Then there is a $u_{n+1} \in U$ such that $\pi^*(a^*) = \pi_U(u_{n+1})$ and for all $a \in A_{n+1}$, $d^*(a^*, a) = d_U(u_{n+1}, a)$. For all $a \in A$ there is $a_i \in A_{n+1}$ such that $|d^*(a^*, a) - d(u_{n+1}, a)| \leq 2d_U(a_i, a) \leq 2^{-n}$. Moreover, there is $a_j \in A_n$ such that $|d^*(a^*, a) - d_U(u_n, a)| \leq 2d_U(a_j, a) \leq 2^{1-n}$ which implies $d_U(u_{n+1}, u_n) = \sup_{a\in A_{n+1}} |d^*(a^*, a) - d_U(u_n, a)| \leq 2^{1-n}$. □

The following proposition states that the Urysohn functional space $\mathcal{U}$ has a stronger form of homogeneity in terms of its compact subspaces (see Theorem 2.2.5).

Proposition 2.2.14. *Any isometry $\phi : \mathcal{A} \to \mathcal{B}$ between compact function subspaces of the Urysohn functional space $\mathcal{U}$ extends to an isometry of $\mathcal{U}$*

Proof. Let $\{u_1, u_2, \dots\}$ be a dense subset of $U \setminus (A \cup B)$. With a back-and-forth argument as in the proof of homogeneity in Theorem 2.2.5, ϕ can be extended to an isometry of $\mathcal{U}$. □

2.3 Genericity of Urysohn Functional Space $\mathcal{U}_B$

In this section, we introduce the space of augmented distance matrices (equipped with the weak topology) and discuss its basic properties. This is the functional counterpart of a similar construction for metric spaces (see 2.1.2).

Let B be a Polish space and $\mathcal{R}_B$ be the set of the following augmented distance matrices:

$$\mathcal{R} := \{\ [r_{ij}|b_i]_{i,i=1}^{\infty}:\ r_{ij} \geq 0,\ r_{ij} = r_{ji},\ r_{ii} = 0,\ r_{ij} \leq r_{ik} + r_{jk},\ \text{and}\ d_B(b_i, b_j) \leq r_{ij}\}$$

Each $[r|b] \in \mathcal{R}_B$ corresponds to a functional space in the following sense: r defines a pseudometric on natural numbers $\mathbb{N}$ and $b = (b_i)_{i\in\mathbb{N}}$ defines a 1-Lipschitz map, with respect to r, on $\mathbb{N}$. We show this space by $(\mathbb{N}, [r|b])$. On the other hand, given a functional space (X, B, π), let $(x_i)_{i=1}^{\infty}$ be any dense countable subset of X. Then $[d_X(x_i, x_j)|\pi_X(x_i)]_{i,j=1}^{\infty}$ is in $\mathcal{R}_B$.

The set of augmented distance matrices, $\mathcal{R}_B$, can be seen as a closed subspace of the product space $\mathbb{R}^{\mathbb{N}^2} \times B^{\mathbb{N}}$ with the weak topology. Unless otherwise stated, all matrices are considered augmented distance matrices throughout this section. Similarly, let $\mathcal{R}_B^n$ be the set of all $n \times n$ augmented distance matrices. Define the projection maps $P_n^{n+1} : \mathcal{R}_B^{n+1} \to \mathcal{R}_B^n$, sending $[r|b]_{i,j=1}^{n+1} \in \mathcal{R}_B^{n+1}$ to $[r|b]_{i,j=1}^{n} \in \mathcal{R}_B^n$. Also $P_n : \mathcal{R}_B \to \mathcal{R}_B^n$, mapping $[r|b]_{i,j=1}^{\infty}$ to $[r|b]_{i,j=1}^{n}$. In fact, P_n^{n+1} deletes the point $n+1$ in $(\{1, 2, ..., n, n+1\}, [r|b])$; and P_n keeps the first n points in $(\mathbb{N}, [r|b])$ and deletes the rest.

Under these projection maps, $\mathcal{R}_B$ can be seen as the inverse limit of towers $\mathcal{R}_B^n$:

$$(B \simeq)\ \mathcal{R}_B^1 \xleftarrow{P_1^2} (\mathbb{R}^+ \times B \simeq)\ \mathcal{R}_B^2 \xleftarrow{P_2^3} \mathcal{R}_B^3 \xleftarrow{P_3^4} \dots \xleftarrow{P_{n-1}^n} \mathcal{R}_B^n \xleftarrow{P_{n+1}^n} \dots$$

Next definition formulates the one-point extension of a functional metric space in terms of augmented distance matrices.

Definition 2.3.1. An augmented vector $(q, a) := \big((q_1, q_2, ..., q_n)|a\big) \in \mathbb{R}^n \times B$ is called an admissible vector for $[r|b] \in \mathcal{R}_B^n$ if $[r^q|b^a] \in \mathcal{R}_B^{n+1}$, where $b^a := (b_1, b_2, \dots, b_n, a) \in B^n \times B$ and

$$r^q := \left[\begin{array}{cccc|c} 0 & r_{12} & \dots & r_{1n} & q_1 \\ r_{12} & 0 & \dots & r_{2n} & q_2 \\ \vdots & \vdots & \ddots & \vdots & \\ r_{1n} & r_{2n} & \dots & 0 & q_n \\ \hline q_1 & q_2 & \dots & q_n & 0 \end{array}\right]$$

The augmented distance matrix $[r^q|b^a]$ may be interpreted as a one-point functional metric extension of $(\{1, 2, \dots n\}, [r|b])$. Precisely, each q_i is the distance between $(n+1)$ and i in $(\{1, 2, \dots n, n+1\}, [r^q|b^a])$, and a is the value of the 1-Lipschitz map at $n+1$.

The set of all admissible vectors for $[r|b] \in \mathcal{R}_B^n$ is denoted by $A[r|b]$ and we have:

$$A[r|b] = \{(q, a) \in \mathbb{R}^n \times B : |q_i - q_j| \leq r_{ij} \leq q_i + q_j \ \text{ and }\ d_B(a, b_i) \leq q_i, \ \ \forall i, j \leq n\}$$

Now we are ready to define the notion of universal distance matrix.

Definition 2.3.2. (Universal Augmented Matrix). $[r|f] \in \mathcal{R}$ is called universal if for each natural number n, $\{\big((r_{1,j}, r_{2,j}, ..., r_{n,j})|b_j\big) : j \geq n+1\}$ is dense in $A(P_n[r|b])$.

Remark 2.3.3. (Properties of universal augmented distance matrices)

1. A universal augmented distance matrix induces a functional space with the approximate one-point extension property and, by Theorem 2.2.8 and 2.1.14, its completion is a Urysohn functional space.

2. As a direct consequence of the definition of universality for augmented distance matrices, $[r|b] \in \mathcal{R}_B$ is universal if for each $n \in \mathbb{N}$, the set of augmented distance sub-matrices of r is dense in $\mathcal{R}_B^n$.

Theorem 2.3.4. *Universal augmented distance matrices exist.*

Proof. Let (U, B, π) be the Urysohn functional space (constructed in the previous section), and let $(u_i)_{i=1}^{\infty}$ be any dense subset of U. Then $((u_i)_{i=1}^{\infty}, B, \pi_U)$ has the approximate one-point extension property and therefore, $[d_U(u_i, u_j)|\pi_U(u_i)]_{i=1}^{\infty}$ is a universal distance matrix. □

The following lemma shows that any finite augmented distance matrix can be extended to a universal augmented distance matrix.

Lemma 2.3.5. *Let n be a natural number and $r \in \mathcal{R}_B^n$. There exists a universal augmented distance matrix s such that $P_n(s) = r$*

Proof. Let $\mathcal{X}_r = (\{1, 2, \ldots, n\}, [r|b])$ denote the functional space induced by r, and suppose $(\{u_1,\ u_2,\ \ldots,\ u_n\},\ B,\ \pi_U)$ is an isometric copy of $\mathcal{X}_r$ in the Urysohn functional space (U, B, d_U). Let $\{u_1,\ u_2,\ \ldots,\ u_n,\ u_{n+1}, \ldots\}$ be an extension of $\{u_1,\ u_2,\ \ldots,\ u_n\}$ to a dense subset of U. Then $s := [d_U(u_i, u_j)|\pi_U(u_i)]_{i,j\in\mathbb{N}}$ is a universal augmented distance matrix such that $P_n(s) = r$. □

Theorem 2.3.6 (Genericity of Urysohn space)**.** *The set of universal matrices, M, is a dense G_δ-subset of $\mathcal{R}_B$.*

Proof. Recall that $\mathcal{R}_B$ is endowed with the weakest topology such that projection maps $P_n : \mathcal{R}_B \to \mathcal{R}_B^n$ are continuous. Let $r \in \mathcal{R}_B$ and let U be an element of the basis of the weak topology that contains r. That is, $U = \cap_{k=1}^{m} P_{n_k}^{-1}(U_{n_k})$ where U_{n_k} is an open set in $\mathcal{R}_B^{n_k}$ for every $k \leq m$. We show there is a universal augmented distance matrix s in U. Let $q = \max\{n_k :\ 1 \leq k \leq m\}$, and r^q denote the NW-corner of r of size q. By Lemma 2.3.5 r^q extends to a universal augmented distance matrix s. It follows that $\forall\ 1 \leq k \leq m, \quad P_{n_k}(s) \in U_{n_k}$. In other words, for all $1 \leq k \leq m$ the NW-corner of s of size n_k is in U_{n_k}, and thus, $s \in U$. This proves the density of the set M.

Recall that a subset M of a topological space X is called G_δ if it is a countable intersection of open subsets in X. let D be a dense subset of B and $\mathbb{Q}$ denote the set of non-negative rational numbers. For any n, fix an order for $\mathbb{Q}^n \times D = \{(q_j, a_j) = \big((q_j^1, q_j^2, \ldots, q_j^n | a_j\big) : \ j \in \mathbb{N}\}$. Also let ∂ denote the max distance on $\mathbb{R} \times B$. By definition of universality, we have $M = \cap_{n,k,j\in\mathbb{N}} M_{n,k,j}$, where $M_{n,k,j}$ is the set of augmented distance matrices such that

$$\{[r|b] \in \mathcal{R} : \ (q_j, p_j) \notin A(P_n[r|b]) \ \text{ or } \ \exists m \in \mathbb{N} \text{ s.t } \forall i \leq n \ \partial((q_j, p_j), (r_{im}, b_m)) < \frac{1}{k}\}$$

Also we have $M_{n,k,j} = \cup_{m\in\mathbb{N}} M^m_{n,kj}$, where

$$M^m_{n,k,j} := \{[r|b] \in \mathcal{R} : (q_j, p_j) \notin A(\pi_n[r|b]) \quad \text{or} \quad max_{i\leq n}\ \partial((q_j, p_j), (r_{im}, b_m)) \ < \frac{1}{k}\}$$

Note that the latter set, $M^m_{n,k,j}$, is open and therefore M is a G_δ-subset of $\mathcal{R}_B$. □

2.4 The Gromov-Hausdorff Distance

The Gromov-Hausdorff distance measures dissimilarity between compact metric spaces. This section introduces this notion for compact functional spaces.

Definition 2.4.1. (The Functional Gromov-Hausdorff Distance $d_{\mathrm{GH,B}}$) Let $\mathcal{X} = (X, B, \pi_X)$ and $\mathcal{Y} = (Y, B, \pi_Y)$ be compact functional spaces. The Gromov-Hausdorff distance $d_{\mathrm{GH,B}}(\mathcal{X}, \mathcal{Y})$ is defined by

$$d_{\mathrm{GH,B}}(\mathcal{X}, \mathcal{Y}) = \inf_{Z,\phi,\psi} d_H^Z(\phi(X), \psi(Y)),$$

where the infimum is taken over all compact functional space $\mathcal{Z} = (Z, B, \pi_Z)$ and isometric embeddings $\phi : \mathcal{X} \to \mathcal{Z}$ and $\psi : \mathcal{Y} \to \mathcal{Z}$.

Remark 2.4.2. Similar to the case of compact metric spaces, the set $\mathcal{F}_B$ of isometry classes of compact functional spaces with the Gromov-Hausdorff distance is Polish ([6], Section. 7.3).

The following lemma shows how the Gromov-Hausdorff distance can be computed in the Urysohn functional space $\mathcal{U}_B$.

Lemma 2.4.3. *Let $\mathcal{X} = (X, B, \pi_X)$ and $\mathcal{Y} = (Y, B, \pi_Y)$ be compact functional spaces, and $\mathcal{U}_B = (U, B, \pi_U)$ be the Urysohn functional space. Then*

$$d_{\mathrm{GH,B}}(\mathcal{X}, \mathcal{Y}) := \inf_{\phi,\psi} d_H^U(\phi(X), \psi(Y))$$

Proof. Let $\epsilon > 0$. We show that there are isometric embeddings $\phi : \mathcal{X} \to \mathcal{U}_B$ and $\psi : \mathcal{Y} \to \mathcal{U}_B$ such that $d_H^U(\phi(X), \psi(Y)) \leq d_{\mathrm{GH,B}}(\mathcal{X}, \mathcal{Y}) + \epsilon$. By definition of the Gromov-Hausdorff distance, there is a compact functional space $\mathcal{Z}$ and isometric embeddings $\phi' : \mathcal{X} \to \mathcal{Z}$ and $\psi' : \mathcal{Y} \to \mathcal{Z}$, such that

$$d_H^Z(\phi'(X), \psi'(Y)) \leq d_{\mathrm{GH}}(\mathcal{X}, \mathcal{Y}) + \epsilon$$

by the universality of $\mathcal{U}_B$, there is an isometric embedding $\rho : \mathcal{Z} \to \mathcal{U}_B$. Let $\phi := \rho \circ \phi'$ and $\psi := \rho \circ \psi'$. Note that $\phi : \mathcal{X} \to \mathcal{U}_B$ and $\psi : \mathcal{Y} \to \mathcal{U}_B$ are isometric embeddings of functional spaces. We have

$$d_H^U(\phi(X), \psi(Y)) = d_H^Z(\phi'(X), \psi'(Y)) \leq d_{\mathrm{GH}}(\mathcal{X}, \mathcal{Y}) + \epsilon$$

□

Gromov ([15], P. 83) discussed that the collection of compact subsets of the Urysohn universal space with Hausdorff distance (after removing the action of the isometry group of the Urysohn space) is isometric to the space of the isometric classes of compact metric spaces with Gromov-Hausdorff distance. The following theorem is an analogy of this result for functional setting.

Theorem 2.4.4. *Let $\mathcal{I}_B$ be the space of isometry classes of compact functional spaces over the base B, endowed with $d_{\mathrm{GH,B}}$. Let $\mathbf{G}_B$ denote the group of automorphisms of $\mathcal{U}_B$ and $\mathcal{F}_B$ denote the set of compact subspaces of $\mathcal{U}_B$, endowed with the Hausdorff distance d_H^U. Then, we have the following isometry*

$$(\mathcal{I}_B, d_{\mathrm{GH,B}}) \simeq (\mathcal{F}_B/\mathbf{G}_B, d_{\mathcal{F}}), \tag{2.7}$$

where $d_{\mathcal{F}}$ is the quotient metric.

Proof. Due to the universality of $\mathcal{U}_B$, with no loss of generality, assume that any compact functional space is a compact subspace of $\mathcal{U}_B$. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are compact functional spaces. By ([6], Lemma 3.6.6), the quotient metric is given by $d_{\mathcal{F}}(\mathcal{X}, \mathcal{Y}) = \inf_{\varphi \in \mathbf{G}_B} d_H^U(\mathcal{X}, \varphi(\mathcal{Y}))$. Lemma 2.4.3, implies that

$$d_{\mathrm{GH,B}}(\mathcal{X}, \mathcal{Y}) = \inf_{\psi} d_H^U(\mathcal{X}, \psi(\mathcal{Y}))$$

where $\psi : \mathcal{Y} \to \mathcal{U}_B$ is an isometric embeddings. By Proposition 2.2.14, ψ can be extended to an isometry $\varphi : \mathcal{U}_B \to \mathcal{U}_B$. Therefore,

$$d_{\mathrm{GH,B}}(\mathcal{X}, \mathcal{Y}) = \inf_{\varphi \in \mathbf{G}_B} d_H^U(X, \varphi(Y)) = d_F(\mathcal{X}, \mathcal{Y})$$

□

CHAPTER 3

FUNCTIONAL METRIC MEASURE SPACES

3.1 Preliminaries

Measurable structures give us tools to assign probabilities to *events*. For this purpose, we need a family of events, closed under finite intersections, unions, and complements (algebra). If, in addition, it is closed under a countable intersection, it is called σ-algebra. This last property makes it possible to talk about, for example, *the law of large numbers*. In particular, one would need a *stability result* for the average of outcomes obtained from a large number of trials.

Definition 3.1.1. Let X be a set and Σ be a collection of subsets of X. Σ is called a σ-algebra if

1. $X \in \Sigma$
2. $A \in \Sigma \iff X \setminus A \in \Sigma$
3. if $A_1, A_2, \ldots$ are in Σ, then $\cup_{i\in\mathbb{N}} A_i \in \Sigma$

A set X endowed with a σ-algebra of its subsets Σ is called a *measurable space* and elements of Σ are measurable sets. If the underlying set comes with a topological structure, we would like the open sets (and hence the closed sets) to be measurable. The smallest σ-algebra generated by open sets is called *the Borel σ-algebra* and is denoted by B_X. A real-valued function μ defined on the Borel σ-algebra B_X, is called a *Borel probability measure* on X if it is non-negative, countably additive and $\mu(X) = 1$. The support of μ is the smallest closed set C such that $\mu(X \setminus C) = 0$ and is denoted by $\mathrm{supp}(\mu)$. A Borel probability measure μ on X is called *regular* if for every Borel subsets $A \subset X$:

$$\text{(inner regularity):} \quad \mu(A) = \sup\{\mu(K): \quad K \text{ is compact set contained in A}\}$$

And

$$\text{(outer regularity):} \quad \mu(A) = \inf\{\mu(C): \quad C \text{ is an open set containing A}\}$$

In other words, one can approximate the measure of a Borel subset by open subsets from ”outside” and compact subsets from ”inside”.

A map $\phi : (X, B_X) \to (Y, B_Y)$ between (Borel) measurable spaces is a *(Borel) measurable map* if for every $A \in B_Y$, $\phi^{-1}(A) \in B_X$. A measurable map ϕ is called an isomorphism if it is bijective with a measurable inverse.

Definition 3.1.2. (Push-Forward) Let $f : X \to Y$ be a measurable map from a probability measure space (X, B_X, μ_X) to a topological space Y with its Borel σ-algebra B_Y. The setting $f_*\mu_X(A) := \mu(f^{-1}(A))$ defines a (Borel) probability measure on Y and is called the *push-forward* measure.

A class of measurable spaces suitable for many applications was introduced and studied by V.A. Rohlin ([5], 9.4). He assumed the underlying set X comes with a Polish structure. Such a measurable space with a probability measure on it is called *Standard Probability Space* or *Rochlin-Lebesgue Space*. Polish structures make the measure space suitable for application: many geometric objects of interest, such as manifolds and separable Banach spaces, are Polish. Also, the setting is convenient for probability theory. For instance, any probability measure on a Polish space is regular, or a countable product of standard Borel spaces turns out to be standard Borel (read more in ([5], 9.4) and [20]). Moreover, two standard Borel spaces are isomorphic if and only if they are of the same cardinality. Precisely, a probability space is standard if it is Borel isomorphic to an interval with Lebesgue measure, a finite or countable set of atoms, or a disjoint union of both:

Theorem 3.1.3. *(Classification of Standard Probability Space) Let I be the unit interval [0,1] equipped with Lebesgue measure λ. Any standard probability measure (X, μ) is Borel isomorphic (mod 0) to I with the measure $\nu := m\lambda + \Sigma_{n=1}^{\infty}\alpha_n\delta_{\frac{1}{n}}$. Where $m = 1 - \Sigma_{n=1}^{\infty}\alpha_n$, $\alpha_n = \mu(a_n)$, $\{a_n\}$ the set of atoms of μ, and $\delta_{\frac{1}{n}}$ is the Dirac's delta measure at $\frac{1}{n}$ defined by*

$$\delta_{\frac{1}{n}}(A) = \begin{cases} 1, & \frac{1}{n} \in A, \\ 0, & o.w \ . \end{cases}$$

Proof. ([5], Theorem 9.4.7) □

Throughout this section, all measurable spaces are standard, and by a probability measure on X, we mean a Borel probability measure defined on the Borel subsets of the Polish space X.

Definition 3.1.4. Let (X, μ) be a measure space. Denote by $C_b(X)$ the set of continuous and bounded real-valued functions on X. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is equidistributed (or uniformly distributed) if for any $f \in C_b(X)$:

$$\lim_{n\to\infty} \frac{1}{n}\Sigma_{i=1}^{n} f(x_i) = \int_X f(x)dx$$

Lemma 3.1.5. *([22], Lemma 2.4) Let (X, μ) be a measurable space and E denote the set of equidistributed sequences in X. E, viewed as a subset of X^∞, is measurable and $\mu^\infty(E) = 1$.*

Let (X, d) be Polish, and denote the set of probability measures on X by $P(X)$. We always consider $P(X)$ with the weak topology, that is, the smallest topology on $P(X)$ such that every function $\mu \to \int f d\mu$ is continuous for every $f \in C_b(X)$.

Definition 3.1.6. (Weak Convergence) Let (μ_n) be a sequence of probability measures on X. We say μ_n converges weakly to a probability measure $\mu \in P(X)$ if

$$\int f d\mu_n \to \int f d\mu \quad \forall f \in C_b(X).$$

Definition 3.1.7. A collection M of probability measures in $P(X)$ is called uniformly tight, if for every $\epsilon > 0$ there is a compact set $K_\epsilon \subset X$ such that for all $\mu \in M$, $\mu(X \setminus K_\epsilon) \leq \epsilon$.

Theorem 3.1.8. *(Prokhorov [10, Theorem 11.5.4]) A collection $M \subset P(X)$ is uniformly tight if and only if the closure of M is compact in $P(X)$.*

The weak topology can be expressed in terms of *Prokhorov* or *Wasserstein* distance and $P(X)$ equipped with these distance functions becomes a Polish metric space ([37], Theorem 6.9).

Definition 3.1.9. (Prokhorov Distance) Let (X, d) be Polish and $\mu, \nu \in P(X)$. Given $a \geq 0$, we define

$$d_{P,a}(\mu, \nu) := \inf\{\epsilon : \mu(A) \leq \nu(A^\epsilon) + a\epsilon \;\text{ for all Borel set } A \subset X\} \tag{3.1}$$

where $A^\epsilon := \cup_{a \in A} B_\epsilon(a)$. The parameter a gives us extra flexibility and different choices of $a \gneq 0$ results in equivalent distances. In fact, given $0 \lneq a \leq a'$ we have $d_{P,a'} \leq d_{P,a} \leq \frac{a}{a'} d_{P,a'}$. Note that $d_{P,a} \leq \frac{1}{a}$.

Definition 3.1.10. Let (X, d_X) and (Y, d_Y) be Polish, and μ_X and μ_Y be two probability measures on X and Y, respectively. A probability measure $\gamma \in P(X \times Y)$ is called a coupling between μ_X and μ_Y if for any $A \in B_X$ and $B \in B_Y$:

$$\gamma(A \times Y) = \mu_X(A) \quad \text{and} \quad \gamma(X \times B) = \mu_Y(B)$$

The set of couplings of μ_X and μ_Y is denoted by $\sqcup(\mu_X, \mu_Y)$. Note that $\sqcup(\mu_X, \mu_Y)$ is not empty as the product measure $\mu_X \times \mu_Y$ is a coupling.

Lemma 3.1.11. $\sqcup(\mu_X, \mu_Y)$ *is compact with respect to weak topology.*

By [10, Theorem 9.3.7], $\sqcup(\mu_X, \mu_Y)$ is closed under the weak convergence. We show $\sqcup(\mu_X, \mu_Y)$ is uniformly tight. Let $\epsilon > 0$. Since μ_X and μ_Y are regular measures (X and Y are Polish), there are compact subspaces K_X and K_Y of X and Y respectively so that $\mu_X(K_X) > 1 - \epsilon/2$ and $\mu_Y(K_Y) > 1 - \epsilon/2$. Then, for any P in $\sqcup(\mu_X, \mu_Y)$, we have

$$\begin{aligned} P(K_X \times K_Y) &= P((K_X \times Y) \cap (X \times K_Y)) \\ &\geq P(K_X \times Y) + P(X \times K_Y) - 1 = \mu_X(K_X) + \mu_Y(K_Y) - 1 \geq 1 - \epsilon. \end{aligned} \tag{3.2}$$

There is another formulation of Prokhorov distance in terms of *couplings* of measures.

Theorem 3.1.12. *(Strassen's theorem) Let (X, d) be Polish, μ and ν in $P(X)$ and $a \geq 0$. then*

$$d_{P,a(\mu,\nu)} = \inf\big\{\epsilon : \ \exists\gamma \in \sqcup(\mu,\nu) \ s.t \ \gamma\big(\{(x,y) \in X \times X \ : \ d(x,y) \geq \epsilon\}\big) \leq a\epsilon\big\} \tag{3.3}$$

([10], theorem 11.6.2)

Definition 3.1.13. (Wasserstein distance) Let (X, d) be a Polish space and let $P(X)$ be the set of probability measures on X. The Wasserstein distance on $P(X)$ is defined by:

$$d_{W,p}(\mu,\nu) := \Big(\inf_{\gamma\in\sqcup(\mu,\nu)} \int_{X\times X} d^p(x,y)d\gamma(x,y)\Big)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$
$$d_{W,\infty}(\mu,\nu) := \inf_{\gamma\in\sqcup(\mu,\nu)} \Bigg(\sup\{d(x,y) : \quad x, y \in \operatorname{supp}\gamma\}\Bigg).$$

Remark 3.1.14. Note that, $(P(X), d_{\mathrm{W},p})$ is an extended metric space. We also, due to Hölder's inequality, have

$$p \leq q \quad \Longleftrightarrow \quad d_{\mathrm{W},p} \leq d_{\mathrm{W},q}$$

(for further reading see [37])

Definition 3.1.15. (mm-spaces) A metric measure space (mm-space) is a triple (X, d_X, μ_X) where (X, d_X) is Polish and μ_X is a full support (i.e., $supp\mu_X = X$) Borel probability measure on X. In this w

Two mm-spaces (X, d_X, μ_X) and (Y, d_Y, μ_Y) are called isometric if there is a map $\phi : \operatorname{supp}(\mu_X) \to \operatorname{supp}(\mu_Y)$ such that

$$d_X(x, x') = d_Y(\phi(x), \phi(x')) \ , \quad \forall x, x' \in X \quad \text{and} \quad \phi_*\mu_X = \mu_Y.$$

In this work, all probability measures considered full support, that is $\mu_X()$ The map ϕ is called an isometry.

There are several ways to define a distance between mm-spaces. We start with the definition of Gromov's Box distance.

Definition 3.1.16. (The Box Distance)

Suppose (X, d_X, μ_X) and (Y, d_Y, μ_Y) are mm-spaces, λ is the Lebesgue measure on the unit interval $[0, 1]$, and $a \geq 0$. The *box distance* $\square_a(X, Y)$ is the infimum of $\epsilon > 0$ such that there exist measure-preserving maps $\phi\colon [0, 1] \to X$ and $\psi\colon [0, 1] \to Y$ and a Borel set $J \subseteq [0, 1]$ such that:

(i) $\lambda(J) \leq a\epsilon$;

(ii) $|d_X(\phi(s), \phi(t)) - d_Y(\psi(s), \psi(t))| \leq \epsilon$, for any $s, t \in [0, 1] \setminus J$;

The box distance $\square$ defines a metric on the set of isometry classes of mm-spaces. ([31], Theorem 4.10).

The Prokhorov distance compares probability measures defined on the same Polish space X. The Gromov-Prokhorov distance generalizes this idea to probability measures defined on possibly different metric spaces.

Definition 3.1.17. (Gromov-Prokhorov Distance) Let (X, d_X, μ_X) and (Y, d_Y, μ_Y) be mm-spaces. The Gromov-Prokhorov distance between X and Y, $d_{GP,a}(X, Y)$, is the infimum of $\epsilon > 0$ such that there exists a Polish space (Z, d_Z) and isometric embeddings $\phi : X \to Z$ and $\psi : Y \to Z$ such that the Prokhorov distance $d_{P,a}(\phi_*(\mu_X), \psi_*(\mu_Y)) < \epsilon$.

The following theorem shows that the box distance and Gromov-Prokhorov distance are equivalent.

Theorem 3.1.18. *(Löhr[23]) Let* (X, d_X, μ_X) *and* (Y, d_Y, μ_Y) *be mm-spaces. Then for any* $a \geq 0$.

$$\square_a(X, Y) = 2d_{GP,2a}(X, Y)$$

Similarly, the Gromov-Wasserstein, a generalization of Wasserstein distance, compares two probability measures defined on possibly different metric spaces.

Definition 3.1.19. (Gromov-Wasserstein Distance) Suppose (X, d_X, μ_X) and (Y, d_Y, μ_Y) are mm-spaces, and $m_{X,Y}(x, y, x', y') := |d_X(x, x') - d_Y(y, y')|$. The Gromov-Wasserstein $d_{\mathrm{GW},p}(X, Y)$ is defined by

$$d_{\mathrm{GW},p}(X, Y) := \inf_{\gamma \in \sqcup(\mu_X, \mu_Y)} \left(\frac{1}{2} \int m_{X,Y}^p \mathrm{d}(\gamma \otimes \gamma) \right)^{1/p} \quad 1 \leq p < \infty$$
$$d_{\mathrm{GW},\infty}(X, Y) := \inf_{\gamma \in \sqcup(\mu_X, \mu_Y)} \left(\frac{1}{2} \sup_{\mathrm{supp}(\gamma \otimes \gamma)} m_{X,Y} \right).$$

Theorem 3.1.20. *([26], Theorem 5.1) Suppose $p \in [1, \infty]$. Then,*

1. *$d_{\mathrm{GW},p}$ defines a metric on the set of isometry class of mm-spaces.*

2. *Let $p \in [1, \infty)$. Suppose $X_n = \{x_1, x_2, \ldots, x_n\} \subset X$ is a set of n random variables $x_i : \Omega \to X$ defined on some probability space (Ω, P) with law μ_X. For $\omega \in \Omega$, define $\mu_n(w, .) := \frac{1}{n}\Sigma_{i=1}^{n}\delta_{x_i(\omega)}$ and consider (X_n, d_X, μ_n) as a mm-space, then for μ_X-almost all ω,*

$$(X_n, d_X, \mu_n) \xrightarrow{d_{\mathrm{GW},p}} (X, d_X, \mu_X).$$

3.2 The Space of Functional Metric Measure Spaces

This section introduces functional metric measure spaces. We extend the box distance, Gromov-Prokhorov distance, and Gromov-Wasserstein distance to the functional setting and study their relations. The fact that these extensions are metrics can be argued as in the case of mm-spaces.

Definition 3.2.1. A *functional metric measure space (fmm-space)* is a quadruple (X, B, π, μ), where X and B are Polish metric spaces, $\pi \colon X \to B$ is a 1-Lipschitz map, and μ is a Borel probability measure on X. Two fmm-spaces $\mathcal{X} = (X, B, \pi_X, \mu_X)$ and $\mathcal{Y} = (Y, B, \pi_Y, \mu_Y)$ are called isomorphic, denoted by $\mathcal{X} \simeq \mathcal{Y}$, if there is an isomorphism $\phi : (X, d_X, \mu_X) \to (Y, d_Y, \mu_Y)$ such that $\pi_X = \pi_Y \circ \phi_X$.

Next definition is the functional version of the box distance (see 3.1.16).

Definition 3.2.2 (Functional Box Distance)**.** Suppose $\mathcal{X} = (X, B, \pi_X, \mu_X)$ and $\mathcal{Y} = (Y, B, \pi_Y, \mu_Y)$ are fmm-spaces, λ is the Lebesgue measure on $[0, 1]$ and $a \geq 0$. The *box distance* $\square_a(\mathcal{X}, \mathcal{Y})$ is the infimum of $\epsilon > 0$ such that there exist measure-preserving maps $\phi \colon [0, 1] \to X$ and $\psi \colon [0, 1] \to Y$ and a Borel set $J \subseteq [0, 1]$ such that:

(i) $\lambda(J) \leq a\epsilon$;

(ii) $|d_X(\phi(s), \phi(t)) - d_Y(\psi(s), \psi(t))| \leq \epsilon$, for any $s, t \in [0, 1] \setminus J$;

(iii) $d_B(\pi_X(\phi(t)), \pi_Y(\psi(t)) \leq \frac{\epsilon}{2}$, for any $t \in [0, 1] \setminus J$.

The functional box distance defines a metric on the set of isomorphism classes of fmm spaces. Moreover, for $a \neq 0$, $\square_a(\mathcal{X}, \mathcal{Y}) \leq \frac{1}{a}$.

Now we define the functional version of Gromov-Prokhorov Distance (see 3.1.17).

Definition 3.2.3 (Functional Gromov-Prokhorov Distance)**.** Suppose $\mathcal{X} = (X, B, \pi_X, \mu_X)$ and $\mathcal{Y} = (Y, B, \pi_Y, \mu_Y)$ are fmm-spaces. Given $a \geq 0$, the *functional Gromov-Prokhorov distance* $d_{GP,a}(\mathcal{X}, \mathcal{Y})$ is the infimum of $\epsilon > 0$ such that there exist a 1-Lipschitz map $\psi\colon Z \to B$, defined on a Polish space Z, and isometric embeddings $\imath\colon X \hookrightarrow Z$ and $\jmath\colon Y \hookrightarrow Z$ such that:

(i) $\pi_X = \psi \circ \imath$ and $\pi_Y = \psi \circ \jmath$,

(ii) $d_{P,a}(\imath_*(\mu_X), \jmath_*(\mu_Y)) \leq \epsilon$.

The functional Gromov-Prokhorov distance defines a metric on the set of isomorphism classes of fmm-space. The next result is a functional counterpart of Theorem 3.1.18.

Proposition 3.2.4. *Let $\mathcal{X} = (X, B, \pi_X, \mu_X)$ and $\mathcal{Y} = (Y, B, \pi_Y, \mu_Y)$ be fmm-spaces. For any $a \geq 0$, we have*

$$\square_a(\mathcal{X}, \mathcal{Y}) = 2 d_{GP,2a}(\mathcal{X}, \mathcal{Y}) .$$

Proof. We begin with the inequality $\square_a(\mathcal{X}, \mathcal{Y}) \leq 2 d_{GP,2a}(\mathcal{X}, \mathcal{Y})$. We show that if $d_{GP,2a}(\mathcal{X}, \mathcal{Y}) < \frac{\epsilon}{2}$, then $\square_a(\mathcal{X}, \mathcal{Y}) \leq \epsilon$. Let $\imath\colon X \to Z$ and $\jmath\colon Y \to Z$ be isometric embeddings as in definition 3.2.3. By Strassen's Theorem 3.3, there exists a coupling ν between $\imath_*(\mu_X)$ and $\jmath_*(\mu_Y)$ such that

$$\nu(\{(z, z') \in Z \times Z \colon d_Z(z, z') > \epsilon/2\}) \leq a\epsilon . \tag{3.4}$$

As $\imath(X) \times \jmath(Y)$ has full measure in $(Z \times Z, \nu)$, ν may be viewed as a coupling between μ_X and μ_Y. Thus,

$$\nu(\{(x, y) \in X \times Y \colon d_Z(\imath(x), \jmath(y)) > \epsilon/2\}) \leq a\epsilon . \tag{3.5}$$

Viewing $(Z \times Z, \nu)$ as a metric measure space, there is a measure-preserving map $(\phi, \phi')\colon ([0,1], \lambda) \to (X \times Y, \nu)$. Note that this implies $\phi\colon [0,1] \to X$ and $\phi'\colon [0,1] \to Y$ are also measure preserving. Letting

$$J = \left\{ t \colon d_Z(\imath(\phi(t)), \jmath(\phi'(t))) > \frac{\epsilon}{2} \right\} , \tag{3.6}$$

we have that $\lambda(J) \leq a\epsilon$. Condition (iii) of definition 3.2.2 follows from the definition of J and the fact that $\psi\colon Z \to B$ is 1-Lipschitz. Thus, it remains to show (ii). For t, s in $[0,1] \setminus J$, we have

$$\begin{aligned} |d_X(\phi(s), \phi(t)) - d_Y(\phi'(s), \phi'(t))| &= |d_Z(\imath(\phi(s)), \imath(\phi(t))) - d_Z(\imath(\phi'(s)), \imath(\phi'(t)))| \\ &\leq d_Z(\imath(\phi(s)), \jmath(\phi'(s))) + d_Z(\imath(\phi(t)), \jmath(\phi'(t))) \\ &\leq \epsilon , \end{aligned} \tag{3.7}$$

which implies $\square_a(\mathcal{X}, \mathcal{Y}) \leq \epsilon$.

To prove the converse inequality $2d_{GP,2a}(\mathcal{X},\mathcal{Y}) \leq \Box_a(\mathcal{X},\mathcal{Y})$, assume that $\Box_a(\mathcal{X},\mathcal{Y}) < \epsilon$. We need to show that $d_{GP,2a}(\mathcal{X},\mathcal{Y}) \leq \frac{\epsilon}{2}$. Let $\phi\colon [0,1] \to X$, $\phi'\colon [0,1] \to Y$ and $J \subseteq I$ be as in definition 3.2.2. Let $Z = X \sqcup Y$ be the disjoint union of X and Y, and $d_Z\colon Z \times Z \to \mathbb{R}$ be the extension of d_X and d_Y given by

$$d_Z(x,y) = \inf_{t\in[0,1]\setminus J} d_X(x,\phi(t)) + \frac{\epsilon}{2} + d_Y(y,\phi'(t))\,, \tag{3.8}$$

for $x \in X$ and $y \in Y$. The symmetry and definiteness of d_Z are straightforward and the triangle inequality follows from (ii) of definition 3.2.2. Let $\psi\colon Z \to B$ be the function extending π_X and π_Y. By (iii) of definition 3.2.2, $\psi\colon Z \to B$ is 1-Lipschitz. Let ν be the coupling between μ_X and μ_Y on Z induced by (ϕ,ϕ'). The set $\{t\colon d_Z(\phi(t),\phi'(t)) > \frac{\epsilon}{2}\}$ is contained in J, so

$$\nu\{(z,z') : d_Z(z,z') > \frac{\epsilon}{2}\} \leq \lambda(J) \leq a\epsilon\,. \tag{3.9}$$

By Strassen's Theorem, $d_{GP,2a}(\imath_*(\mu_X),\jmath_*(\mu_Y)) \leq \epsilon/2$. This completes the proof. □

Next definition, is the functional counterpart of the Gromov-Wasserstein distance for mm-spaces.

Definition 3.2.5 (Functional Gromov-Wasserstein Distance)**.** suppose $\mathcal{X} = (X,B,\pi_X,\mu_X)$ and $\mathcal{Y} = (Y,B,\pi_Y,\mu_Y)$ are *fmm*-spaces, $m_{X,Y}(x,y,x',y') := |d_X(x,x') - d_Y(y,y')|$ and $d_{X,Y}(x,y) := d_B(\pi_X(x),\pi_Y(y))$. The *functional Gromov-Wasserstein distance* is defined by:

$$d_{\mathrm{GW},p}(\mathcal{X},\mathcal{Y}) := \inf_{\gamma\in\sqcup(\mu_X,\mu_Y)} \max\left\{\frac{1}{2}\left(\int m^p_{X,Y}\mathrm{d}(\gamma\otimes\gamma)\right)^{1/p}, \left(\int d^p_{X,Y}\mathrm{d}\gamma\right)^{1/p}\right\} \quad (1 \leq p < \infty)$$

$$d_{\mathrm{GW},\infty}(\mathcal{X},\mathcal{Y}) := \inf_{\gamma\in\sqcup(\mu_X,\mu_Y)} \max\left\{\frac{1}{2}\sup_{\mathrm{supp}(\gamma\otimes\gamma)} m_{X,Y},\ \sup_{\mathrm{supp}\gamma} d_{X,Y}\right\}.$$

The fact that $d_{\mathrm{GW},p}$ and $d_{\mathrm{GW},\infty}$ are metrics can be argued as in the case of metric-measure spaces (see 3.1.20).

Proposition 3.2.6. *For each $1 \leq p \leq \infty$, $d_{\mathrm{GW},p}(\mathcal{X},\mathcal{Y})$ is realizaed by a coupling. Furthermore,*

$$\lim_{p\to\infty} d_{\mathrm{GW},p}(\mathcal{X},\mathcal{Y}) = d_{\mathrm{GW},\infty}(\mathcal{X},\mathcal{Y}).$$

Proof. For each integer $n \geq 1$, there exists a coupling γ_n in $\sqcup(\mu_X,\mu_Y)$ such that the right hand side expression in the definition of $d_{\mathrm{GW},p}(\mathcal{X},\mathcal{Y})$ is less than or equal to $d_{\mathrm{GW},p}(\mathcal{X},\mathcal{Y}) + 1/n$. By Lemma 3.1.11, without loss of generality, we can assume that γ_n weakly converges to a coupling γ. This also implies that $\gamma_n \otimes \gamma_n$ weakly converges to $\gamma\otimes\gamma$ ([4, Theorem 2.8]).

Case 1"$1 \leq p < \infty$":

Since $m_{X,Y}$ and $d_{X,Y}$ are continous and bounded below, by [37, Lemma 4.3] we have

$$\begin{aligned} \int m_{X,Y}^p \mathrm{d}(\gamma \otimes \gamma) &\leq \liminf_n m_{X,Y}^p \mathrm{d}(\gamma_n \otimes \gamma_n) \\ \int d_{X,Y}^p \mathrm{d}\gamma &\leq \liminf_n \int d_{X,Y}^p \mathrm{d}\gamma_n. \end{aligned} \tag{3.10}$$

For any real sequences (a_n) and (b_n), one can show that

$$\max(\liminf a_n, \liminf b_n) \leq \liminf \max(a_n, b_n).$$

Hence, we have

$$\begin{aligned} d_{\mathrm{GW},p}(\mathcal{X}, \mathcal{Y}) &\leq \max\left\{ \frac{1}{2}\left(\int m_{X,Y}^p \mathrm{d}(\gamma \otimes \gamma) \right)^{1/p}, \left(\int d_{X,Y}^p \mathrm{d}\gamma \right)^{1/p} \right\} \\ &\leq \liminf_n \max\left\{ \frac{1}{2}\left(\int m_{X,Y}^p \mathrm{d}(\gamma_n \otimes \gamma_n) \right)^{1/p}, \left(\int d_{X,Y}^p \mathrm{d}\gamma_n \right)^{1/p} \right\} \\ &\leq \liminf_n d_{\mathrm{GW},p}(\mathcal{X}, \mathcal{Y}) + \frac{1}{n} = d_{\mathrm{GW},p}(\mathcal{X}, \mathcal{Y}) \end{aligned} \tag{3.11}$$

Case 2 "$p = \infty$":

Note that if $1 \leq q \leq q' < \infty$, then $d_{\mathrm{GW},q}(\mathcal{X}, \mathcal{Y}) \leq d_{\mathrm{GW},q'}(\mathcal{X}, \mathcal{Y}) \leq d_{\mathrm{GW},\infty}(\mathcal{X}, \mathcal{Y})$. Hence we have

$$\lim_{q \to \infty} d_{\mathrm{GW},q}(\mathcal{X}, \mathcal{Y}) = \sup_q d_{\mathrm{GW},q}(\mathcal{X}, \mathcal{Y}) \leq d_{\mathrm{GW},\infty}(\mathcal{X}, \mathcal{Y}).$$

Let (q_n) be a sequence of real numbers greater than 1 converging to ∞. Let γ_n be the optimal coupling realizing $d_{\mathrm{GW},q_n}(\mathcal{X}, \mathcal{Y})$. Without loss of generality γ_n converges to γ weakly. This also implies that $\gamma_n \otimes \gamma_n$ converges to $\gamma \otimes \gamma$.

Let $0 \leq r < \max\left\{ \frac{1}{2} \sup_{\mathrm{supp}(\gamma \otimes \gamma)} m_{X,Y}, \sup_{\mathrm{supp}\gamma} d_{X,Y} \right\}$. Let $U := \{m_{X,Y}/2 > r\}$ and $V := \{d_{X,Y} > r\}$. Either $\gamma \otimes \gamma(U) > 0$ or $\gamma(V) > 0$. For now, let us assume that $\gamma \otimes \gamma(U) = 2m > 0$. By Portmanteau Theorem [10, Theorem 11.1.1]

$$2m \leq \liminf \gamma_n \otimes \gamma_n(U).$$

By passing to a subsequence if necessary, we can assume that $\gamma_n \otimes \gamma_n(U) \geq m > 0$ for all n. We have

$$\begin{aligned} d_{\mathrm{GW},q_n}(\mathcal{X}, \mathcal{Y}) &\geq \frac{1}{2}\left(\int m_{X,Y}^{q_n} \mathrm{d}(\gamma_n \otimes \gamma_n) \right)^{1/q_n} \\ &\geq r\,(\gamma_n \otimes \gamma_n(U))^{1/q_n} \geq r\, m^{1/q_n}. \end{aligned} \tag{3.12}$$

Hence, we get

$$\lim_{p\to\infty} d_{\mathrm{GW},p}(\mathcal{X},\mathcal{Y}) = \lim_{n} d_{\mathrm{GW},q_n}(\mathcal{X},\mathcal{Y}) \geq r. \tag{3.13}$$

Since $r < \max\left\{\frac{1}{2}\sup_{\mathrm{supp}(\gamma\otimes\gamma)} m_{X,Y},\ \sup_{\mathrm{supp}\gamma} d_{X,Y}\right\}$ was arbitrary, we get

$$\begin{aligned}\max\left\{\frac{1}{2}\sup_{\mathrm{supp}(\gamma\otimes\gamma)} m_{X,Y},\ \sup_{\mathrm{supp}\gamma} d_{X,Y}\right\} &\geq d_{\mathrm{GW},\infty}(\mathcal{X},\mathcal{Y})\\ &\geq \lim_{p\to\infty} d_{\mathrm{GW},p}(\mathcal{X},\mathcal{Y})\\ &\geq \max\left\{\frac{1}{2}\sup_{\mathrm{supp}(\gamma\otimes\gamma)} m_{X,Y},\ \sup_{\mathrm{supp}\gamma} d_{X,Y}\right\}.\end{aligned} \tag{3.14}$$

The case $P(V) > 0$ is handled similarly. □

To compare probability measures, which do not lie in the same functional space, one can consider a functional space Z containing an isometric copy of the spaces in question and use the Wasserstein distance in Z. The following proposition shows that the Gromov-Wasserstein is a lower bound for the Wasserstein distance (in Z).

Proposition 3.2.7. *Let $\mathcal{X} = (X, B, \pi_X, \mu_X)$ and $\mathcal{Y} = (Y, B, \pi_Y, \mu_Y)$ be* fmm*-spaces. Suppose that $\psi\colon Z \to B$ is 1-Lipschitz and $\imath\colon X \to Z$ and $\jmath\colon Y \to Z$ are isometric embeddings satisfying $\pi_X = \psi\circ\imath$ and $\pi_Y = \psi\circ\jmath$. Then, for any $1 \leq p \leq \infty$, we have*

$$d_{\mathrm{GW},p}(\mathcal{X},\mathcal{Y}) \leq d_{\mathrm{W},p}(\iota_*(\mu), \iota'_*(\mu_Y))\,.$$

Proof. Let γ be a coupling between $\imath_*(\mu_X)$ and $\imath_*(\mu_Y)$. Since $\imath(X)\times\jmath(Y)$ has full measure in $(Z\times Z, \gamma)$, we may view γ as a measure in $X\times Y$. Then, for $1 \leq p < \infty$, we have

$$\begin{aligned}&\frac{1}{2}\left(\iint |d_X(x,x') - d_Y(y,y')|^p d\gamma(x,y)d\gamma(x',y')\right)^{1/p}\\ &\leq \frac{1}{2}\left(\iint (d_Z(x,y) + d_Z(y,y'))^p \mathrm{d}\gamma(x,y)d\gamma(x',y')\right)^{1/p}\\ &\leq \frac{1}{2}\left(\iint d_Z(x,y)^p d\gamma(x,y)d\gamma(x',y')\right)^{1/p} + \frac{1}{2}\left(\iint d_Z(x,x')^p d\gamma(x,y)d\gamma(x',y')\right)^{1/p}\\ &= \left(\int d_Z(x,x')^p d\gamma(x,y)\right)^{1/p}.\end{aligned} \tag{3.15}$$

We also have

$$\begin{aligned}\left(\int d_B(\pi_X(x),\pi_Y(y))^p d\gamma(x,y)\right)^{1/p} &= \left(\int d_B(\psi(z),\psi(z'))^p d\gamma(z,z')\right)^{1/p}\\ &\leq \left(\int d_Z(z,z')^p d\gamma(z,z')\right)^{1/p}.\end{aligned} \tag{3.16}$$

Thus, for $1 \leq p < \infty$, the claim follows from (3.15) and (3.16) by taking the infimum over γ.

For $p = \infty$, we have

$$\frac{1}{2} \sup_{\text{supp}(\gamma\otimes\gamma)} |d_X(x,y) - d_{X'}(x',y')| \leq \frac{1}{2} \sup_{\text{supp}(\gamma\otimes\gamma)} (d_Z(x,x') + d_Z(y,y')) = \sup_{\text{supp}(\gamma\otimes\gamma)} d_Z(z,z'), \quad (3.17)$$

and

$$\sup_{\text{supp}\gamma} d_B(\pi_X(x), \pi_Y(x')) = \sup_{\text{supp}\gamma} d_B(\psi(x), \psi(x')) \leq \sup_{\text{supp}\gamma} d_Z(z,z'). \quad (3.18)$$

The last two inequalities complete the proof since the coupling γ between μ_X and μ_Y is arbitrary.

□

The following proposition relates the functional Gromov-Wasserstein and the functional box distances.

Proposition 3.2.8. *Let $M = \max\{\sup(m_{X,Y}), 2\sup(d_{X,Y})\}$. Then,*

$$d_{\text{GW},p}(\mathcal{X}, \mathcal{Y}) \leq \frac{1}{2} (\square_a(\mathcal{X}, \mathcal{Y}) + a^{1/p} M^{(p+1)/p}),$$

for all $p \geq 1$ and $a \geq 0$.

Proof. Assume $1 \leq p < \infty$ and $\square_a(\mathcal{X}, \mathcal{Y}) < \epsilon$. Let $\phi_X : [0,1] \to X$, $\phi_Y : [0,1] \to Y$ and $J \subseteq I$ be as in Definition 3.2.2. Let $P = (\phi_X, \phi_Y)_*(\lambda)$, which is a coupling between μ_X and μ_Y. We have

$$\begin{aligned}
\left(\int m^p_{X,Y} \mathrm{d}(\gamma\otimes\gamma) \right)^{1/p} &= \left(\iint m^p_{X,Y}(\phi_X(s), \phi_Y(s), \phi_X(t), \phi_Y(t) d\lambda(s) d\lambda(t)) \right)^{1/p} \\
&\leq \left(\iint_{[0,1]\times[0,1]-J\times J} m^p_{X,Y}(\phi_X(s), \phi_Y(s), \phi_X(t), \phi_Y(t) d\lambda(s) d\lambda(t)) \right)^{1/p} \\
&+ \left(\iint_{J\times J} m^p_{X,Y}(\phi_X(s), \phi_Y(s), \phi_X(t), \phi_Y(t) d\lambda(s) d\lambda(t)) \right)^{1/p} \\
&\leq (\lambda\otimes\lambda)([0,1]\times[0,1] \setminus J\times J)^{1/p}\epsilon + (\lambda\otimes\lambda)(J\times J)^{1/p} M \\
&\leq \epsilon + \lambda(J)^{1/p} M \leq \epsilon + (a\epsilon)^{1/p} M.
\end{aligned} \quad (3.19)$$

We also have

$$\begin{aligned}
\left(\int d^p_{X,Y} \mathrm{d}\gamma \right)^{1/p} &= \left(\int d^p_{X,Y}(\phi_X(s), \phi_Y(s)) d\lambda(s) \right)^{1/p} \\
&\leq \left(\int_{[0,1]-J} d^p_{X,Y}(\phi_X(s), \phi_Y(s)) d\lambda(s) \right)^{1/p} + \left(\int_J d^p_{X,Y}(\phi_X(s), \phi_Y(s)) d\lambda(s) \right)^{1/p} \\
&\leq \frac{1}{2}(\epsilon + (a\epsilon)^{1/p} M)
\end{aligned} \quad (3.20)$$

Therefore,

$$d_{\mathrm{GW},p}(\mathcal{X},\mathcal{Y}) \leq \frac{1}{2}(\epsilon + (a\epsilon)^{1/p}M). \tag{3.21}$$

Since $\epsilon > \Box_a(\mathcal{X},\mathcal{Y})$ was arbitrary and $\Box_a(\mathcal{X},\mathcal{Y}) \leq M$, we have

$$d_{\mathrm{GW},p}(\mathcal{X},\mathcal{Y}) \leq \frac{1}{2}(\Box_a(\mathcal{X},\mathcal{Y}) + (a\Box_a(\mathcal{X},\mathcal{Y}))^{1/p}M) \leq \frac{1}{2}(\Box_a(\mathcal{X},\mathcal{Y}) + a^{1/p}M^{(p+1)/p}). \tag{3.22}$$

□

The following theorem characterizes $d_{\mathrm{GW},\infty}(\mathcal{X},\mathcal{Y})$ in terms of uniformly distributed sequences.

Theorem 3.2.9. *Let $\mathcal{X}$ and $\mathcal{Y}$ be fmm-spaces, and U_X and U_Y denote the set of uniformly distributed sequences in X and Y, respectively.*

$$d_{\mathrm{GW},\infty}(\mathcal{X},\mathcal{Y}) = \inf_{(x_n)\in U_X,(y_n)\in U_Y} \max\left\{\frac{1}{2}\sup_{i,j} m_{X,Y}(x_i,y_i,x_j,y_j),\, \sup_i d_{X,Y}(x_i,y_i)\right\}.$$

Furthermore, the infimum in the right-hand side is realized.

Proof. Let us denote the right-hand side by α. Let γ be the optimal coupling realizing $d_{\mathrm{GW},\infty}(\mathcal{X},\mathcal{Y})$. Then,

$$d_{\mathrm{GW},\infty}(\mathcal{X},\mathcal{Y}) = \max\left\{\frac{1}{2}\sup_{\mathrm{supp}(\gamma\otimes\gamma)} m_{X,Y},\, \sup_{\mathrm{supp}\gamma} d_{X,Y}\right\}.$$

Let (x_n, y_n) be an equidistributed sequence with respect to γ in $\mathrm{supp}\gamma$. Then $(x_n) \in U_X$ and $(y_n) \in U_Y$, and we have

$$\begin{aligned}\alpha &\leq \max\left\{\frac{1}{2}\sup_{i,j} m_{X,Y}(x_i,y_i,x_j,y_j),\, \sup_i d_{X,Y}(x_i,y_i)\right\}\\ &\leq \max\left\{\frac{1}{2}\sup_{\mathrm{supp}(\gamma\otimes\gamma)} m_{X,Y},\, \sup_{\mathrm{supp}\gamma} d_{X,Y}\right\} = d_{\mathrm{GW},\infty}(\mathcal{X},\mathcal{Y}).\end{aligned} \tag{3.23}$$

Now, let us prove the inequality in the other direction. Let $p \geq 1$, and $\epsilon > 0$. Let $(x_n) \in U_X$ and $(y_n) \in U_Y$. Let $\mathcal{E}_n$ be the functional metric measure space with underlying set $\{1,\ldots,n\}$ with the normalizing counting measure, (pseudo)-metric given by $d_{E_n}(i,j) = d_X(x_i,x_j)$ and the function $\pi_E(i) = \pi_X(x_i)$. Similarly, define $\mathcal{F}_n$ for using (y_n). By Proposition 3.2.7, we have

$$d_{\mathrm{GW},p}(\mathcal{X},\mathcal{E}_n) \leq d_{\mathrm{W},p}(\mu_X, \sum_{i=1}^n \delta_{x_i}/n), \quad d_{\mathrm{GW},p}(\mathcal{Y},\mathcal{F}_n) \leq d_{\mathrm{W},p}(\mu_Y, \sum_{i=1}^n \delta_{y_i}/n)$$

Since the Wasserstein distances in the above inequalities converge to 0, we can choose n-large enough so that the Gromov-Wasserstein distances in the above inequalities are less than ϵ. If we use diagonal coupling P between $\mathcal{E}_n, \mathcal{Y}_n$ (i.e. $P(i,i) = 1/n$), we get

$$d_{\mathrm{GW},p}(\mathcal{E}_n, \mathcal{F}_n) \leq \max\left\{\frac{1}{2}\sup_{i,j} m_{X,Y}(x_i, y_i, x_j, y_j),\ \sup_i d_{X,Y}(x_i, y_i)\right\}.$$

This in turn implies that

$$d_{\mathrm{GW},p}(\mathcal{X}, \mathcal{Y}) \leq \max\left\{\frac{1}{2}\sup_{i,j} m_{X,Y}(x_i, y_i, x_j, y_j),\ \sup_i d_{X,Y}(x_i, y_i)\right\} + 2\epsilon.$$

Since $(x_n) \in U_X$, $(y_n) \in U_Y$ and $\epsilon > 0$ was arbitrary, we get

$$d_{\mathrm{GW},p}(\mathcal{X}, \mathcal{Y}) \leq \alpha.$$

Since $p \geq 1$ was arbitrary, by Proposition 3.2.6

$$d_{\mathrm{GW},\infty}(\mathcal{X}, \mathcal{Y}) = \lim_{p\to\infty} d_{\mathrm{GW},p}(\mathcal{X}, \mathcal{Y}) \leq \alpha.$$

This shows that the $(x_n) \in U_X$ and $(y_n) \in U_Y$ constructed in the first paragraph of the proof realizes the infimum. □

CHAPTER 4

AUGMENTED MATRIX DISTRIBUTION

The matrix distribution is well known to be an invariant of metric measure spaces (Gromov's mm-reconstruction theorem [15]). Therefore, a distance function on the set of matrix distributions can be interpreted as a measure of dissimilarity between metric measure spaces. This section introduces the functional version of matrix distributions and studies the Wasserstein and Prokhorov distance between them. Recall that $\mathcal{R}_B$ is the set of augmented distance matrices (see you 2.3). $\mathcal{R}_B$ can be endowed with an extended metric given by

$$\partial([r_{i,j}|b_i],[r'_{i,j}|b'_i)] := \max\big(\frac{1}{2}\sup_{i,j}|r_{i,j}-r'_{i,j}|, \sup_i d_B(b_i,b'_i)\big).$$

For a non-negative integer N, $\mathcal{R}_B^N$ and the distance ∂^N defined similarly, except the indices i, j run through 1 to N.

Remark 4.0.1. The extended metric space $(\mathcal{R}_B, \partial)$ is not separable and hence not Polish. Polish structures offer a convenient setting for probability theory. For instance, as discussed in section 3.1, every metric measure space is *almost isomorphic* to an interval with Lebesgue measure, a countable set of atoms, or a disjoint union of both ([5], Theorem 9.4.7). As another example, in computing the Monge–Kantorovich problem, if the underlying sets are Polish spaces, the existence of the optimal coupling that minimizes the total cost (under some conditions on cost function) is guaranteed ([37], Theorem 4.1). That is why, we regard $\mathcal{R}_B$ as a (closed) topological subspace of $\mathbb{R}^{\mathbb{N}\times\mathbb{N}} \times B^{\mathbb{N}}$ with weak topology. This is the coarsest topology for which the projection maps $P_N : \mathcal{R}_B \to (\mathcal{R}_B^N, \partial^N)$ are continuous (see [36], 2.1). Throughout this paper, any probability measure on $\mathcal{R}_B$ is defined on the Borel σ-algebra induced by this topology.

The following lemma shows that ∂ is a lower semicontinuous function with respect to weak topology.

Lemma 4.0.2. *The extended distance function $\partial : \mathcal{R}_B \times \mathcal{R}_{\mathcal{B}} \to [0, \infty]$ is lower semicontinuous.*

Proof. Let $P^N : \mathcal{R}_B \to \mathcal{R}_B^N$ denote the projection map. Note that $\partial^N \circ P^N \uparrow \partial$ pointwise, and ∂ as a pointwise supremum of a sequence of continuous functions is lower semicontinuous. □

Definition 4.0.3 (Augmented Distance Matrix Distribution)**.** Let $\mathcal{X} = (X, B, \pi, \mu)$ be a functional metric measure space. Let $\mathcal{F}_\mathcal{X} : X^\infty \to \mathcal{R}_B$ be the map $(x_i) \mapsto (d_X(x_i, x_j), \pi(x_i))$. The *augmented matrix distribution* of $\mathcal{X}$ is $\mathcal{D}_\mathcal{X} := (\mathcal{F}_\mathcal{X})_*(\mu^\infty)$. For a positive integer N, $\mathcal{F}^N_\mathcal{X} : X^N \to \mathcal{R}^N_B$ is defined similarly and $\mathcal{D}^N_\mathcal{X} := (\mathcal{F}^N_\mathcal{X})_*(\mu^N)$.

Remark 4.0.4. The map $\mathcal{F}_\mathcal{X} : (X^\infty, d^\infty_X) \to (\mathcal{R}_B, \partial)$ (and $\mathcal{F}^N_\mathcal{X}$) defined in the above definition is 1-Lipschitz.

Theorem 4.0.5. *(Functional Gromov's Reconstruction Theorem)*

$$\mathcal{X} \simeq \mathcal{Y} \textit{ if and only if } \mathcal{D}_\mathcal{X} = \mathcal{D}_\mathcal{Y}.$$

Proof. Clearly, $\mathcal{X} \simeq \mathcal{Y}$ implies that $\mathcal{D}_\mathcal{X} = \mathcal{D}_\mathcal{Y}$. Now, suppose that $\mathcal{D}_\mathcal{X} = \mathcal{D}_\mathcal{Y}$. U_X is Borel measurable set of full measure [22, Lemma 2.4], hence its image $C_X := \mathcal{F}_\mathcal{X}(U_X)$ is an analytical set, which has full measure in the completion of $\mathcal{D}_\mathcal{X}$ by [10, Theorem 13.2.6]. Define C_Y similarly. Therefore, $C_X \cap C_Y$ is full measure in the completion of $\mathcal{D}_\mathcal{X} = \mathcal{D}_\mathcal{Y}$. Let $(x_n) \in U_X$, $(y_n) \in U_Y$ such that $\mathcal{F}_\mathcal{X}((x_n)) = \mathcal{F}_\mathcal{Y}((y_n))$. The setting $\phi(x_i) = y_i$ defines a well defined isometry between (x_n) and (y_n). Note that $\pi_X(x_i) = \pi_Y(y_i)$ and since (x_n) (resp. (y_n)) is dense in X (resp. Y), ϕ extends to an isometry between $\mathcal{X}, \mathcal{Y}$. □

The next theorem relates the Prokhorov distance between functional matrix distributions and the Gromov-Wasserstein distance between their fmm-spaces.

Theorem 4.0.6. *Let $\mathcal{X} = (X, B, \pi, \mu)$ and $\mathcal{X}' = (X', B', \pi', \mu')$ be functional metric measure spaces. For any $a \geq 0$, we have*

$$d_{GP,2a}(\mathcal{X}, \mathcal{X}') \leq d_{P,a}(\mathcal{D}_\mathcal{X}, \mathcal{D}_{\mathcal{X}'}) \leq d_{GP,0}(\mathcal{X}, \mathcal{X}').$$

Furthermore, if $a\, d_{P,a}(\mathcal{D}_\mathcal{X}, \mathcal{D}_{\mathcal{X}'}) < 1$, then

$$d_{P,a}(\mathcal{D}_\mathcal{X}, \mathcal{D}_{\mathcal{X}'}) = d_{P,0}(\mathcal{D}_\mathcal{X}, \mathcal{D}_{\mathcal{X}'}) = d_{GP,0}(\mathcal{X}, \mathcal{X}'). \tag{4.1}$$

In the proof of this theorem, we are going to use the following lemmas:

Lemma 4.0.7. *Let X and Y be metric spaces, μ, μ' be Borel probability measures on X, and $F : X \to Y$ be λ-Lipschitz. For all $a \geq 0$, we have*

$$d_{P,a}(F_*\mu, F_*\mu') \leq \lambda\, d_{P,\lambda a}(\mu, \mu').$$

Proof. Assume $d_{P,\lambda a}(\mu,\mu') < \frac{\epsilon}{\lambda}$. We need to show $d_{P,a}(F_*\mu, F_*\mu') \leq \epsilon$. For any $A \subseteq Y$, $F^{-1}(A^\epsilon)$ contains $(F^{-1}(A))^{\frac{\epsilon}{\lambda}}$. Therefore,

$$F_*\mu(A) = \mu(F^{-1}(A)) \leq \nu((F^{-1}(A))^{\frac{\epsilon}{\lambda}}) + a\epsilon \leq \nu(F^{-1}(A^\epsilon)) + a\epsilon = F_*\nu(A^\epsilon) + a\epsilon.$$

□

Lemma 4.0.8. *Let X be a metric space and μ, μ' be Borel probability measures on X. Then,*

$$d_{P,0}(\mu,\mu') = d_{P,0}(\mu^\infty, \mu'^\infty).$$

Proof. "$d_{P,0}(\mu,\mu') \leq d_{P,0}(\mu^\infty,\mu'^\infty)$": This follows from Lemma 4.0.7 as the projection map $X^\infty \to X$ is 1-Lipschitz.

"$d_{P,0}(\mu^\infty,\mu'^\infty) \leq d_{P,0}(\mu,\mu')$": Let $d_{P,0}(\mu,\mu') < \epsilon$. We need to show for each measurable A in X^∞, $\mu^\infty(A) \leq \mu'^\infty(A^\epsilon)$. It is enough to show this for A of the form $A = A_1 \times A_2 \times \cdots \times A_n \times X^\infty$ where each A_i is measurable in X. Since we are using the sup metric, for such an A, we have $A^\epsilon = A_1^\epsilon \times A_2^\epsilon \times \cdots \times A_n^\epsilon \times X^\infty$. Hence,

$$\mu(A) = \mu(A_1)\ldots\mu(A_n) \leq \mu'(A_1^\epsilon)\ldots\mu'(A_1)^\epsilon = \mu'^\infty(A^\epsilon).$$

□

Lemma 4.0.9. *Let $\mathcal{X} = (X, B, \pi, \mu)$, $\mathcal{X}' = (X', B', \pi', \mu')$ be functional metric measure spaces and (x_n), (x'_n) be uniformly distributed sequence in X and X' respectively. Then we have*

$$d_{GP,0}(\mathcal{X},\mathcal{X}') \leq \partial((r,b),(r',b')).$$

where $(r,b) := (r_{i,j}, b_i)$ and $(r',b') := (r'_{i,j}, b'_i))$ are augmented distance matrices induced by (x_n) and (x'_n) respectively.

Proof. Denote the right side of the inequality by α and let Z be the disjoint union of X and X' endowed with the metric d_Z extending d_X and $d_{X'}$. For x in X and x' in X',

$$d_Z(x,x') = \inf_n d_X(x,x_n) + \alpha + d_{X'}(x',x'_n).$$

Then d_Z is a metric and the map $\psi : Z \to B$ extending π and π' is 1-Lipschitz. Given $A \subseteq Z$, we have

$$\mu(A) = \lim_{n\to\infty} \frac{\#\{i : i \leq n,\, x_i \in A\}}{n} \leq \lim_{n\to\infty} \frac{\#\{i : i \leq n,\, x'_i \in A^\alpha\}}{n} = \mu'(A^\alpha).$$

Therefore, $d_{GP,0}(\mathcal{X},\mathcal{X}') \leq \alpha$. □

Proof of Theorem 4.0.6. "$d_{P,a}(\mathcal{D}_{\mathcal{X}}, \mathcal{D}'_X) \leq d_{GP,0}(\mathcal{X}, \mathcal{X}')$":

Let $d_{GP,0}(\mathcal{X}, \mathcal{X}') < \epsilon$. Let $\psi : Z \to B$, $\iota : X \to Z$ and $\iota : X' \to Z$ be as in Definition 3.2.3. Note that $\mathcal{D}_X = (F_Z)_*(\iota_*^\infty(\mu^\infty))$ and the same holds for $\mathcal{D}_{X'}$. By Lemma 4.0.8, we have

$$d_{P,0}(\iota_*(\mu^\infty), \iota_*(\mu'^\infty)) = d_{P,0}(\iota_*(\mu), \iota_*(\mu')) \leq \epsilon.$$

Since F_Z is 1-Lipschitz, by Lemma 4.0.7 we have $d_{P,a}(\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}'}) \leq d_{P,0}(\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}'}) \leq \epsilon$.

"$d_{GP,2a}(\mathcal{X}, \mathcal{X}') \leq d_{P,a}(\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}'})$:"

The result follows trivially if $d_{P,a}(\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}'}) = \frac{1}{a}$, so without loss of generality we can assume that $a\, d_{P,a}(\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}'}) < 1$. Let $d_{P,a}(\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}'}) < \epsilon' < \epsilon < \frac{1}{a}$. Let $U_X \subseteq X^\infty$ (resp. $U_{X'} \subseteq X'^\infty$) be the set of uniformly distributed sequences in X (resp. X'). By ([22], Lemma 2.4), U_X and $U_{X'}$ has full measure. Let C_X (resp. $C_{X'}$) denote the closure of the image of U_X (resp. $U_{X'}$) under F_X (resp. $F_{X'}$). We have:

$$\mathcal{D}_{\mathcal{X}'}(C_X^{\epsilon'} \cap C_{X'}) = \mathcal{D}_{\mathcal{X}'}(C_X^{\epsilon'}) \geq \mathcal{D}_{\mathcal{X}}(C_X) - a\epsilon' = 1 - a\epsilon' > 0.$$

Therefore, there exists a uniformly distributed sequence (x_n) in X and (x'_n) in X' such that $\partial(F_X((x_n)), F_{X'}((x'_n))) \leq \epsilon$. By lemma 4.0.9, we have $d_{GP,2a}(\mathcal{X}, \mathcal{X}') \leq d_{GP,0}(\mathcal{X}, \mathcal{X}') \leq \epsilon$. □

Proposition 4.0.10. *Let $1 \leq p < \infty$ and (X, d_X, μ_X) be a metric measure space such that μ_X has finite moments of order p. Given an integer $N > 0$, real $\epsilon > 0$ and $1 \leq p < \infty$, let*

$$U_{X,p}^{N,\epsilon} := \{(x_i) \in X^N : d_{\mathrm{W},p}(\mu_X, \sum_{i=1}^N \delta_{x_i}/N) \leq \epsilon\}.$$

For N large enough,

$$\mu^N(U_{X,p}^{N,\epsilon}) \geq 1 - \epsilon.$$

Proof. Let $P_p(X)$ denote the set of Borel probability measures on X with finite moments of order p. It is metrizable by $d_{\mathrm{W},p}$, and the corresponding convergence is the weak convergence [37, Theorem 6.9]. Furthermore, $P_p(X)$ is complete and separable [37, Theorem 6.18]. Let $\pi_N : X^\infty \to X^N$ denote the projection to the first N coordinates.

Let $\psi_N : X^\infty \to P_p(X)$ be the map given by

$$\psi_N((x_n)) := \sum_{i=1}^N \delta_{x_i}/N.$$

This is a continuous map. By Varadarajan Theorem [10, Theorem 11.4.1], (ψ_N) converges to μ_X almost surely. By [10, Theorem 9.2.1], (ψ_N) converges to μ in probability. Hence, for N large enough,

$$\begin{aligned} 1-\epsilon &\leq \mu^{\infty}(d_{\mathrm{W},p}(\mu_X,\psi_N)\leq \epsilon) \\ &= \mu^{\infty}(\pi_N^{-1}(U_{X,p}^{N,\epsilon})) = \mu^N(U_{X,p}^{N,\epsilon}). \end{aligned} \tag{4.2}$$

□

Proposition 4.0.11. *Let $\mathcal{X}$ and $\mathcal{Y}$ be fmm-spaces. Let $U_{X,p}^{N,\epsilon}$ and $U_{Y,p}^{N,\epsilon}$ be defined as in Proposition 4.0.10. Let $(x_n)\in U_{X,p}^{N,\epsilon}$, and $(y_n)\in U_{Y,p}^{N,\epsilon}$ Then,*

$$\partial^N(\mathcal{F}_{\mathcal{X}}^N((x_n)),\mathcal{F}_{\mathcal{Y}}^N((y_n))) \geq d_{\mathrm{GW},p}(\mathcal{X},\mathcal{Y})-2\epsilon.$$

Proof. Let $\mathcal{E}_X$ be the fmm with the underlying set $\{1,\dots,n\}$, with normalizing counting measure, with (pseudo)-metric given by $d_E(i,j)=d_X(x_i,x_j)$ and with the 1-Lipschitz function sending i to $\pi_X(x_i)$. Similarly, define $\mathcal{E}_Y$ using (y_n). Note that, by Proposition 3.2.7, $d_{\mathrm{GW},p}(\mathcal{E}_X,\mathcal{X})\leq\epsilon$ and $d_{\mathrm{GW},p}(\mathcal{E}_Y,\mathcal{Y})\leq\epsilon$. If γ is the diagonal coupling between the measures of $\mathcal{E}_X$ and $\mathcal{E}_Y$, then we have

$$\begin{aligned} \partial^N(\mathcal{F}_{\mathcal{X}}^N((x_n)),\mathcal{F}_{\mathcal{Y}}^N((y_n))) &= \max\left\{\frac{1}{2}\sup_{\mathrm{supp}(\gamma\otimes\gamma)} m_{E_X,E_Y},\ \sup_{\mathrm{supp}\gamma} d_{E_X,E_Y}\right\} \\ &\geq d_{\mathrm{GW},p}(\mathcal{E}_X,\mathcal{E}_Y)\geq d_{\mathrm{GW},p}(\mathcal{X},\mathcal{Y})-2\epsilon. \end{aligned} \tag{4.3}$$

□

The following theorem, relates the Gromov-Wasserstein distance and the Wasserstein distance between their matrix distributions. Although ∂ does not metrize $\mathcal{R}_B$, we are still going to denote the optimal transport cost with respect to ∂ by $d_{\mathrm{W},p}(\mathcal{D}_{\mathcal{X}},\mathcal{D}_{\mathcal{Y}})$.

Theorem 4.0.12. *Let $\mathcal{X}$ and $\mathcal{Y}$ be bounded mm-fields. Then, for any $p\geq 1$, we have*

$$\lim_N (d_{\mathrm{W},p}(\mathcal{D}_{\mathcal{X}}^N,\mathcal{D}_{\mathcal{Y}}^N)) = d_{\mathrm{W},p}(\mathcal{D}_{\mathcal{X}},\mathcal{D}_{\mathcal{X}'}) = d_{\mathrm{GW},\infty}(\mathcal{X},\mathcal{X}')$$

Proof. The projection map $\mathcal{R}_B^{N+1}\to\mathcal{R}_B^N$ is 1-Lipschitz and pushforwards $\mathcal{D}_{\mathcal{X}}^{N+1}$ and $\mathcal{D}_{\mathcal{Y}}^{N+1}$ to $\mathcal{D}_{\mathcal{X}}^N$ and $\mathcal{D}_{\mathcal{Y}}^N$ respectively. This implies that $d_{\mathrm{W},p}(\mathcal{D}_{\mathcal{X}}^N,\mathcal{D}_{\mathcal{Y}}^N)\leq d_{\mathrm{W},p}(\mathcal{D}_{\mathcal{X}}^{N+1},\mathcal{D}_{\mathcal{Y}}^{N+1})$. By a similar argument, we get $d_{\mathrm{W},p}(\mathcal{D}_{\mathcal{X}}^N,\mathcal{D}_{\mathcal{Y}}^N)\leq d_{\mathrm{W},p}(\mathcal{D}_{\mathcal{X}},\mathcal{D}_{\mathcal{Y}})$. Therefore, we have

$$\lim_N d_{\mathrm{W},p}(\mathcal{D}_{\mathcal{X}}^N,\mathcal{D}_{\mathcal{Y}}^N) = \sup_N d_{\mathrm{W},p}(\mathcal{D}_{\mathcal{X}}^N,\mathcal{D}_{\mathcal{Y}}^N) \leq d_{\mathrm{W},p}(\mathcal{D}_{\mathcal{X}},\mathcal{D}_{\mathcal{Y}}). \tag{4.4}$$

Since ∂ is lower semicontinuous by Lemma 4.0.2, by a similar argument as in the proof of Proposition 3.2.6, one can show that $d_{\mathrm{W},\infty}(\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{Y}}) = \lim_{p\to\infty} d_{\mathrm{W},p}(\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{Y}})$. Therefore, without loss of generality, we can assume that $p < \infty$.

Now, let us show that $d_{\mathrm{W},p}(\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{Y}}) \leq d_{\mathrm{GW},\infty}(\mathcal{X}, \mathcal{Y})$. Let γ be the optimal coupling between μ_X and μ_Y realizing $d_{\mathrm{GW},\infty}(\mathcal{X}, \mathcal{Y})$. Let $\psi_X : (X \times Y)^\infty \to \mathcal{R}_B$ be the map given by $(x_n, y_n)_n \mapsto \mathcal{F}_{\mathcal{X}}((x_n)_n)$. Define ψ_Y similarly. Then $Q := (\psi_X, \psi_Y)_*(\gamma^\infty)$ is a coupling between $\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{Y}}$. We have

$$\begin{aligned}
d_{\mathrm{W},p}(\mathcal{X}, \mathcal{Y}) &\leq \left(\int \partial^p dQ\right)^{1/p} \\
&= \left(\int_{\mathrm{supp}\gamma^\infty} \partial^p(\psi_X((x_n, y_n)_n), \psi_Y((x_n, y_n)_n)\mathrm{d}\gamma^\infty((x_n, y_n)_n)\right)^{1/p} \\
&= \left(\int_{(\mathrm{supp}\gamma)^\infty} \max\big(\frac{1}{2}\sup_{i,j} m_{X,Y}(x_i, y_i, x_j, y_j), \sup_i d_{X,Y}(x_i, y_i)\big)^p \mathrm{d}\gamma^\infty((x_n, y_n)_n)\right)^{1/p} \\
&\leq \max\left\{\frac{1}{2}\sup_{\mathrm{supp}(\gamma\otimes\gamma)} m_{X,Y},\ \sup_{\mathrm{supp}\gamma} d_{X,Y}\right\} = d_{\mathrm{GW},\infty}(\mathcal{X}, \mathcal{Y}).
\end{aligned} \tag{4.5}$$

It remains to show that $d_{\mathrm{GW},\infty}(\mathcal{X}, \mathcal{Y}) \leq \lim_N d_{\mathrm{W},p}(\mathcal{D}^N_{\mathcal{X}}, \mathcal{D}^N_{\mathcal{Y}})$. Let $0 < \epsilon < 1/2$. By Proposition 3.2.6, there exist $1 \leq q < \infty$ so that $d_{\mathrm{GW},\infty}(\mathcal{X}, \mathcal{Y}) \leq d_{\mathrm{GW},q}(\mathcal{X}, \mathcal{Y}) + \epsilon$. Let

$$U^{N,\epsilon}_{X,q} := \{(x_i) \in X^N : d_{\mathrm{W},q}(\mu_X, \sum_{i=1}^N \delta_{x_i}/N) \leq \epsilon\}.$$

Define $U^{N,\epsilon}_{Y,q}$ similarly. By Proposition 4.0.10, there exists N large enough so that, $\mu^n_X(U^{N,\epsilon}_{X,q}) \geq 1-\epsilon$ and $\mu^n_Y(U^{N,\epsilon}_{Y,q}) \geq 1-\epsilon$. If we define $C^{N,\epsilon}_{X,q} := \mathcal{F}^N_{\mathcal{X}}(U^{N,\epsilon}_{X,q})$, $C^{N,\epsilon}_{Y,q} := \mathcal{F}^N_{\mathcal{X}}(U^{N,\epsilon}_{Y,q})$, both of this sets are analytical, hence measurable in the completion of $\mathcal{D}^N_{\mathcal{X}}$ and $\mathcal{D}^N_{\mathcal{Y}}$ respectively by [10, Theorem 13.2.6], with measures greater then or equal to $1-\epsilon$. Let Q be the coupling realizing $d_{\mathrm{W},p}(\mathcal{D}^N_{\mathcal{X}}, \mathcal{D}^N_{\mathcal{Y}})$. Note that we have

$$Q(C^{N,\epsilon}_{X,q} \times C^{N,\epsilon}_{Y,q}) \geq 1 - 2\epsilon. \tag{4.6}$$

By Proposition 4.0.11, we also have

$$\partial^N|_{C^{N,\epsilon}_{X,q}\times C^{N,\epsilon}_{Y,q}} \geq d_{\mathrm{GW},q}(\mathcal{X}, \mathcal{Y}) - 2\epsilon \geq d_{\mathrm{GW},\infty}(\mathcal{X}, \mathcal{Y}) - 3\epsilon. \tag{4.7}$$

Therefore,

$$\begin{aligned}
d_{\mathrm{W},p}(\mathcal{D}^N_{\mathcal{X}}, \mathcal{D}^N_{\mathcal{Y}}) &= \left(\int \partial^N \mathrm{d}\gamma\right)^{1/p} \\
&\geq (d_{\mathrm{GW},\infty}(\mathcal{X}, \mathcal{Y}) - 3\epsilon)(1-2\epsilon)^{1/p}.
\end{aligned} \tag{4.8}$$

This implies that

$$\lim_N d_{\mathrm{W},p}(\mathcal{D}^N_{\mathcal{X}}, \mathcal{D}^N_{\mathcal{Y}}) \geq (d_{\mathrm{GW},\infty}(\mathcal{X}, \mathcal{Y}) - 3\epsilon)(1 - 2\epsilon)^{1/p}. \tag{4.9}$$

Since $0 < \epsilon < 1/2$ was arbitrary, we get

$$\lim_N d_{\mathrm{W},p}(\mathcal{D}^N_{\mathcal{X}}, \mathcal{D}^N_{\mathcal{Y}}) \geq d_{\mathrm{GW},\infty}(\mathcal{X}, \mathcal{Y}). \tag{4.10}$$

□

Example 4.0.13. (This example is adopted from ([30], Proposition 1.4.)

Let $\mathcal{X}_1 = (\,\{x\},\, \pi(x) = b,\, \delta_x\,)$ where b is an element of some polish space B. And for $m > 1$, let $\mathcal{X}_m = \big(\,\{x, x'\},\, d(x, x') = 1,\, \pi(x) = \pi(x') = b,\, (1 - \frac{1}{m})\delta_x + \frac{1}{m}\delta_{x'}\,\big)$. Then for every $a \geq 0$ we have

(1) $\square_1(\mathcal{X}_1, \mathcal{X}_m) = 2\, d_{GP,2}(\mathcal{X}_1, \mathcal{X}_m) = \frac{1}{m}$

(2) $d_{P,a}(\mathcal{D}_{\mathcal{X}_1}, \mathcal{D}_{\mathcal{X}_m}) = \frac{1}{a}$

(3) $d_{P,0}(\mathcal{D}_{\mathcal{X}_1}, \mathcal{D}_{\mathcal{X}_m}) = 1$

(4) $\square_0(\mathcal{X}_1, \mathcal{X}_m) = 2\, d_{GP,0}(\mathcal{X}_1, \mathcal{X}_m) = 1$

Proof. (1): For $m \geq 1$ Let ϕ_m be a parameter of $\mathcal{X}_m$. Note that for $m \geq 2$ any parameter ϕ_m of $\mathcal{X}_m$ divides the unit interval into two parts $J_1 := \phi^{-1}(x_1)$ and $J_2 := \phi^{-1}(x_2)$ of length $\frac{1}{m}$ and $1 - \frac{1}{m}$ respectively. For any $r, s \in J_1$ we have $|d_{X_1}(\phi_1(r), \phi_1(r)) - d_{X_m}(\phi_m(r), \phi_m(r))| = \frac{1}{m}$. This completes the proof. Note that the second equality follows from Proposition 3.2.4.

(2): Let $C_N := \{(r_{ij}, b) \in \mathcal{R}^N_B \;:\; \exists\, u, w \leq N \;\; s.t \;\; r_{u,w} = 1\}$. C_N is Borel measurable and we have

$$\begin{aligned}\mathcal{D}^N_{\mathcal{X}_m}(C_N) &= \{(y_1,\, y_2,\, \ldots,\, y_N) \in \mathcal{X}^N_m \;:\; \exists\, u, w \leq N \;\; s.t \;\; y_u \neq y_w\} \\ &= \sum_{k=1}^{N-1} \binom{N}{k} (\frac{1}{m})^k (1 - \frac{1}{m})^{N-k} \\ &= 1 - (\frac{1}{m})^N - (1 - \frac{1}{m})^N\end{aligned}$$

Let $\delta = 1 - (\frac{1}{m})^N - (1 - \frac{1}{m})^N$. We have $\mathcal{D}^N_{\mathcal{X}_1}(C_N) = \mathcal{D}^N_{\mathcal{X}_1}(B_\delta(C_N))$. Therefore, $\mathcal{D}^N_{\mathcal{X}_m}(C_N) \;\leq\; \mathcal{D}^N_{\mathcal{X}_1}(B_\epsilon(C_N) + \; a\epsilon$ if and only if $a\epsilon \;\geq\; \delta$. This shows that for any $N > 1$ $d_{P,a}(\mathcal{D}^N_{\mathcal{X}_1}, \mathcal{D}^N_{\mathcal{X}_m}) \;\geq\; \frac{\delta}{a}$. The result follows from the fact that $d_{P,a}(\mathcal{D}^N_{\mathcal{X}_1}, \mathcal{D}^N_{\mathcal{X}_m})$ is monotone non-decreasing in $N \in \mathbb{N}$ ([30], Lemma 3.3). In particular $\lim_{N\to\infty}(d_{P,a}(\mathcal{D}^N_{\mathcal{X}_1}, \mathcal{D}^N_{\mathcal{X}_m})) = d_{P,a}(\mathcal{D}_{\mathcal{X}_1}, \mathcal{D}_{\mathcal{X}_m})$.

(3) The proof is similar to (2).

(4) The proof is trivial.

□

This example shows that in Theorem 4.0.6, the condition $a\, d_{P,a}(\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}'}) < 1$ is necessary and cannot be removed.

Example 4.0.14. Distance functions defined on the class of fmm-spaces with the base B, $\mathbb{F}_B$ are generalizations of the ones defined on the class of mm-spaces, $\mathbb{M}$. In fact, for any $b \in B$ let $\mathbb{F}_b$ be the subspace of $\mathbb{F}_B$ containing fmm-spaces with constant Lipschitz functions equal to $b : \mathbb{F}_b := \{(X, B, \pi, \mu) : \ \pi(x) = b, \ \forall x \in X\}$. Then $(\mathbb{M}, D)$ is isometric to $(\mathbb{F}_b, D)$ where D is either the Gromov's box distance, the Gromov-Prohorov distance or the Gromov-Wasserstein distance.

CHAPTER 5

SUMMARY

This work studies the class of functional data defined on metric spaces, modeled as 1-Lipschitz maps between Polish spaces. We refer to them as functional metric spaces or simply functional spaces and denote them by triples $\mathcal{X} = (X, B, \pi)$. We construct a unique (up to isometry) Urysohn functional space $\mathcal{U}_B$ for the class of functional spaces over a fixed base space B in Theorem 2.2.11. We also show that a functional space is Urysohn if and only if it is universal and homogeneous (see Theorem 2.1.14).

In Section 2.4 we introduce the notion of the Gromov–Hausdorff distance $d_{\mathrm{GH,B}}$, between compact functional spaces and show that $d_{\mathrm{GH,B}}$ can be interpreted as the Hausdorff distance in $\mathcal{U}_B$, modulo the action of isometries of $\mathcal{U}_B$.

We also study the class of functional spaces where the domain of the 1-Lipschitz map is a metric measure space, referred to as functional metric measure space (fmm-space), and denoted by $\mathcal{X} = (X, B, \pi, \mu)$. We develop a functional analogues of the Gromov box distance $\square_a$, Gromov-Prokhorov distance $d_{GP,a}$, and Gromov-Wasserstein distance $d_{\mathrm{GW},p}$, that have been studied extensively for mm-spaces [15],[26], [37],[23]. Similar to the case of mm-spaces, we have in Proposition 3.2.4, that Gromov's box distance is equivalent to Gromov-Prokhorov distance. Moreover, we show that the Gromov–Wasserstein distance $d_{\mathrm{GW},\infty}(\mathcal{X}, \mathcal{Y})$ is realized by uniformly distributed sequences in Theorem 3.2.9.

We introduce the notion of augmented matrix distribution as an extension of the matrix distribution of mm-spaces to fmm-spaces. The matrix distribution is well known to be an invariant of metric measure spaces (Gromov's mm-reconstruction theorem [15]). We prove a functional version of this theorem that states that the augmented matrix distribution gives a faithful representation of its functional space:

$$\mathcal{X} \simeq \mathcal{Y} \text{ if and only if } \mathcal{D}_{\mathcal{X}} = \mathcal{D}_{\mathcal{Y}}.$$

Therefore, a distance function on the matrix distributions can be regarded as a measure of dissimilarity between fmm-spaces. In particular, we show that the Gromov-Prokhorov distance $d_{GP,0}$, the Gromov's box distance $\square_0$, and the Prokhorov distance $d_{P,a}$ (between matrix distributions) induce the same topology on the class of fmm-spaces (see Theorem 4.0.12 and Proposition 3.1.18).

This result generalizes the work of [30] where the author shows how the box distance between mm-spaces relates to the Prokhorov distance between their matrix distributions. Finally, we prove the following empirical approximation result for Gromov–Wasserstein distance:

$$\lim_{N}(d_{\mathrm{W},p}(\mathcal{D}_{\mathcal{X}}^{N},\mathcal{D}_{\mathcal{Y}}^{N})) = d_{\mathrm{W},p}(\mathcal{D}_{\mathcal{X}},\mathcal{D}_{\mathcal{X}'}) = d_{\mathrm{GW},\infty}(\mathcal{X},\mathcal{X}').$$

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BIOGRAPHICAL SKETCH

Soheil Anbouhi was born in Tehran, Iran. He earned his Bachelor's degree in Mathematics at Shahid Beheshti University, Tehran, Iran, in 2010. He also holds a Master's degree in Mathematics from Tehran University in Iran. In his thesis, he investigated the use of C^*-algebra techniques in problems arising in functional analysis. In August 2016, Soheil began his graduate studies in pure Mathematics at Florida State University (FSU). During his Ph.D. research, he studied universal spaces and the metric geometry of functional data (2019-2022). As a Ph.D. student, he also worked as a teaching assistant (T.A) and a solo instructor (2016-2022) for various undergraduate classes.