A Dynamical System with Many Independent Variables

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Take Classical Mechanics

We need a set of functions $R_1, R_2 \ldots R_n$ of the dyn. variables such that

$$[R_k, R_l] = \sum_m \chi_{klm} R_m \qquad (1)$$

where the χ's are any fns of the dyn. variables that do not take infinite values.

We then restrict ourselves to states for which $R_k \approx 0 \qquad (k = 1, 2, \ldots n) \quad (2)$ (The $\approx$ sign means an equality such that we cannot equate the P.B.'s of both sides with a general variable)

We may suppose the dyn. variables ξ vary according to the law

$$\frac{d\xi}{d\sigma_k} = [\xi, R_k] \qquad (k = 1 \ldots n) \qquad (3)$$

These equations are not in general integrable, since they lead to

$$\frac{d^2\xi}{d\sigma_l d\sigma_k} - \frac{d^2\xi}{d\sigma_k d\sigma_l} = [[\xi\, R_k] R_l] - [[\xi\, R_l] R_k] = [\xi \cdot [R_k\, R_l]] = \sum_m [\xi, \chi_{klm} R_m] \approx \sum_m \chi_{klm} \frac{d\xi}{d\sigma_m}$$

Thus applying $\dfrac{d^2}{d\sigma_l d\sigma_k} - \dfrac{d^2}{d\sigma_k d\sigma_l}$ to the dyn. variables results in a permissible change (non-zero in general) of the dyn. variables.

All the permissible changes in the dyn. variables are well-defined, but the labelling of those changes with changes in the σ's is not well-defined.

We may replace the R's by any indep. lin. fns of them, $\sum_l \alpha_{kl} R_l = R_k^*$, the α's being any fns of the dyn. variables that do not take on infinite values.

The R^*'s have similar properties to the R's, i.e. $R_k^* \approx 0$, follows from (2)

$$[R_k^*, R_l^*] = \sum_m \chi^*_{klm} R_m^* \qquad \text{" " (1)}$$

They define an equivalent dyn. system.

If the χ's are numbers, the conditions (1) are just the condition for the R's to be the elements of a cont. group.

In general, we may look upon the R's and χR's, with the χ's including all those coefficients that can be obtained by forming PB's, as the elements of a cont. group

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If the dyn. variables contain, e.g. the three ang. momenta m_x, m_y, m_z and the eqns (3) all leave $m_x^2 + m_y^2 + m_z^2$ invariant i.e. $[m_x^2 + m_y^2 + m_z^2, R_k] \approx 0$, we can take new R's such that $R_1 - m_x, R_2 - m_y, R_3 - m_z, R_4, R_5 \ldots, R_n$ all have their P.B's with m_x, m_y, m_z and with each other ≈ 0, and $[R_1 - m_x, R_2 - m_y] = m_z - R_3 + \sum_{l=4}^{n} a_{3l} R_l, \ldots$.

The R's, with suitable χR's, now form the elements of a group containing the rotation group as a factor group.