Four Component Spinor Notation

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The usual spinor notation requires 2 equations for one electron, 2^n equations for n electrons interacting.
The present — 1 — , 1 — .

We use suffixes $\alpha, \beta \ldots$ to have each four values, $1, 2, \dot{1}, \dot{2}$.

$\varepsilon_{\alpha\beta} = 0$ for α, β one dotted and one undotted and has the usual values otherwise.

First rule $A_\alpha B^\alpha = -A^\alpha B_\alpha$ holds as with usual spinor notation.

Second rule $X_{\alpha\beta} - X_{\beta\alpha} = \frac{1}{2} X_\gamma{}^\gamma \varepsilon_{\alpha\beta}$ holds under special conditions, namely

$$X_{\alpha\beta} = X_{\beta\alpha} \text{ for } \alpha, \beta \text{ one dotted and one undotted}$$

$$\text{and } X_1{}^1 + X_2{}^2 = X_{\dot{1}}{}^{\dot{1}} + X_{\dot{2}}{}^{\dot{2}}$$

A four-vector $A_{\alpha\beta}$ has the components $\begin{pmatrix} 0 & 0 & A_{1\dot{1}} & A_{1\dot{2}} \\ 0 & 0 & A_{2\dot{1}} & A_{2\dot{2}} \\ A_{\dot{1}1} & A_{\dot{1}2} & 0 & 0 \\ A_{\dot{2}1} & A_{\dot{2}2} & 0 & 0 \end{pmatrix}$ so that $A_{\alpha\beta} = A_{\beta\alpha}$

$$A_\alpha{}^\beta = \begin{pmatrix} 0 & 0 & A_1{}^{\dot{1}} & A_1{}^{\dot{2}} \\ 0 & 0 & A_2{}^{\dot{1}} & A_2{}^{\dot{2}} \\ A_{\dot{1}}{}^1 & A_{\dot{1}}{}^2 & 0 & 0 \\ A_{\dot{2}}{}^1 & A_{\dot{2}}{}^2 & 0 & 0 \end{pmatrix}$$

Its tensor components are $A_0 = A_{1\dot{1}} + A_{2\dot{2}}$ $\quad A_1 = A_{1\dot{2}} + A_{\dot{2}1}$ $\quad A_2 = -i(A_{1\dot{2}} - A_{\dot{2}1})$ $\quad A_3 = A_{1\dot{1}} - A_{2\dot{2}}$

$$(AB) = A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3 = \tfrac{1}{2}(A_0 + A_3)(B_0 - B_3) + \tfrac{1}{2}(A_0 - A_3)(B_0 + B_3) - \tfrac{1}{2}(A_1 + iA_2)(B_1 - iB_2) - \tfrac{1}{2}(A_1 - iA_2)(B_1 + iB_2)$$

$$= 2\left(A_{1\dot{1}} B_{2\dot{2}} + A_{2\dot{2}} B_{1\dot{1}} - A_{1\dot{2}} B_{\dot{2}1} - A_{\dot{2}1} B_{1\dot{2}}\right)$$

$$= 2\left(A_{1\dot{1}} B^{1\dot{1}} + A_{2\dot{2}} B^{2\dot{2}} + A_{1\dot{2}} B^{1\dot{2}} + A_{\dot{2}1} B^{\dot{2}1}\right) = A_{\alpha\beta} B^{\alpha\beta}$$

$$A_{\alpha\beta} A^\beta{}_\gamma = A_{\beta\alpha} A_\gamma{}^\beta = -A_{\gamma\beta} A^\beta{}_\alpha = \tfrac{1}{2}\left(A_{\alpha\beta} A^\beta{}_\gamma - A_{\gamma\beta} A^\beta{}_\alpha\right)$$

Second rule is now applicable, so

$$A_{\alpha\beta} A^\beta{}_\gamma = \tfrac{1}{4} A_{\delta\beta} A^{\beta\delta} \varepsilon_{\alpha\gamma} = \tfrac{1}{4} A^2 \varepsilon_{\alpha\gamma}$$

more generally

$$A_{\alpha\beta} B^\beta{}_\gamma - A_{\gamma\beta} B^\beta{}_\alpha = \tfrac{1}{2} A_{\delta\beta} B^{\beta\delta} \varepsilon_{\alpha\gamma} = \tfrac{1}{2}(AB)\, \varepsilon_{\alpha\gamma}$$

Besides the spinor $\varepsilon_{\alpha\beta}$, we need also the fundamental spinor $\omega_{\alpha\beta}$, defined to have the components

$$\omega_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \omega_\alpha{}^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Thus $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$. Also $\omega_{\alpha\beta} = \varepsilon_{\alpha\beta}$ for α, β undotted
$= -\varepsilon_{\alpha\beta}$ for α, β dotted

$$\omega_\alpha{}^\beta \omega_\beta{}^\gamma = \delta_\alpha^\gamma$$

$$A_\alpha{}^\beta \omega_\beta{}^\gamma = -\omega_\alpha{}^\beta A_\beta{}^\gamma$$

With $\alpha = 1, 2, \dot{1}, \dot{2}$, define $\dot{\alpha} = \dot{1}, \dot{2}, 1, 2$. Then condition for fourvector A to be real is $\overline{A_{\alpha\beta}} = A_{\dot{\alpha}\dot{\beta}}$

We have $\omega_{\alpha\beta} = -\omega_{\dot{\alpha}\dot{\beta}}$.