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The 1-Type of Algebraic K-Theory as a Multifunctor

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THE 1-TYPE OF ALGEBRAIC K-THEORY AS A MULTIFUNCTOR

By

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TABLE OF CONTENTS

ABSTRACT

It is known that the category of Waldhausen categories is a closed symmetric multicategory and algebraic K-theory is a functor from the category of Waldhausen categories to the category of spectra. By assigning to any Waldhausen category the fundamental groupoid of the 1-type of its K-theory spectrum, we get a functor from the category of Waldhausen categories to the category of Picard groupoids, since stable 1-types are classified by Picard groupoids. We prove that this functor is a multifunctor to a corresponding multicategory of Picard groupoids.

CHAPTER 1

INTRODUCTION

Given any Waldhausen category, $\mathcal{W}$, we can construct its K-theory spectrum, $K(\mathcal{W})$, and for any exact functor of Waldhausen categories $\mathcal{W} \to \mathcal{V}$, we get a morphism of spectra $K(\mathcal{W}) \to K(\mathcal{V})$. It is well known that given an infinite loop space X its 1-type can be realized as a Picard groupoid, $P_1(X)$ [5]. Therefore, realizing the 1-type of the K-theory of a Waldhausen category as a Picard groupoid gives us a functor from the category of Waldhausen categories to the category of Picard groupoids:

$$\mathbf{Wald} \xrightarrow{P_1 K} \mathbf{Pic}$$
$$\mathcal{W} \mapsto P_1(K(\mathcal{W}))$$

We also know that although the category of Waldhausen categories does not have a symmetric monoidal structure, it does have a (closed) symmetric multicategorical structure. (Blumberg and Mandell gave a proof of this in [4], as well as Zakharevich in [2].) We call the multifunctors of Waldhausen categories multiexact functors.

Similarly, we can show that Picard groupoids also form a (closed) symmetric multicategory when equipped with multimonoidal functors. Therefore, since P_1K is a functor between categories underlying multicategories, a natural question to ask is if it extends to a multifunctor. Our main result answers this question in the affirmative, namely we prove the following theorem:

Theorem 9.1. $\mathbf{Wald} \xrightarrow{P_1 K} \mathbf{Pic}$ *extends to a multifunctor.*

To be clear, if a map between multicategories is a just a functor between their underlying categories, then we say it "*extends to a multifunctor*" if there is a multifunctor between the multicategories such that when seen as a functor between the underlying categories, it agrees with the original one. This is defined in Definition 3.3.

This theorem is based on two main ingredients: (1) the existence of a universal determinant functor

$$\det_{\mathcal{W}} : w\mathcal{W} \to P_1K(\mathcal{W})$$

that satisfies certain properties, described in detail in section 8.1, where $w\mathcal{W}$ is the subcategory of weak equivalences of $\mathcal{W}$; and (2) the closed multicategorical structures on **Wald** and **Pic**. The former is known thanks to [2], [4] and we give some attention to the latter.

With these premises, our key ingredient is that the functor P_1K "respects the closed structures" in the following sense:

$$P_1K(\mathbf{Wald}(\mathcal{W},\mathcal{V})) \to \mathbf{Pic}(P_1K(\mathcal{W}), P_1K(\mathcal{V}))$$

which allows us to prove the theorem.

An application of this result is to use this multifunctor to study the algebraic structures on the crossed modules, which model Picard groupoids, coming from multiexact maps of Waldhausen categories. Aldrovandi shows there is an equivalence between bimonoidal functors and biextensions of crossed modules, which in fact extends to multimonoidal functors and multiextensions [3]. Therefore, an application of our results is to study the multiextensions corresponding to the multimonoidal functors of Picard groupoids under the multifunctor, $PD_{\bullet}$.

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* *

This thesis is organized as follows: we start with reminding the reader of relevant algebraic structures such as Waldhausen categories, Picard groupoids, and stable quadratic modules. We cover universal determinant functors as they play a major role in proving the main result. We also recall the K-theory construction for a Waldhausen category by forming the S-construction. Finally, we end with stating and proving two propositions that allow us to prove the main theorem. Certain cumbersome but necessary computations are in the appendix.

CHAPTER 2

PRELIMINARIES

Definition 2.1. Let $[n]$ denote the finite, nonempty, totally ordered set $\{0 < 1 < \cdots < n\}$ considered as a category. This means the objects are $0, 1, \ldots, n$ and

$$\mathrm{Hom}_{[n]}(j,k) = \begin{cases} \emptyset & j > k \\ \mathrm{id}_j & j = k \\ j \to k & j < k \end{cases}$$

Definition 2.2. For any two categories $\mathcal{C}$ and $\mathcal{D}$, we can form another category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ called the *functor category* whose objects are functors $F : \mathcal{C} \to \mathcal{D}$ and whose morphism are natural transformations between these functors.

Definition 2.3. The *arrow category* of some category $\mathcal{C}$ is $\mathrm{Ar}\,\mathcal{C} := \mathrm{Fun}([1], \mathcal{C})$. This is truly the category of arrows in $\mathcal{C}$ because any object in $\mathrm{Ar}\,\mathcal{C}$ is a functor from $[1] \to \mathcal{C}$ and such a functor just chooses two objects in $\mathcal{C}$ and a morphism between them since there is only two objects in $[1]$ and one non-identity morphism.

One particular arrow category used in this thesis is $\mathrm{Ar}[n]$.

Definition 2.4. The *ordinal category* Δ is the category whose objects are the ordered sets $[n]$ and whose morphisms are order preserving maps. Specifically, $f : [m] \to [n]$ is a morphism in Δ if whenever $i < j$ in $[m]$, then $f(i) \leq f(j)$ in $[n]$.

There are special maps in Δ called the *coface* and *codegeneracy* maps. For $n \geq 0$, the $n+1$ coface maps are the injections $d^i : [n-1] \to [n]$, which send $j \mapsto j$ for all $j < i$ and $j \mapsto j+1$ for $j \geq i$. For $n \geq 0$, the $n+1$ codegeneracy maps are the surjections $s^i : [n+1] \to [n]$, which send $j \mapsto j$ for all $j \leq i$ and $j \mapsto j-1$ for $j > i$. These morphisms satisfy several relations and it can be verified that every morphism in Δ is a composite of coface and codegeneracy maps. A nice reference on this is in [16].

Definition 2.5. A *simplicial set* is a contravariant functor from the ordinal category to the category of sets. More generally, for any category $\mathcal{C}$, a *simplicial object in* $\mathcal{C}$, X, is a contravariant functor from the ordinal category to $\mathcal{C}$

$$X : \Delta^{\mathrm{op}} \to \mathcal{C}$$

So the data of a simplicial object in $\mathcal{C}$ comprises of an object in $\mathcal{C}$ for every natural number and *face* and *degeneracy* maps, denoted d_i and s_i respectively, which are the images of the coface and codegeneracy maps in Δ. The image of $[n]$ under this functor is an object X_n in $\mathcal{C}$. If $\mathcal{C} = \mathbf{Set}$ the category of sets, then X_n can be thought of as a set of *simplices*.

The category of simplicial sets will be denoted **sSet**.

Definition 2.6. A simplex $x \in X_n$ is called *nondegenerate* if $x \neq s_i(y)$ for any $y \in X_{n-1}$ for all i.

Definition 2.7. The *topological n-simplex* Δ^n is a subspace of $\mathbb{R}^{n+1}$ whose underlying set is

$$\Delta^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} x_i = 1 \text{ and } \forall i, x_i \geq 0\}$$

whose topology is the subspace topology induced from the standard topology in $\mathbb{R}^{n+1}$. This is homeomorphic to the n-dimensional ball.

The *i-th n-face* is the inclusion

$$D^i : \Delta^n \to \Delta^{n+1}$$

$$(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_i, 0, x_{i+1}, \ldots, x_n)$$

and the *i-th degenerate n-simplex* is the surjective map

$$S^i : \Delta^n \to \Delta^{n-1}$$

$$(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_i + x_{i+1}, \ldots, x_n)$$

Definition 2.8. The *geometric realization* of a simplicial set X is

$$|X| := \coprod_{n=0}^{\infty} \Delta^n \times X_n / \sim$$

where $\sim$ is the equivalence relation generated by the relations $(p, d_i(x)) \sim (D_i(p), x)$ and the relations $(p, s_i(y)) \sim (S_i(p), y)$ for all $p \in \Delta^n, x \in X_{n+1}$, and $y \in X_{n-1}$.

The topological space $|X|$, as Milnor showed in [17], is a CW-complex with a topological n-simplex for every nondegenerate simplex in X_n.

Definition 2.9. For a topological space X, the *loop space* of X at some chosen basepoint $x \in X$ is the space of maps

$$\mathrm{Map}((S^1, *), (X, x))$$

Definition 2.10. A *pointed category* is a category $\mathcal{C}$ with a zero object $*$. This means for every $x \in \mathrm{Ob}\,\mathcal{C}$, there are unique maps from and to the zero object: $* \to x,\ x \to *$

Definition 2.11. A *wide subcategory* of a category $\mathcal{C}$ is a subcategory whose objects are all the objects of $\mathcal{C}$.

CHAPTER 3

(CLOSED, SYMMETRIC) MULTICATEGORIES

Multicategories generalize categories in that they allow for morphisms with multiple objects in the source. If one thinks of morphisms in a category as functions, then morphisms in a multicategory can be thought of as multi-variable functions. A functor between multicategories, called a multifunctor, should preserve these multiple arity morphisms. We formally define these concepts now.

3.1 Definitions

Definition 3.1. A *symmetric multicategory*, $\mathcal{M}$ consists of a class of objects, $\operatorname{Ob}\mathcal{M}$, and for any integer $k \geq 0$ and $A_1, \ldots, A_k, B \in \operatorname{Ob}\mathcal{M}$, a set of *k-morphisms*, $\operatorname{Hom}_{\mathcal{M}}(A_1, \ldots, A_k; B)$.

There is a distinguished identity morphism $1_A \in \operatorname{Hom}_{\mathcal{M}}(A; A)$ for all $A \in \operatorname{Ob}\mathcal{M}$. There is also a right action of the symmetric group Σ_k on the set of k-morphisms:

$$\sigma^* : \operatorname{Hom}_{\mathcal{M}}(A_1, \ldots, A_k; B) \to \operatorname{Hom}_{\mathcal{M}}(A_{\sigma(1)}, \ldots, A_{\sigma(k)}; B)$$

Finally, there is a composition law

$$\circ : \operatorname{Hom}_{\mathcal{M}}(B_1, \ldots, B_k; C) \times \prod_{i=1}^{k} \operatorname{Hom}_{\mathcal{M}}(A_{i1}, \ldots, A_{il_i}; B_i) \to \operatorname{Hom}_{\mathcal{M}}(A_{11}, \ldots, A_{kl_k}; C)$$

subject to the following axioms:

associativity

$$\forall\ \theta_i^j \in \operatorname{Hom}_{\mathcal{M}}(A_{ij1}, \ldots, A_{ijm_{ij}}; A_{ij}),\ \ \theta_i \in \operatorname{Hom}_{\mathcal{M}}(A_{i1}, \ldots, A_{ik_i}; B_i)$$

and

$$\forall \theta \in \operatorname{Hom}_{\mathcal{M}}(B_1, \ldots, B_n; C)$$

we have

$$\theta \circ (\theta_1 \circ (\theta_1^1, \ldots, \theta_1^{k_1}), \ldots, \theta_n \circ (\theta_n^1, \ldots, \theta_n^{k_n})) = (\theta \circ (\theta_1, \ldots, \theta_n)) \circ (\theta_1^1, \ldots, \theta_1^{k_1}, \ldots, \theta_n^1, \ldots, \theta_n^{k_n})$$

identity for every $\theta \in \operatorname{Hom}_{\mathcal{M}}(A_1, \ldots, A_k; B)$, we have

$$\theta \circ (1_{A_1}, \ldots, 1_{A_k}) = \theta = 1_B \circ \theta$$

compatibility $\forall \sigma \in \Sigma_k, \forall \tau_i \in \Sigma_{l_i}$ for $1 \leq i \leq k$ and

$$\forall \theta \in \mathrm{Hom}_{\mathcal{M}}(A_1, \ldots, A_k; B),\ \forall \theta_i \in \mathrm{Hom}_{\mathcal{M}}(A_{i1}, \ldots, A_{il_i}; B_i)$$

we have

$$\sigma^*(\theta \circ (\theta_1, \ldots, \theta_k)) = \sigma^*(\theta) \circ (\theta_{\sigma(1)}, \ldots, \theta_{\sigma(k)})$$

and

$$\theta \circ (\tau_1^*(\theta_1), \ldots \tau_k^*(\theta_k)) = (\tau_1 \oplus \cdots \oplus \tau_k)^*(\theta \circ (\theta_1, \ldots, \theta_k))$$

Definition 3.2. A *multifunctor* F between symmetric multicategories $\mathcal{M}$, $\mathcal{N}$ is a map sending each object in $\mathcal{M}$ to an object in $\mathcal{N}$ and $\forall k$ each k-morphism $f : A_1 \times \cdots \times A_k \to B$ to a k-morphism $F(f) : F(A_1) \times \cdots \times F(A_k) \to F(B)$ that preserves the units, the σ_k action on the collection of morphisms, and the composition of morphisms.

There is a forgetful functor from the category of (symmetric) multicategories to the category of categories by simply forgetting the set of $k-$morphisms for $k > 1$. The image of any multicategory $\mathcal{M}$ under this functor is called the *underlying category* of $\mathcal{M}$.

Definition 3.3. Let $\mathcal{M}, \mathcal{N}$ be symmetric multicategories and $F : \mathcal{M} \to \mathcal{N}$ be a functor between their underlying categories. Then we say F *extends to a multifunctor* if there is a multifunctor $F' : \mathcal{M} \to \mathcal{N}$ so that its image under the forgetful functor from the category of multicategories to the category of categories is F.

Definition 3.4. A symmetric multicategory $\mathcal{M}$ is *closed* if $\forall A_1, \ldots, A_k, B \in \mathcal{M}$, there is an *internal hom object*, $\mathcal{M}(A_1, \ldots, A_k; B)$, and an *evaluation map*

$$\mathrm{ev}_{A_1,\ldots,A_k;B} \in \mathrm{Hom}_{\mathcal{M}}(A_1, \ldots, A_k, \mathcal{M}(A_1, \ldots, A_k; B); B)$$

such that the following holds:

i) $\forall A_1, \ldots, A_k, C_1, \ldots, C_l, B \in \mathcal{M}$ there is a bijection

$$\mathrm{Hom}_{\mathcal{M}}(C_1, \ldots, C_l; \mathcal{M}(A_1, \ldots, A_k; B)) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{M}}(A_1, \ldots, A_k, C_1, \ldots, C_l; B)$$

which depends on these objects and is defined by sending

$$f \in \mathrm{Hom}_{\mathcal{M}}(C_1, \ldots, C_l; \mathcal{M}(A_1, \ldots, A_k; B))$$

to the composite

$$\mathrm{ev}_{A_1,\ldots,A_k;B} \circ (\mathrm{id}_{A_1}, \ldots, \mathrm{id}_{A_k}, f)$$

ii) The following diagram commutes $\forall(\sigma,\tau) \in \Sigma_k \times \Sigma_l$:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{M}}(C_1,\dots,C_l;\mathcal{M}(A_1,\dots,A_k;B)) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{M}}(A_1,\dots,A_k,C_1,\dots,C_l;B) \\
\downarrow{\scriptstyle f\mapsto\sigma\circ(f\circ\tau)} & & \downarrow{\scriptstyle g\mapsto g\cdot(\sigma\times\tau)} \\
\mathrm{Hom}_{\mathcal{M}}(C_{\tau(1)},\dots,C_{\tau(l)};\mathcal{M}(A_{\sigma(1)},\dots,A_{\sigma(k)};B)) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{M}}(A_{\sigma(1)},\dots,A_{\sigma(k)},C_{\tau(1)},\dots,C_{\tau(l)};B)
\end{array}$$

Definition 3.5. If $\mathcal{M},\mathcal{N}$ are closed multicategories and $F:\mathcal{M}\to\mathcal{N}$ is a functor between their underlying categories then we say F *respects the closed structure* if the following morphism $\varphi_{A,B}$ exists for any objects A,B of $\mathcal{M}$:

$$\varphi_{A,B}:\ F(\mathcal{M}(A,B)) \to \mathcal{N}(F(A),F(B))$$

3.2 Examples and Properties

Example 3.6. We can view any (symmetric) monoidal category (definition 5.5), $(\mathcal{M},\otimes)$, as a (symmetric) multicategory by defining the k-morphisms to be $\mathrm{Hom}_{\mathcal{M}}(A_1,\dots,A_k;B) := \mathrm{Hom}_{\mathcal{M}}(A_1\otimes\cdots\otimes A_k, B)$.

If $\mathcal{M}$ is non-strict, then we have to be careful about how we choose $A_1\otimes\cdots\otimes A_k$ (see, for instance, [10]). One way to deal with this ambiguity is to choose a certain bracketing; for example, for $n=5$ choose $(((A_1\otimes A_2)\otimes A_3)\otimes A_4)\otimes A_5$. In this case, the multicategory is called a *representable multicategory*. We can similarly view any category with finite products as a multicategory.

Example 3.7. An operad is a multicategory with one object.

Remark 3.8. Not every multicategory is a representable multicategory. In fact, one of the main categories we will deal with in this paper, the category of Waldhausen categories, is an example of a non-representable multicategory (proof given in [2]). However, there is a nice relationship between multicategories and monoidal categories by adjoint functors. For any (symmetric) multicategory $\mathcal{M}$, even non-representable ones, we can construct a (symmetric) monoidal category whose objects (respectively, morphisms) are lists of objects (respectively, morphisms) in $\mathcal{M}$ where the monoidal structure is given by concatenation. This functor from the category of multicategories to the category of monoidal categories is left adjoint to the representable functor that takes any monoidal category to its representable multicategory.

CHAPTER 4

WALDHAUSEN CATEGORIES

Waldhausen defined what we now call *Waldhausen categories*. These generalize exact categories and are the most general input for K-theory which agrees with the classical K-theory notions. We will discuss this in more detail later, but we begin with formally defining Waldhausen categories.

4.1 Definitions and Examples

Definition 4.1. A *Waldhausen category*, $\mathcal{W}$, is a pointed category with two wide subcategories, $w\mathcal{W}$ (whose morphisms $A \xrightarrow{\sim} A'$ are called *weak equivalences*) and $co\mathcal{W}$ (whose morphisms $A \rightarrowtail B$ are called *cofibrations*) subject to the following axioms:

i) All isomorphisms are cofibrations and weak equivalences

ii) the unique map from the zero object to any object A is a cofibration: $* \rightarrowtail A \in co\mathcal{W}$

iii) Push-outs over cofibrations exist, meaning every such diagram can be completed with the bottom morphism being a cofibration:

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \vdots \\ C & \rightarrowtail & B \coprod_A C \end{array}$$

iv) For any such diagram:

$$\begin{array}{ccccc} C & \longleftarrow & A & \rightarrowtail & B \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ C' & \longleftarrow & A' & \rightarrowtail & B' \end{array}$$

the unique map from $B \coprod_A C \overset{\sim}{\rightarrowtail} B' \coprod_{A'} C'$ is a weak equivalence.

Remark 4.2. 1. Every Waldhausen category has coproducts by pushing out along the unique morphism from the zero object to any object, which is required to be a cofibration by axiom iii). More specifically, we have $A \coprod B = A \coprod_{*} B$ by:

$$\begin{array}{ccc} * & \rightarrowtail & B \\ \downarrow & & \vdots \\ A & \rightarrowtail & A \coprod_{*} B \end{array}$$

2. Waldhausen categories have *cofibration sequences* for every cofibration $A \rightarrowtail B$:

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \vdots \\ * & \rightarrowtail & * \coprod_{A} B \end{array}$$

Letting $B/A = * \coprod_{A} B$, then

$$A \rightarrowtail B \twoheadrightarrow B/A$$

is a cofibration sequence.

Definition 4.3. An *exact functor* of Waldhausen categories, $F : \mathcal{W} \to \mathcal{V}$, is a functor that preserves the zero object, weak equivalences, cofibrations, and push-out diagrams.

Definition 4.4. The *southern arrow condition* of a commutative diagram of cofibrations

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & D \end{array}$$

is that the unique map from $B \coprod_{A} C$ to D is also a cofibration:

$$\begin{array}{ccccc} A & & \rightarrowtail & & B \\ \downarrow & & & \swarrow & \downarrow \\ & & B \coprod_{A} C & & \\ \downarrow & \nearrow & & \searrow{\scriptstyle \exists!} & \downarrow \\ C & & \rightarrowtail & & D \end{array}$$

Definition 4.5. A *biexact functor*f of Waldhausen categories, $F : W_1 \times W_2 \to V$ is a functor which is exact in each factor and for any pair of cofibrations, $f_1 : A \rightarrowtail A'$ and $f_2 : B \rightarrowtail B'$ in $\mathcal{W}_1$ and $\mathcal{W}_2$ respectively, we can form the cube, $[(f_1, f_2)]_F$:

$$\begin{array}{ccc} F(A,B) & \xrightarrow{F(f_1\,B)} & F(A',B) \\ {\scriptstyle F(A\,f_2)}\downarrow & & \downarrow{\scriptstyle F(A'\,f_2)} \\ F(A,B') & \xrightarrow{F(f_1\,B')} & F(A',B') \end{array}$$

This cube should satisfy the southern arrow condition.

Definition 4.6. A *multiexact functor* generalizes a biexact functor; namely, it should be exact in each factor and for each cofibration in each factor, there is a n-cube that should satisfy a generalized version of the southern arrow condition above. Full details on this are in [2].

The (multi)category of Waldhausen categories will be denoted **Wald**.

Example 4.7. The category of pointed finite sets is a Waldhausen category by letting the weak equivalences be the bijections and the cofibrations be the inclusions. We require pointed sets so that there is a zero object, namely the singletons.

Example 4.8. A typical example of a Waldhausen category is any abelian category, or more generally, any Quillen exact category where one views the admissible monomorphisms as cofibrations and the isomorphisms as weak equivalences. The category of finitely generated projective modules over a ring is a Quillen exact category and therefore Waldhausen. (Notice, this is not abelian since kernels may not exist, for example, if the ring is non-Noetherian.)

Example 4.9. The typical non-exact example of a Waldhausen category is the example motivating Waldhausen in [5]: the category $R(X)$ of retractive spaces over X, namely the category of spaces having X as a retract. The cofibrations here are the typical cofibration of spaces and the weak equivalences are weak homotopy equivalences (or classes of maps that induce isomorphisms in any homology theory). Technically, we have to take the objects in $R(X)$ to be spaces Y that are finite CW complexes relative to X. This way, the inclusion from the zero object, X, is a cofibration as it should be. More generally, one can view a full subcategory of cofibrant objects in a pointed model category as a Waldhausen category.

4.2 The closed structure on Wald

The *closed structure* on **Wald** is given by defining the internal hom

$$\mathbf{Wald}(\mathcal{W}_1, \ldots, \mathcal{W}_k; \mathcal{V})$$

to be the Waldhausen category whose objects are k-exact functors $F : \mathcal{W}_1 \times \cdots \times \mathcal{W}_k \to \mathcal{V}$ and whose morphisms are natural transformations between k-exact functors. The weak equivalences are the natural weak equivalences between k-exact functors (which means they induce weak equivalences for all objects) and the cofibrations are the natural transformations $\alpha : F \to G$ such that for all cofibrations $f_i \in co\mathcal{W}_i$ $(1 \leq i \leq k)$, we can construct a $k+1$ cube which should satisfy a generalized version of the southern arrow condition.

Details for the general case can be found in [2]; the case for $k = 1$ is needed for the proof of Proposition 9.3 so we define it now.

The internal Hom object $\mathbf{Wald}(\mathcal{W};\mathcal{V})$ as a Waldhausen category has as its objects exact functors $\varphi : \mathcal{W} \to \mathcal{V}$ and as its morphisms natural transformations between such exact functors. The weak equivalences are defined to be the natural weak equivalences and the cofibrations are the natural transformations, $\alpha : \varphi_0 \Longrightarrow \varphi_1 : \mathcal{W} \longrightarrow \mathcal{V}$, such that for every cofibration $f : X \rightarrowtail X'$ in $\mathcal{W}$, all the maps along with the southern arrow of the following diagram are cofibrations:

$$\begin{array}{ccc} \varphi_0(X) & \xrightarrow{\varphi_0(f)} & \varphi_0(X') \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_{X'}} \\ \varphi_1(X) & \xrightarrow{\varphi_1(f)} & \varphi_1(X') \end{array}$$

(Zakharevich proves that this is well-defined in [2].)

4.3 The S_* Construction

Waldhausen's S_*-construction, which generalizes Quillen's Q-construction for exact categories, was defined in [5]. This is used to define K-theory for a Waldhausen category.

Definition 4.10. Let $\mathcal{W}$ be a Waldhausen category. The *S_*-construction* on $\mathcal{W}$, denoted $S_n(\mathcal{W})$, is the category whose objects are functors:

$$F_n : \mathrm{Ar}[n] \longrightarrow \mathcal{W}$$
$$(i,j) \mapsto A_{i,j} \in \mathrm{Ob}(\mathcal{W})$$

such that

i) $F_n(j,j) = 0$

ii) $A_{ij} \rightarrowtail A_{ik} \twoheadrightarrow A_{jk}$ is a cofibration sequence $\forall i \leq j \leq k$

The idea is for any $A \in S_n(\mathcal{W})$, $A := A_{01} \rightarrowtail A_{02} \rightarrowtail \ldots \rightarrowtail A_{0n}$ is a sequence of cofibrations with choice of sub-quotients $A_{ij} = A_{0j}/A_{0i}$.

An arrow in $S_n(\mathcal{W})$ is a diagram of the form

$$\begin{array}{ccccccc} A_{01} & \rightarrowtail & A_{02} & \rightarrowtail & \ldots & \rightarrowtail & A_{0n} \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & & & \downarrow{\scriptstyle \alpha_n} \\ A'_{01} & \rightarrowtail & A'_{02} & \rightarrowtail & \ldots & \rightarrowtail & A'_{0n} \end{array} \tag{4.1}$$

We let $S_n(\mathcal{W})$ be a category with cofibrations and weak equivalences as follows. The diagram (4.1) above is:

1. a cofibration if $\forall i \in \{1, \ldots, n-1\}$, each α_i is a cofibration in $\mathcal{W}$ and each unique map

   $$A'_{0i} \bigsqcup_{A_{0i}} A_{0(i+1)} \rightarrowtail A'_{0(i+1)}$$

   is a cofibration in $\mathcal{W}$;

2. a weak equivalence if $\forall i \in \{1, \ldots, n\}$, each α_i is a weak-equivalence in $\mathcal{W}$.

It is easily verified that $S_n(\mathcal{W})$, with the above notions of cofibrations and weak equivalences, becomes a Waldhausen category.

It is also clear by the definition of $S_n(\mathcal{W})$ that

$$S_*(\mathcal{W}) : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Wald}$$
$$[n] \mapsto S_n(\mathcal{W})$$

is a simplicial Waldhausen category, as shown by Waldhausen in [5]. The face map $d_i : S_n(\mathcal{W}) \to S_{n-1}(\mathcal{W})$ omits A_{0i} for $i = 1, \ldots, n$ and quotients everything by A_{01} for $i = 0$. The degeneracy map $s_i : S_n(\mathcal{W}) \to S_{n+1}(\mathcal{W})$ extends A_{0i} to $A_{0i} \xrightarrow{\mathrm{id}} A_{0i}$ for $i = 1, \ldots, n$ and includes $* \rightarrowtail A_{01}$ at beginning for $i = 0$.

4.4 K-theory for a Waldhausen Category

Before we can define K-theory for a Waldhausen category, we need to recall some definitions.

Definition 4.11. For any category $\mathcal{C}$, we can construct its *nerve* which is a simplicial set

$$\mathrm{Ner}_* \mathcal{C} : \Delta^{\mathrm{op}} \to \mathbf{Set}$$

where $\mathrm{Ner}_n \mathcal{C}$ is a set of n-composable morphisms between $n+1$ objects in $\mathcal{C}$ where the face maps d_i ignore the i^{th} object X_i and replaces the sequence

$$X_{i-1} \to X_i \to X_{i+1}$$

with the composition

$$X_{i-1} \to X_{i+1}$$

and the degeneracy maps s_i replaces each object X_i with the identity morphism $X_i \xrightarrow{=} X_i$.

Let **ssSet** be the category of bisimplicial sets, i.e. the category of contravariant functors $\Delta^{\mathrm{op}} \to$ **sSet**. There is a *diagonal functor*

$$\mathrm{Diag} : \mathbf{ssSet} \to \mathbf{sSet}$$

by precomposing with

$$(\mathrm{id}, \mathrm{id}) : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$$

Therefore, the set of n-simplices of the diagonal of a bisimplicial set $X_{*,*}$ is $X_{n,n}$.

Waldhausen gave a definition of K-theory as an infinite loop space as well as a connective spectrum (spectra whose negative homotopy groups vanish). We will give both definitions here.

Definition 4.12. For any Waldhausen category $\mathcal{W}$, its K-theory as an infinite loop space is defined as:

$$K(\mathcal{W}) = \Omega |\,\mathrm{Diag}(\mathrm{Ner}_* \, wS_*\mathcal{W})|$$

where

$$|\cdot| : \mathbf{sSet} \to \mathbf{Top}$$

is the geometric realization of simplicial sets and Ω is the loop space functor.

We can define the K-groups as the homotopy groups of this space:

$$K_i(\mathcal{W}) = \pi_i(K(\mathcal{W})) = \pi_{i+1}(|\,\mathrm{Diag}(\mathrm{Ner}_* \, wS_*\mathcal{W})|)$$

Definition 4.13. For any Waldhausen category $\mathcal{W}$, its K-theory as a connective spectrum of spaces is:

$$K(\mathcal{W}) = \Omega^\infty |\,\mathrm{Diag}(\mathrm{Ner}_* \, wS_*^\infty\mathcal{W})|$$

where S_*^n is the multi-simplicial Waldhausen category obtained by iterating the S-construction to the simplicial Waldhausen category S_*.

We could also define the spectrum as a connective spectrum of simplicial sets where:

$$K(\mathcal{W}) = \mathrm{Diag}(wS_*^\infty \mathcal{W})$$

So the n-th simplicial set is $\mathrm{Diag}(wS_*^n \mathcal{W})$

This agrees with the classical definitions of K-groups as well as the higher K-theory defined for symmetric monoidal categories and Quillen exact categories. However, if one is interested in the lower K-groups K_0 and K_1, then computing them via this construction may be tedious and complicated. This is where Picard groupoids and determinant functors play a role since the 1-type of this K-theory space will have the same homotopy groups (hence K-groups) in dimensions 0 and 1. These 1-types are known to be realized by Picard groupoids and there are many algebraic models which allow us to construct these 1-types directly from the Waldhausen category itself. These algebraic models are a lot nicer and easier to construct than the S_*-construction so it gives a better route for computing K_0 and K_1.

4.5 K-theory Simplices from Total Simplicial Set

There is another way to go from bisimplicial sets to simplicial sets, namely the functor

$$\mathrm{Tot} : \mathbf{ssSet} \to \mathbf{sSet}$$

originally defined by Artin and Mazur in [14]. It is known that there is a natural weak equivalence between the diagonal simplicial set and the *total simplicial set* of any bisimplicial set. Although the diagonal functor is used to define the K-theory of a Waldhausen category, any weakly equivalent model will work just as well. In particular, the simplices and relations that Muro and Tonks [1] use to construct their model for the 1-type of the K-theory of a Waldhausen category arise from the total simplicial set, rather than the diagonal simplicial set.

We remind the reader of the construction along with the face and degeneracy maps which is given in [14].

$$(\mathrm{Tot}\, X)_n = \prod_{i=1}^{n} X_{i,n-i} \Big/ d_v^0 x_i = d_h^{i+1} x_{i+1}$$

where $(x_0, x_1, \ldots, x_n) \in \prod_{i=0}^{n} X_{i,n-i}$. The face and degeneracy maps (for $j = 0, \ldots, n$) are:

$$\begin{aligned} D^j : (\operatorname{Tot} X)_n &\to (\operatorname{Tot} X)_{n-1} \\ (x_0, \ldots, x_n) &\mapsto (d_v^j x_0, d_v^{j-1} x_1, \ldots, d_v^1 x_{j-1}, d_h^j x_{j+1}, \ldots, d_h^j x_n) \\ S^j : (\operatorname{Tot} X)_n &\to (\operatorname{Tot} X)_{n+1} \\ (x_0, \ldots, x_n) &\mapsto (s_v^j x_0, s_v^{j-1} x_1, \ldots, s_v^0 x_j, s_h^j x_j, \ldots, s_h^j x_n) \end{aligned}$$

Take the bisimplicial set $\mathrm{Ner}_* \, wS_*\mathcal{W}$, given by the nerve of the simplicial Waldhausen category given by the S-construction for a Waldhausen category, $\mathcal{W}$. (Draw the bisimplicial set so S_* changes horizontally.) The set of 0-simplices is the trivial set and the set of 1-simplices is the set of objects in $\mathcal{W}$. The 2-simplices are

$$(*, A \xrightarrow{\sim} A', A' \rightarrowtail B')$$

which can be drawn as

$$\begin{array}{ccc} A & & \\ \big\downarrow{\sim} & & \\ A' & \rightarrowtail & B' \end{array}$$

whose face maps are $D_0 = B'/A'$, $D_1 = B'$, and $D_2 = A$.

The elements of the third simplicial set $\operatorname{Tot}(\mathrm{Ner}_* \, \mathrm{w} \, S_*\mathcal{W})_3$ are

$$\left(*, A \xrightarrow{\sim} A' \xrightarrow{\sim} A\text{"}, \begin{array}{ccc} A' & \rightarrowtail & B' \\ \big\downarrow{\sim} & & {\sim}\big\downarrow \\ A\text{"} & \rightarrowtail & B\text{"} \end{array}, A\text{"} \rightarrowtail B\text{"} \rightarrowtail C\text{"} \right)$$

Instead, we can draw this 3-simplex as:

$$\begin{array}{ccccc} A & & & & \\ \big\downarrow{\sim} & & & & \\ A' & \rightarrowtail & B' & & \\ \big\downarrow{\sim} & & \big\downarrow{\sim} & & \\ A\text{"} & \rightarrowtail & B\text{"} & \rightarrowtail & C\text{"} \end{array}$$

And its faces are:

$$D^0 : \begin{array}{ccc} B'/A' & & \\ \downarrow\wr & & \\ B''/A'' & \rightarrowtail & C''/A'' \end{array} \qquad D^2 : \begin{array}{ccc} A & & \\ \downarrow\wr & & \\ A'' & \rightarrowtail & C'' \end{array}$$

$$D^1 : \begin{array}{ccc} B' & & \\ \downarrow\wr & & \\ B'' & \rightarrowtail & C'' \end{array} \qquad D^3 : \begin{array}{ccc} A & & \\ \downarrow\wr & & \\ A' & \rightarrowtail & B' \end{array}$$

Similarly, to go one step further, we can draw a 4-simplex and define its face maps as follows:

$$\begin{array}{ccccccc} A & & & & & & \\ \downarrow\wr & & & & & & \\ A' & \rightarrowtail & B' & & & & \\ \downarrow\wr & & \downarrow\wr & & & & \\ A'' & \rightarrowtail & B'' & \rightarrowtail & C'' & & \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\ A''' & \rightarrowtail & B''' & \rightarrowtail & C''' & \rightarrowtail & E''' \end{array}$$

The faces of this 4-simplex are:

$$D^0 : \begin{array}{ccccc} B'/A' & & & & \\ \downarrow\wr & & & & \\ B''/A'' & \rightarrowtail & C''/A'' & & \\ \downarrow\wr & & \downarrow\wr & & \\ B'''/A''' & \rightarrowtail & C'''/A''' & \rightarrowtail & E'''/A''' \end{array}$$

$$D^1 : \begin{array}{ccccc} B' & & & & \\ \downarrow\wr & & & & \\ B'' & \rightarrowtail & C'' & & \\ \downarrow\wr & & \downarrow\wr & & \\ B''' & \rightarrowtail & C''' & \rightarrowtail & E''' \end{array}$$

$$D^2 : \begin{array}{ccccc} A & & & & \\ \downarrow\wr & & & & \\ A'' & \rightarrowtail & C'' & & \\ \downarrow\wr & & \downarrow\wr & & \\ A''' & \rightarrowtail & C''' & \rightarrowtail & E''' \end{array}$$

$$D^3 : \begin{array}{ccccc} A & & & & \\ \downarrow\wr & & & & \\ A' & \rightarrowtail & B' & & \\ \downarrow\wr & & \downarrow\wr & & \\ A''' & \rightarrowtail & B''' & \rightarrowtail & E''' \end{array}$$

$$D^4 : \begin{array}{ccccc} A & & & & \\ \downarrow\wr & & & & \\ A' & \rightarrowtail & B' & & \\ \downarrow\wr & & \downarrow\wr & & \\ A'' & \rightarrowtail & B'' & \rightarrowtail & C'' \end{array}$$

It is easy to see how this generalizes for higher simplices.

4.6 Classical K-theory of a Ring

Classical K-theory more or less began with Grothendieck's K-group for a ring or topological space; this is what we now call K_0. Other lower K-groups were then defined, such as Milnor's Whitehead group K_1, until Quillen defined higher K-theory as a connective spectrum of spaces (or simplicial sets) whose homotopy groups computed the classical lower K-groups and more; this construction was done for Quillen exact categories. Waldhausen generalized this construction for Waldhausen categories which is what we have described above. Now we can compute K_0 of a ring using Grothendieck's definition or Waldhausen's generalization.

Using Grothendieck's method, we construct the abelian group K_0R for some ring R by taking Proj_R^{fg} to be the set of isomorphism classes of finitely generated projective modules; this is actually an abelian monoid where the monoidal structure is given by:

$$[P] + [Q] = [P \oplus Q],\ 0 = [0_R]$$

(In fact, this is a semi-ring since tensoring two finitely generated projective modules over R is still a finitely generated projective module.) Now we can define K_0R:

$$K_0R = Gr(\mathrm{Proj}_R^{fg})$$

where Gr is the group completion of a module. Recall that this group satisfies an obvious universal property and is constructed by forming the free group on the monoid as a set and identifying the new monoidal structure from the free group construction with the original monoidal structure of the monoid.

Alternatively, we can use Waldhausen's K-theory construction to define K_0R. Abusing notation a bit, take Proj_R^{fg} now to be the category whose objects are finitely generated projective modules over R. This is clearly an exact category, therefore it is Waldhausen. Now, we can take K_0R to be $\pi_0 K(\mathrm{Proj}_R^{fg})$, which is equivalently the fundamental group of the geometric realization of the diagonal of the bisimplicial set formed by the nerve of the $wS_* \,\mathrm{Proj}_R^{fg}$. These two groups are isomorphic, although it may not be obvious right now. We will use the 1-type to get a better sense of what K_0R is in terms of this K-theory construction. We will continue this discussion in example 8.5 in the section of Universal Determinant Functors after we have carefully constructed a model for the 1-type.

CHAPTER 5

PICARD GROUPOIDS

We will now define the structures that make-up a *Picard groupoid*. Recall that a groupoid is a category for which every morphism is an isomorphism. The *Picard* structure means it has a *braided, symmetric, group-like* structure. In particular, this means Picard categories have a monoidal structure, so we start with formally defining *monoidal categories*. A nice reference for these notions is [11].

5.1 Definitions and Examples

Definition 5.1. A *monoidal category* is a category $\mathcal{M}$ equipped with the following data:

a) an object $1 \in \operatorname{Ob} \mathcal{M}$, called the "unit object"

b) a functor: $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ called the "tensor product"

c) a natural isomorphism: $\alpha_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\cong} x \otimes (y \otimes z)$ called the "associator"

d) a natural isomorphism: $l_x : x \otimes 1 \xrightarrow{\cong} x$, called the "left unitor"

e) a natural isomorphism: $r_x : 1 \otimes x \xrightarrow{\cong} x$, called the "right unitor"

such that the following diagrams commute:

pentagon identity

$$\begin{array}{ccccc} & & (w \otimes x) \otimes (y \otimes z) & & \\ & \nearrow^{\alpha_{w \otimes x, y, z}} & & \searrow^{\alpha_{w,x,y \otimes z}} & \\ ((w \otimes x) \otimes y) \otimes z & & & & w \otimes (x \otimes (y \otimes z)) \\ \downarrow{\scriptstyle \alpha_{w,x,y} \otimes \mathrm{id}_z} & & & & \uparrow{\scriptstyle \mathrm{id}_w \otimes \alpha_{x,y,z}} \\ (w \otimes (x \otimes y)) \otimes z & & \xrightarrow{\alpha_{w, x \otimes y, z}} & & w \otimes ((x \otimes y) \otimes z) \end{array}$$

triangle identity

$$\begin{array}{ccc} (x \otimes 1) \otimes y & \xrightarrow{\alpha_{x,1,y}} & x \otimes (1 \otimes y) \\ & {\scriptstyle l_x \otimes \mathrm{id}_y} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_x \otimes r_y} & \\ & x \otimes y & \end{array}$$

Definition 5.2. A *group-like* groupoid $\mathcal{G}$ is a monoidal groupoid such that for every object x of $\mathcal{G}$, the functors

$$x \otimes - : \mathcal{G} \to \mathcal{G}$$

$$- \otimes x : \mathcal{G} \to \mathcal{G}$$

are equivalences of categories.

In particular, this means for every object x of $\mathcal{G}$, we can find an "inverse" object x', so there are isomorphisms

$$x \otimes x' \xrightarrow{\cong} 1$$

$$x' \otimes x \xrightarrow{\cong} 1$$

Remark 5.3. In any group-like category $\mathcal{P}$ we get the following bijection

$$\mathrm{Hom}_{\mathcal{P}}(x \otimes z^{-1} \otimes y, x) \cong \mathrm{Hom}_{\mathcal{P}}(y, z)$$

for all objects x, y, z in $\mathcal{P}$. This means that if we want to define a monoidal functor $F : \mathcal{P} \to \mathcal{G}$ on $\mathrm{Hom}_{\mathcal{P}}(x \otimes z^{-1} \otimes y, x)$, we only need to define it on $\mathrm{Hom}_{\mathcal{P}}(y, z)$. This will be useful later.

Definition 5.4. A *braided monoidal category* is a monoidal category equipped additionally with a natural isomorphism

$$\beta_{x,y} : x \otimes y \xrightarrow{\cong} y \otimes x$$

called the "braiding" such that the following diagrams commute (called the hexagon identities):

$$\begin{array}{ccccc}
(x \otimes y) \otimes z & \xrightarrow{\alpha_{x,y,z}} & x \otimes (y \otimes z) & \xrightarrow{B_{x,y\otimes z}} & (y \otimes z) \otimes x \\
\downarrow{\scriptstyle \beta_{x,y}\otimes \mathrm{id}_z} & & & & \downarrow{\scriptstyle \alpha_{y,z,x}} \\
(y \otimes x) \otimes z & \xrightarrow{\alpha_{y,x,z}} & y \otimes (x \otimes z) & \xrightarrow{\mathrm{id}_y \otimes B_{x,z}} & y \otimes (z \otimes x)
\end{array}$$

and

$$\begin{array}{ccccc}
x \otimes (y \otimes z) & \xrightarrow{\alpha^{-1}_{x,y,z}} & (x \otimes y) \otimes z & \xrightarrow{B_{x\otimes y,z}} & z \otimes (x \otimes y) \\
\downarrow{\scriptstyle \mathrm{id}_x \otimes \beta_{y,z}} & & & & \downarrow{\scriptstyle \alpha^{-1}_{z,x,y}} \\
x \otimes (z \otimes y) & \xrightarrow{\alpha^{-1}_{x,z,y}} & (x \otimes z) \otimes y & \xrightarrow{B_{x,z}\otimes \mathrm{id}_y} & (z \otimes x) \otimes y
\end{array}$$

Definition 5.5. A *symmetric monoidal category* is a braided monoidal category such that the braiding must satisfy the condition

$$\beta_{y,x} \circ \beta_{x,y} = \mathrm{id}_{x\otimes y}$$

for all objects x, y in the category.

A consequence of this condition is that the two hexagon identities now become one.

Definition 5.6. A *Picard groupoid* $\mathcal{P}$ is a symmetric, group-like groupoid.

Remark 5.7.

1. A nice way to think of Picard groupoids is that they categorify abelian groups.
2. One should note that the symmetric (or more generally the braided) structure and group-like structure in a Picard groupoid are independent of one another.

We denote the set of isomorphism classes of objects in $\mathcal{P}$ by $\pi_0\mathcal{P}$; the Picard structure gives this set an abelian group structure. The automorphisms of the identity object always form an abelian group denoted $\pi_1\mathcal{P}$. (See [11] for details.)

Definition 5.8. A *monoidal functor* between Picard groupoids is a functor $F : \mathcal{P} \to \mathcal{Q}$ such that the following exist:

a) an isomorphism in $\mathcal{Q}$: $\epsilon : 1_{\mathcal{Q}} \xrightarrow{\cong} F(1_{\mathcal{P}})$

b) a natural isomorphism: $\mu_{x,y} : F(x) \otimes_{\mathcal{P}} F(y) \xrightarrow{\cong} F(x \otimes_{\mathcal{Q}} y)$

making the following diagrams commute:

associativity

$$\begin{array}{ccc}
(F(x) \otimes_{\mathcal{Q}} F(y)) \otimes_{\mathcal{Q}} F(z) & \xrightarrow{\alpha_{F(x),F(y),F(z)}} & F(x) \otimes_{\mathcal{Q}} (F(y) \otimes_{\mathcal{Q}} F(z)) \\
\downarrow{\scriptstyle \mu_{x,y}\otimes \mathrm{id}_{F(z)}} & & \downarrow{\scriptstyle \mathrm{id}_{F(x)} \otimes \mu_{y,z}} \\
F(x \otimes_{\mathcal{P}} y) \otimes_{\mathcal{Q}} F(z) & & F(x) \otimes_{\mathcal{Q}} (F(y \otimes_{\mathcal{P}} z)) \\
\downarrow{\scriptstyle \mu_{x\otimes y,z}} & & \downarrow{\scriptstyle \mu_{x,y\otimes z}} \\
F((x \otimes_{\mathcal{P}} y) \otimes_{\mathcal{P}} z) & \xrightarrow{F(\alpha_{x,y,z})} & F(x \otimes_{\mathcal{P}} (y \otimes_{\mathcal{P}} z))
\end{array}$$

unitality

$$\begin{array}{ccc}
1_{\mathcal{Q}} \otimes_{\mathcal{Q}} F(x) & \xrightarrow{\epsilon\otimes \mathrm{id}_{F(x)}} & F(1_{\mathcal{P}}) \otimes_{\mathcal{Q}} F(x) \\
\downarrow{\scriptstyle r_{F(x)}} & & \downarrow{\scriptstyle \mu_{1,x}} \\
F(x) & \xleftarrow[F(r_x)]{} & F(1_{\mathcal{P}} \otimes_{\mathcal{P}} x)
\end{array}$$

$$\begin{array}{ccc}
F(x) \otimes_{\mathcal{Q}} 1_{\mathcal{Q}} & \xrightarrow{\mathrm{id}_{F(x)} \otimes \epsilon} & F(x) \otimes_{\mathcal{Q}} F(1_{\mathcal{P}}) \\
\downarrow{\scriptstyle l_{F(x)}} & & \downarrow{\scriptstyle \mu_{x,1}} \\
F(x) & \xleftarrow[F(l_x)]{} & F(x \otimes_{\mathcal{P}} 1_{\mathcal{P}})
\end{array}$$

symmetry

$$\begin{array}{ccc}
F(x) \otimes_{\mathcal{Q}} F(y) & \xrightarrow{B_{F(x),F(y)}} & F(y) \otimes_{\mathcal{Q}} F(x) \\
\downarrow{\scriptstyle \mu_{x,y}} & & \downarrow{\scriptstyle \mu_{y,x}} \\
F(x \otimes_{\mathcal{P}} y) & \xrightarrow{F(B_{x,y})} & F(y \otimes_{\mathcal{P}} x)
\end{array}$$

We will denote the (multi)category of Picard groupoids by **Pic**. This is also a 2-category if we let the 2-arrows be the natural transformations between monoidal functors. We form the homotopy category Ho(**Pic**) by identifying all naturally isomorphic monoidal functors and discarding the natural transformations.

Definition 5.9. A *bimonoidal functor* of Picard groupoids is a bifunctor $F : \mathcal{P}_1 \times \mathcal{P}_2 \to \mathcal{Q}$ such that:

a) it is monoidal in each factor, meaning the following isomorphisms exist:

$$F(x,y) \otimes_{\mathcal{Q}} F(x',y) \xrightarrow{\mu_{x,x'},\mathrm{id}_y} F(x \otimes_{\mathcal{P}_1} x', y), \quad F(x,y) \otimes_{\mathcal{Q}} F(x,y') \xrightarrow{\mathrm{id}_x,\mu_{y,y'}} F(x, y \otimes_{\mathcal{P}_2} y')$$

b) it has a compatibility condition between the monoidal structure in each factor, meaning the following diagram commutes

$$\begin{array}{ccc}
(F(x,y) \otimes F(x'y)) \otimes (F(x,y') \otimes F(x'y')) & \xrightarrow{\mu_{x,x'},\mathrm{id}_y \otimes \mu_{x,x'},y'} & F(x \otimes x', y) \otimes F(x \otimes x', y') \\
& & \downarrow{\scriptstyle \mathrm{id}_{x \otimes x'},\mu_{y,y'}} \\
\downarrow{\scriptstyle B_{F(x',y),F(x,y')}} & & F(x \otimes x', y \otimes y') \\
& & \uparrow{\scriptstyle \mu_{x,x'},\mathrm{id}_{y \otimes y'}} \\
(F(x,y) \otimes F(x,y')) \otimes (F(x',y) \otimes F(x'y')) & \xrightarrow{\mathrm{id}_x,\mu_{y,y'} \otimes \mathrm{id}_{x'},\mu_{y \otimes y'}} & F(x, y \otimes y') \otimes F(x', y \otimes y')
\end{array}$$

c) The two ways to form the isomorphism $F(1_{\mathcal{P}_1}, 1_{\mathcal{P}_2}) \xrightarrow{\cong} 1_{\mathcal{Q}}$ by using the monoidal structure in each factor must coincide, meaning the following diagram must commute

$$F(1_{\mathcal{P}_1}, 1_{\mathcal{P}_2}) \underset{F(-\ 1_{\mathcal{P}_2})}{\overset{F(1_{\mathcal{P}_1}\ -)}{\rightrightarrows}} 1_{\mathcal{Q}}$$

For general n, an *n-monoidal* or *multimonoidal functor* of Picard groupoids is a multifunctor $F : \mathcal{P}_1 \times \cdots \times \mathcal{P}_n \to \mathcal{Q}$ that generalizes the notion of bimonoidal functors, meaning fixing all but two factors produces a bimonoidal functor.

Example 5.10. If X is a scheme then $\mathcal{L}(X)$ is the groupoid of all line bundles on X with the usual monoidal structure, $\otimes$.

Example 5.11. Take Pic_R to be the category of invertible R-modules over some commutative ring R. This means the objects are rank 1 R-modules M such that there is an R-module N with $M \otimes_R N \cong R$. (These are always projective.) This is a Picard groupoid under the tensor product $\otimes_R$. Notice:

$$\pi_0(\mathrm{Pic}_R) \cong \mathrm{Pic}(\mathbf{Spec}R)$$
$$\pi_1(\mathrm{Pic}_R) \cong R^\times$$

5.2 The closed structure on Pic

The *closed structure* on **Pic** is given by defining the internal hom $\mathbf{Pic}(\mathcal{P}_1, \ldots, \mathcal{P}_k; \mathcal{Q})$ to be the Picard groupoid whose objects are k-monoidal functors $F : \mathcal{P}_1 \times \cdots \times \mathcal{P}_k \to \mathcal{Q}$ and whose morphisms are natural isomorphism between k-monoidal functors. The "unit object" is $1_\mathcal{Q}$ where

$$\begin{aligned} 1_\mathcal{Q} : \mathcal{P}_1 \times \cdots \times \mathcal{P}_k &\to \mathcal{Q} \\ (x_1, \ldots, x_k) &\mapsto 1_\mathcal{Q} \end{aligned}$$

and the tensor product of $F, G \in \mathrm{Ob}\,\mathbf{Pic}(\mathcal{P}_1, \ldots, \mathcal{P}_k; \mathcal{Q})$ is

$$\begin{aligned} F \otimes G : \mathcal{P}_1 \times \cdots \times \mathcal{P}_k &\to \mathcal{Q} \\ (x_1, \ldots, x_k) &\mapsto F(x_1, \ldots, x_k) \otimes \mathcal{Q}(x_1, \ldots, x_k) \end{aligned}$$

This makes $\mathbf{Pic}(\mathcal{P}_1, \ldots, \mathcal{P}_k; \mathcal{Q})$ a Picard groupoid because for any $F : \mathcal{P}_1 \times \cdots \times \mathcal{P}_k \to \mathcal{Q}$, there is an inverse $F^{-1} : \mathcal{P}_1 \times \cdots \times \mathcal{P}_k \to \mathcal{Q}$ such that:

$$F(x_1, \ldots, x_k) \otimes_\mathcal{Q} F^{-1}(x_1, \ldots, x_k) \xrightarrow{\cong} 1_\mathcal{Q}$$

Stable homotopy theory can be done in the category of spectra, **Spec**. *Stable 1-types* are the objects of the full subcategory of **Spec** that have trivial homotopy groups in all levels except 0 and

1, $\mathbf{Spec}_0^1$. Picard groupoids are known to classify stable 1-types under the classifying space and fundamental groupoid functors. Specifically, we have an equivalence of categories:

$$\mathrm{Ho}(\mathbf{Spec}_0^1) \rightleftarrows \mathrm{Ho}(\mathbf{Pic})$$

CHAPTER 6

STABLE QUADRATIC MODULES

We are particularly interested in the 1-type of the K-theory of a Waldhausen category, which has many algebraic models. Muro and Tonks [1] construct a model for this 1-type by ways of *stable quadratic modules* which we define now.

6.1 Definitions and Properties

Definition 6.1. A *stable quadratic module*, C_*, is a diagram of group homomorphisms

$$C_0^{\mathrm{ab}} \otimes C_0^{\mathrm{ab}} \xrightarrow{\langle \cdot,\cdot \rangle} C_1 \xrightarrow{\partial} C_0$$

such that $\forall a_i, b_i \in C_i$ for $i = 0, 1$,

- $\partial\langle a_0, b_0\rangle = [b_0, a_0] = -b_0 - a_0 + b_0 + a_0$
- $\langle \partial a_1, \partial b_1\rangle = [b_1, a_1]$
- $\langle a_0, b_0\rangle + \langle b_0, a_0\rangle = 0$

The *homotopy groups* of C_* are $\pi_1 C_* = \ker \partial$ and $\pi_0 C_* = \operatorname{coker} \partial$.

Remark 6.2. 1. The third condition is the stability condition. If we instead only require:

$$\langle \partial a_1, b_0\rangle + \langle b_0, \partial a_1\rangle = 0$$

then C_* would be a *reduced quadratic module.*

2. For any stable quadratic module C_*, we get a group action

$$\begin{aligned} C_0 \times C_1 &\to C_1 \\ (c_0, c_1) &\mapsto c_1^{c_0} := c_1 + \langle c_0, \partial c_1\rangle \end{aligned}$$

Definition 6.3. A *morphism between stable quadratic modules*, $f : C_* \to D_*$ is given by group homomorphisms $f_i : C_i \to D_i$, $i = 0, 1$, such that $\forall c_i, d_i \in C_i$

- $\partial_{D_*} f_1(c_1) = f_0 \partial_{C_*}(c_1)$

- $\langle f_0(c_0), f_0(d_0)\rangle_{D_*} = f_1\langle c_0, d_0\rangle_{C_*}$

Weak equivalences of stable quadratic modules are morphisms that induce isomorphisms on π_i for $i = 0, 1$.

We will denote the category of stable quadratic modules as **squad** and its homotopy category, Ho **squad**, formally inverts the weak equivalences.

The following are some simple examples, see [7]:

Example 6.4. Take any group homomorphism between abelian groups: $f : A \to B$. Then

$$B \otimes B \xrightarrow{\langle \cdot,\cdot \rangle} A \xrightarrow{\partial} B$$

where $w = 0_A$ and $\partial = f$ is a stable quadratic module. Notice the action of B on A here is trivial and $\pi_0 = \operatorname{coker} f$ and $\pi_1 = \ker f$.

Example 6.5. Take $C_0 = \mathbb{Z}$ and $C_1 = \mathbb{Z}/2\mathbb{Z}$. Let $\mathbb{Z}$ be generated by γ, then $\mathbb{Z}/2\mathbb{Z}$ is generated by $\langle \gamma, \gamma \rangle$. The map $\partial : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ is the zero map. Notice the action here is again trivial and $\pi_0 = \mathbb{Z}$ and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$. This is an algebraic model for the 1-type of the sphere spectrum.

Example 6.6. A generalization of the last example is

$$\mathbb{Z} \otimes \mathbb{Z} \xrightarrow{\langle \cdot,\cdot \rangle} F^\times \xrightarrow{0} \mathbb{Z}$$

with $\langle 1, 1 \rangle = -1$. The action again is trivial and $\pi_0 = \mathbb{Z}$ and $\pi_1 = F^\times$. This is an algebraic model for the K-theory of a field (or more generally, a commutative local ring).

6.2 An Algebraic Model for the 1-type of $K(\mathcal{W})$ as a Stable Quadratic Module

We briefly recall Muro and Tonks constructed a stable quadratic module, $D_*\mathcal{W}$, from a Waldhausen category, $\mathcal{W}$, by letting the group $D_0\mathcal{W}$ be the group of nilpotency class 2 of the free group generated by objects of $\mathcal{W}$, and $D_1\mathcal{W}$ be the group generated by the weak equivalences and cofibrations of $\mathcal{W}$ subject to several relations outlined below. ($D_1\mathcal{W}$ will also end up being a group of nilpotency class 2.) There is a functor (details in section 7.2)

$$P : \textbf{squad} \to \textbf{Pic}$$

which induces an equivalence of categories

$$P : \mathrm{Ho}\,\mathbf{squad} \to \mathrm{Ho}\,\mathbf{Pic}$$

between the homotopy category of stable quadratic modules and the homotopy category of Picard groupoids. In particular,

$$\pi_i C_* \cong \pi_i(PC_*),\ i = 0, 1$$

Therefore, as Picard groupoids classify stable 1-types, we can alternatively use stable quadratic modules as algebraic models for stable 1-types. We denote as $PD_{\bullet}\mathcal{W}$ the Picard groupoid corresponding to the stable quadratic module $D_{\bullet}\mathcal{W}$ and, by composition, we get a functor:

$$PD_{\bullet} : \mathbf{Wald} \to \mathbf{Pic}$$

Definition 6.7. Given a Waldhausen category, define a stable quadratic module $D_*\mathcal{W}$ as follows:

$D_0\mathcal{W}$ is generated by $[A]$ $\forall A \in \mathcal{W}$ with $[*] = 0$

$D_1\mathcal{W}$ is generated by $[A \xrightarrow{\sim} A']$ and $[A \rightarrowtail B \twoheadrightarrow B/A]$, for all $A \xrightarrow{\sim} A'$ weak equivalences and $A \rightarrowtail B$ cofibrations in $\mathcal{W}$ subject to the following relations:

a) $\partial[A \xrightarrow{\sim} A'] = -[A'] + [A]$

b) $\partial[A \rightarrowtail B \twoheadrightarrow B/A] = -[B] + [B/A] + [A]$

c) $[A \xrightarrow{id_A} A] = 0$

d) $[A \overset{id_A}{\rightarrowtail} A \twoheadrightarrow *] = [* \rightarrowtail A \twoheadrightarrow A] = 0$

e) For all composable weak equivalences $A \xrightarrow{\sim} B \xrightarrow{\sim} C$ we have

$$[A \xrightarrow{\sim} C] = [B \xrightarrow{\sim} C] + [A \xrightarrow{\sim} B]$$

f) For all commutative diagrams:

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & B/A \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ A' & \rightarrowtail & B' & \twoheadrightarrow & B'/A' \end{array}$$

we have

$$[A \xrightarrow{\sim} A'] + [B/A \xrightarrow{\sim} B'/A'] + \langle [A], -[B'/A'] + [B/A] \rangle$$
$$= -[A' \rightarrowtail B' \twoheadrightarrow B'/A'] + [B \xrightarrow{\sim} B'] + [A \rightarrowtail B \twoheadrightarrow B/A]$$

g) For all commutative diagrams:

$$\begin{array}{ccccc} & & & & C/B \\ & & & & \uparrow \\ & & B/A & \rightarrowtail & C/A \\ & & \uparrow & & \uparrow \\ A & \rightarrowtail & B & \rightarrowtail & C \end{array}$$

we have the relations

$$[B \rightarrowtail C \twoheadrightarrow C/B] \;+\; [A \rightarrowtail B \twoheadrightarrow B/A] = [A \rightarrowtail C \twoheadrightarrow C/A] + [B/A \rightarrowtail C/A \twoheadrightarrow C/B] + \langle [A], -[C/A] + [C/B] + [B/A] \rangle$$

h) $\langle [A], [B] \rangle = -[B \overset{i_2}{\rightarrowtail} A \bigsqcup B \overset{p_1}{\twoheadrightarrow} A] + [A \overset{i_1}{\rightarrowtail} A \bigsqcup B \overset{p_2}{\twoheadrightarrow} B]$

We have a functor:

$$D_{\bullet} : \mathbf{Wald} \to \mathbf{squad}$$

since for any exact functor $F : \mathcal{W} \to \mathcal{V}$, we get a morphism of stable quadratic modules

$$D_{\bullet} F : D_{\bullet}\mathcal{W} \to D_{\bullet}\mathcal{V}$$

defined on generators by:

$$(D_0 F)[A] = [F(A)]$$

$$(D_1 F)[A \overset{\sim}{\to} A'] = [F(A) \overset{\sim}{\to} F(A')]$$

$$(D_1 F)[A \rightarrowtail B \twoheadrightarrow B/A] = [F(A) \rightarrowtail F(B) \twoheadrightarrow F(B/A)]$$

Notice $\pi_0 D_{\bullet}\mathcal{W}$ is the group generated by $[A]$ for all objects A in $\mathcal{W}$ and whose relations are generated by $[A] = [A']$ for all weak equivalences $A \overset{\sim}{\to} A'$ and by $[B/A] + [A] = [B]$ for all cofibrations $A \rightarrowtail B \twoheadrightarrow B/A$.

$\pi_1 D_{\bullet}\mathcal{W}$ is the group generated by generators of automorphisms: $[A \to A]$.

CHAPTER 7

CROSSED MODULES

7.1 Definitions and Examples

Definition 7.1. A *crossed module* is a group homomorphism $C_1 \xrightarrow{\partial} C_0$ along with a right action $C_1 \times C_0 \to C_1$ (where $(a_1, a_0) \mapsto a_1^{a_0}$) such that

- $\partial(a_1^{a_0}) = -a_0 + \partial(a_1) + a_0 \ \forall a_i \in C_i,\ i = 0, 1$
- $a_1^{\partial(b_1)} = -b_1 + a_1 + b_1 \ \forall a_1, b_1 \in C_1$

Again, we can define $\pi_i C_*$ $(i = 0, 1)$ for crossed modules as we did for stable quadratic modules.

Definition 7.2. A *morphism between crossed modules*, $f : C_* \to D_*$, is given by group homomorphisms $f_i : C_i \to D_i$, $i = 0, 1$, such that $\forall c_i \in C_i,\ f_1(c_1^{c_0}) = f_1(c_1)^{f_0(c_0)}$.

We will denote the category of crossed modules as **cross**.

Example 7.3.

i) For any normal subgroup N of a group G, the inclusion

$$i : N \hookrightarrow G$$

is a crossed module where the action is given by conjugation.

ii) For any central extension of groups

$$1 \to A \to E \to G \to 1$$

the (surjective) map $E \to G$ is a crossed module with the action of G on E.

iii) For a pointed pair of spaces (A, X, x), where A is a subspace of X with $x \in A$, the homotopy boundary from the second relative homotopy group to the fundamental group of A is a crossed module:

$$d : \pi_2(X, A, x) \to \pi_1(A, x)$$

7.2 Properties

Lemma 7.4. **squad** *is a full reflective subcategory of* **cross**. *(Lemma 4.15 in [1])*

Any crossed module, C_*, gives rise to a Picard groupoid, PC_*, by letting the objects of PC_* be the group C_0 and letting the morphisms be the semidirect product $C_0 \ltimes C_1$. This means for any pair $(c_0, d_1) \in C_0 \ltimes C_1$, we take this to be a morphism in $\mathrm{Hom}_{PC_*}(c_0 + \partial d_1, c_0)$. The isomorphism between $c_0 \otimes d_0$ and $d_0 \otimes c_0$ is given by the morphism $(d_0 \otimes c_0, \langle d_0, c_0 \rangle)$. A morphism of crossed modules corresponds to a functor of Picard groupoids in a functorial way and the homotopy groups of C_* will agree with the homotopy groups of PC_*. Altogether, this gives us a functor

$$P : \mathbf{cross} \to \mathbf{Pic}$$

which reduces to a functor on **squad**.

Remark 7.5. Any crossed module C_* has a classifying space BC_* whose homotopy groups in dimension 1 is $\pi_0 C_*$, in dimension 2 is $\pi_1 C_*$, and is 0 for all dimensions above 2. This is used to prove that crossed modules classify homotopy 2-types.

Example 7.6. Now that we have defined a stable quadratic module (hence a crossed module) $D_\bullet \mathcal{W}$ for any Waldhausen category $\mathcal{W}$, we can define the corresponding Picard groupoid.

$PD_\bullet \mathcal{W}$ is the Picard groupoid whose objects is a group generated by $[X]$ for every $X \in \mathcal{W}$ and whose morphisms are generated by

$$([X], [f : Y \xrightarrow{\sim} Y']) : [X] - [Y'] + [Y] \to [X]$$

and

$$([X], [g : Y \rightarrowtail Z]) : [X] - [Z] + [Z/Y] + [Y] \to [X]$$

for every weak equivalence $Y \xrightarrow{\sim} Y'$ and every cofibration $Y \rightarrowtail Z$.

In the proof of Proposition 9.3, we need to define a monoidal functor on $PD_\bullet \mathcal{W}$ for a particular Waldhausen category $\mathcal{W}$. By the remark 5.3, defining the functor on the generator

$$([X], [f : Y \xrightarrow{\sim} Y']) : [X] - [Y'] + [Y] \to [X]$$

is equivalent to defining it on the generator

$$([Y'], [f : Y \xrightarrow{\sim} Y']) : [Y] \to [Y']$$

and defining it on the generator

$$([X], [g : Y \rightarrowtail Z]) : [X] - [Z] + [Z/Y] + [Y] \to [X]$$

is equivalent to defining it on the generator

$$([Z], [g : Y \rightarrowtail Z]) : [Z/Y] + [Y] \to [Z]$$

This will significantly simplify the definition of the functor which is essential for the main theorem.

CHAPTER 8

DETERMINANT FUNCTOR

Knudsen and Mumford categorified the determinant of invertible matrices as a *determinant functor* in [15]. Deligne in [9] axiomatized these properties to define a determinant functor on any exact category. He also explicitly constructed a Picard groupoid, which is the target of a universal determinant functor on an exact category. Muro and Tonks generalized this definition and construction tp Waldhausen categories in [7]. We now recall determinant functors on Waldhausen categories.

8.1 Definitions and Examples

Definition 8.1. Let $\mathcal{W}$ be a Waldhausen category and $\mathcal{P}$ a Picard groupoid. A *determinant* is a functor

$$\det : \mathrm{w}\,\mathcal{W} \to \mathcal{P}$$

equipped, for every cofibration sequence $\Delta : X \rightarrowtail Y \twoheadrightarrow Y/X$ in $\mathcal{W}$, with an isomorphism

$$\det(\Delta) : \det(Y/X) \otimes \det(X) \to Y$$

which is natural with respect to commutative diagrams in $\mathcal{W}$ of the following form:

$$\begin{array}{ccccc} X & \rightarrowtail & Y & \twoheadrightarrow & Y/X \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ X' & \rightarrowtail & Y' & \twoheadrightarrow & Y'/X' \end{array}$$

and further subject to the following axioms:

associativity For any commutative diagram consisting of four cofibration sequences

$$\begin{array}{ccccc} & & & & Z/Y \\ & & & & \uparrow \\ & & Y/X & \rightarrowtail & Z/X \\ & & \uparrow & & \uparrow \\ X & \rightarrowtail & Y & \rightarrowtail & Z \end{array}$$

we get the following commutative diagram in $\mathcal{P}$

$$\begin{array}{ccc} & \det(Z) & \\ \nearrow & & \nwarrow \\ \det(Z/Y)\otimes\det(Y) & & \det(Z/X)\otimes\det(X) \\ \uparrow & & \uparrow \\ \det(Z/Y)\otimes(\det(Y/X)\otimes\det(X)) & \longrightarrow & (\det(Z/Y)\otimes\det(Y/X))\otimes\det(X) \end{array}$$

commutativity For any two objects $X, Y \in \mathcal{W}$, we have the following cofibration sequences

$$\Delta_1 : X \rightarrowtail X \bigsqcup Y \twoheadrightarrow Y, \quad \Delta_2 : Y \rightarrowtail X \bigsqcup Y \twoheadrightarrow X$$

and the following diagram in $\mathcal{P}$ commutes:

$$\begin{array}{ccc} & \det(X \bigsqcup Y) & \\ \nearrow & & \nwarrow \\ \det(Y)\otimes\det(X) & \longrightarrow & \det(X)\otimes\det(Y) \end{array}$$

Definition 8.2. A *universal determinant functor* is a determinant functor, $\det : \mathcal{W} \to \mathrm{V}(\mathcal{W})$, such that any determinant functor $\det' : \mathcal{W} \to \mathcal{P}$ must factor through det in an essentially unique way. Specifically, there exists a monoidal functor $f : \mathrm{V}(\mathcal{W}) \to \mathcal{P}$ and a natural transformation $\alpha : f \circ \det \Rightarrow \det'$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\det} & \mathrm{V}(\mathcal{W}) \\ & \searrow^{\det'} \;\; \Swarrow \alpha & \downarrow f \\ & & \mathcal{P} \end{array}$$

The uniqueness of f and α are detailed in [7].

Example 8.3. Fix some field F. Take Vect_F^{fd} to be the category of finite dimensional vector spaces over F. This is an exact category which we can view as an exact category. The universal determinant functor on this category is given by

$$\begin{aligned} \det : \mathrm{Vect}_F^{fd} &\to \mathrm{lines}_{\mathbb{Z}} \\ V &\mapsto (\wedge^{\dim V} V, \dim V) \end{aligned}$$

where $\text{lines}_{\mathbb{Z}}$ is the category of $\mathbb{Z}$-graded 1-dimensional vector spaces over F. This means an object is given by a pair, $(L,\ n)$ for some 1-dimensional vector space L and some integer n. This is a Picard groupoid by $(L,n)\otimes(K,m)=(L\otimes_F K, n+m)$.

The determinant is defined on weak equivalences, which in this case means isomorphisms. For any isomorphisms $f : V \xrightarrow{\cong} W$ of finite dimensional vector spaces where $\{v_1,\dots,v_n\}$ is a basis for V and $\{w_1,\dots,w_n\}$ is a basis for W, we get a morphism of the $\mathbb{Z}$-graded lines $(\wedge^n V, n)\to(\wedge^n W, n)$ by defining the morphism

$$\wedge^n V \to \wedge^n W$$
$$v_1\wedge\cdots\wedge v_n \mapsto \det A_f(w_1\wedge\cdots\wedge w_n)$$

where A_f is the matrix corresponding to the isomorphism f. The determinant also tells us that for any short exact sequence of finite dimensional vector spaces:

$$0\to U\to V\to W\to 0$$

we get an isomorphism in $\text{lines}_{\mathbb{Z}}$:

$$(\wedge^n U, n)\otimes(\wedge^m W, m)\to(\wedge^{n+m}V, n+m)$$

8.2 Universal Determinant Functor for a Waldhausen Category

Theorem 8.4 (Muro and Tonks [1]). *The following is a universal determinant functor:*

$$\det : \mathcal{W}\to PD_{\bullet}\mathcal{W}$$
$$X\mapsto [X]$$
$$X\xrightarrow{\sim} X'\mapsto([X'],[X\to X'])$$
$$\Delta\mapsto([Y],[\Delta])$$

for any cofibration sequence $\Delta:\ X\rightarrowtail Y\twoheadrightarrow Y/X$

In fact, any model for the 1-type of the K-theory of $\mathcal{W}$, $P_1K(\mathcal{W})$, is part of a universal determinant functor. This is very useful because now we can compute K_0, K_1 algebraically from $\mathcal{W}$ itself.

$$K_0(\mathcal{W})=\pi_0 PD_{\bullet}\mathcal{W}=\pi_0 D_{\bullet}\mathcal{W}$$
$$K_1(\mathcal{W})=\pi_1 PD_{\bullet}\mathcal{W}=\pi_1 D_{\bullet}\mathcal{W}$$

Example 8.5. Fix a (commutative) ring R. Let Proj_R^{fg} be the category of finitely generated projective modules over R. This is an exact category, hence a Waldhausen category. Every exact category of finitely generated projective modules splits and the weak equivalences are the isomorphisms. For the rest of this example, let P be a finitely generated projective module over R. The group $D_0 \text{Proj}_R^{fg}$ is generated by $[P]$ for every P where $[0_R] = 0$ and the group $D_1 \text{Proj}_R^{fg}$ is generated by the isomorphisms and split exact sequences.

Therefore, defining $K_0(\text{Proj}_R^{fg})$ as $\pi_0 D_\bullet \text{Proj}_R^{fg}$, we get the abelian group generated by classes of $[P]$ where $[P] = [P']$ for all isomorphisms $P \xrightarrow{\cong} P'$ and $[P \oplus Q] = [P] + [Q]$ for every short exact sequence isomorphic to $0 \to P \to P \oplus Q \to Q \to 0$. Recalling the Grothendieck group of a ring from 4.6, we see this is precisely $K_0(R)$.

Similarly, defining $K_1(\text{Proj}_R^{fg})$ as $\pi_1 D_\bullet \text{Proj}_R^{fg}$ we get the abelian group that is generated by classes of isomorphisms $[P \xrightarrow{\cong} P]$. $K_1(R)$, defined by Milnor, is the abelianization of the stable general linear group over R, $GL(R) = \bigsqcup_n GL_n(R)$ where

$$GL_1(R) \hookrightarrow GL_2(R) \hookrightarrow \cdots \hookrightarrow GL_n(R) \hookrightarrow GL_{n+1}(R) \hookrightarrow \cdots$$

and every $A \in GL_n(R)$ is identified with $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ in $GL_{n+1}(R)$.

Whitehead's lemma states that the commutator subgroup of $GL(R)$ is exactly the subgroup generated by *elementary matrices* where an elementary matrix differs from the identity matrix by some non-diagonal. Every matrix in $K_1(R)$ can be represented by a diagonal matrix which represents an isomorphism $R^n \xrightarrow{\cong} R^n$. This is precisely $K_1(\text{Proj}_R^{fg})$.

Weibel explains this in more detail in [18].

CHAPTER 9

P_1K AS A MULTIFUNCTOR

The foregoing and Muro and Tonks' results imply there exists a functor

$$PD_{\bullet} : \mathbf{Wald} \longrightarrow \mathbf{Pic}$$

Both **Wald** and **Pic** are closed, symmetric multicategories. We have the following theorem:

Theorem 9.1. *$PD_{\bullet}$ extends to a multifunctor.*

This theorem follows from two propositions: Proposition 9.2 below and Proposition 9.3 on page 37.

Proposition 9.2. *Given closed, multicategories, $\mathcal{M}$, $\mathcal{N}$, and a functor $F : \mathcal{M} \to \mathcal{N}$, if there is a morphism*

$$F(\mathcal{M}(A, B)) \to \mathcal{N}(F(A), F(B))$$

$\forall A, B \in \operatorname{Ob} \mathcal{M}$, then we get a morphism

$$\operatorname{Hom}_{\mathcal{M}}(A_1, \ldots, A_k; B) \to \operatorname{Hom}_{\mathcal{N}}(F(A_1), \ldots, F(A_k); F(B))$$

$\forall A_1, \ldots, A_k, B \in \mathcal{M}$ and $\forall k \in \mathbb{N}$.

Proof. (We will prove this inductively on k.) First, we want to define the morphism:

$$F : \operatorname{Hom}_{\mathcal{M}}(A, B; C) \to \operatorname{Hom}_{\mathcal{N}}(FA, FB; FC)$$

Since $\mathcal{M}$ and $\mathcal{N}$ are closed, multicategories, we have the following bijections:

$$\varphi_{A,B;C} : \operatorname{Hom}_{\mathcal{M}}(B; \mathcal{M}(A, C)) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{M}}(A, B; C)$$

$$\varphi_{FA,FB;FC} : \operatorname{Hom}_{\mathcal{N}}(FB; \mathcal{N}(FA, FC)) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{N}}(FA, FB; FC)$$

Since $\operatorname{Hom}_{\mathcal{M}}(B; \mathcal{M}(A, C))$ is a set of 1-morphisms, we can apply F and get the following morphism:

$$\operatorname{Hom}_{\mathcal{M}}(B; \mathcal{M}(A, C)) \to \operatorname{Hom}_{\mathcal{N}}(FB; F\mathcal{M}(A, C))$$

So, all together, we have the following diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{M}}(B, \mathcal{M}(A;C)) & \xrightarrow{F} & \mathrm{Hom}_{\mathcal{N}}(FB, F(\mathcal{M}(A;C))) \\
\Big\downarrow \cong & & \mathrm{Hom}_{\mathcal{N}}(FB, \mathcal{N}(FA;FC)) \\
 & & \Big\downarrow \cong \\
\mathrm{Hom}_{\mathcal{M}}(A, B; C) & \dashrightarrow & \mathrm{Hom}_{\mathcal{N}}(FA, FB; FC)
\end{array}$$

In order to get the bottom dotted map necessary for F to preserve the set of 2-morphisms, we need to a morphism

$$\omega : \mathrm{Hom}_{\mathcal{N}}(FB, F(\mathcal{M}(A;C))) \to \mathrm{Hom}_{\mathcal{N}}(FB, \mathcal{N}(FA;FC))$$

By Yoneda's lemma, this is equivalent to having a morphism

$$\overline{\omega} : F(\mathcal{M}(A;C)) \to \mathcal{N}(FA;FC)$$

This is precisely what Proposition 9.2 requires. Now we can extend the same argument to define a map

$$F : \mathrm{Hom}_{\mathcal{M}}(A_1, \ldots, A_k; B) \to \mathrm{Hom}_{\mathcal{N}}(FA_1, \ldots, FA_k; FB)$$

by studying the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{M}}(A_2, \ldots, A_{k+1}; \mathcal{M}(A_1;B)) & \xrightarrow{F} & \mathrm{Hom}_{\mathcal{N}}(FA_2, \ldots, A_{k+1}; F(\mathcal{M}(A_1;B))) \\
\Big\downarrow \cong & & \mathrm{Hom}_{\mathcal{N}}(FA_2, \ldots, A_{k+1}; \mathcal{N}(FA_1;FB)) \\
 & & \Big\downarrow \cong \\
\mathrm{Hom}_{\mathcal{M}}(A_1, \ldots, A_{k+1}; B) & \dashrightarrow & \mathrm{Hom}_{\mathcal{N}}(FA_1, \ldots, FA_{k+1}; FB)
\end{array}$$

Again, by Yoneda's lemma, the missing gap is determined by a morphism

$$\overline{\omega} : F(\mathcal{M}(A_1, B)) \to \mathcal{N}(FA_1, FB)$$

$\square$

Proposition 9.3. *There exists a functor of Picard groupoids:*

$$PD_{\bullet}(\mathbf{Wald}(\mathcal{W}, \mathcal{V})) \to \mathbf{Pic}(PD_{\bullet}\mathcal{W}; PD_{\bullet}(\mathcal{V}))$$

Proof. We will start with explicitly describing each of the Picard groupoids in the functor. First, let's describe the internal Hom object $\mathbf{Wald}(\mathcal{W};\mathcal{V})$ as a Waldhausen category. Its objects are exact functors $\varphi : \mathcal{W} \to \mathcal{V}$ and its morphisms are natural transformations between exact functors. The weak equivalences are defined to be the natural weak equivalences and the cofibrations are the natural transformations, $\alpha : \varphi_0 \Longrightarrow \varphi_1 : \mathcal{W} \longrightarrow \mathcal{V}$, such that for every cofibration $f : X \rightarrowtail X'$ in $\mathcal{W}$, all the maps along with the southern arrow of the following diagram are cofibrations:

$$\begin{array}{ccc} \varphi_0(X) & \xrightarrow{\varphi_0(f)} & \varphi_0(X') \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_{X'}} \\ \varphi_1(X) & \xrightarrow{\varphi_1(f)} & \varphi_1(X') \end{array}$$

(Zakharevich proves that this is well-defined in [2].) We can construct the stable quadratic module, $D_\bullet(\mathbf{Wald}(\mathcal{W};\mathcal{V}))$ where $D_0(\mathbf{Wald}(\mathcal{W};\mathcal{V}))$ is the group generated by exact functors and $D_1(\mathbf{Wald}(\mathcal{W};\mathcal{V}))$ is generated by the weak equivalence and cofibration natural transformations defined above, subject to some relations. Finally, $PD_\bullet(\mathbf{Wald}(\mathcal{W};\mathcal{V}))$ is the Picard groupoid corresponding to this stable quadratic module. In particular, the objects are $D_0(\mathbf{Wald}(\mathcal{W};\mathcal{V}))$ and the morphisms are given by the semi-direct product:

$$D_0(\mathbf{Wald}(\mathcal{W};\mathcal{V})) \ltimes D_1(\mathbf{Wald}(\mathcal{W};\mathcal{V})) = \{([\varphi],[\alpha]) : [\varphi] + \partial[\alpha] \longrightarrow [\varphi]\}$$

The group structure for the Hom sets in $PD_\bullet(\mathbf{Wald}(\mathcal{W};\mathcal{V}))$ is given by:

$$([\varphi_0],[\alpha_0]) + ([\varphi_1],[\alpha_1]) = ([\varphi_0]+[\varphi_1], [\alpha_0]^{[\varphi_1]} + [\alpha_1])$$

and

$$1_{[\varphi]} = ([\varphi], 0)$$

We need to describe $\mathbf{Pic}(PD_\bullet\mathcal{W}; PD_\bullet\mathcal{V})$ as an internal Hom object in $\mathbf{Pic}$. Its objects are monoidal functors $F : PD_\bullet\mathcal{W} \longrightarrow PD_\bullet\mathcal{V}$ and its morphisms are natural transformations between such monoidal functors. The monoidal structure on the set of objects is given by computing the monoidal operation pointwise in $PD_\bullet\mathcal{V}$. Specifically, for every object X in $\mathcal{W}$, we have

$$\begin{aligned} F + G : PD_\bullet\mathcal{W} &\longrightarrow PD_\bullet\mathcal{V} \\ [X] &\mapsto F([X]) + G([X]) \end{aligned}$$

We are now ready to define the functor:

$$\overline{\omega} : PD_{\bullet}(\mathbf{Wald}(\mathcal{W};\mathcal{V})) \longrightarrow \mathbf{Pic}(PD_{\bullet}\mathcal{W}; PD_{\bullet}\mathcal{V})$$

For every generator $[\varphi : \mathcal{W} \longrightarrow \mathcal{V}]$ in $PD_{\bullet}(\mathbf{Wald}(\mathcal{W};\mathcal{V}))$, $\overline{\omega}([\varphi : \mathcal{W} \longrightarrow \mathcal{V}]) = PD_{\bullet}\varphi$. Therefore:

$$\begin{aligned} \overline{\omega}([\varphi]) : PD_{\bullet}\mathcal{W} &\longrightarrow PD_{\bullet}\mathcal{V} \\ [X] &\mapsto [\varphi(X)] \\ [X]+[Y] &\mapsto [\varphi(X)]+[\varphi(Y)] \\ ([X],[Y \xrightarrow{f} Y']) &\mapsto ([\varphi(X)],[\varphi(Y) \xrightarrow{\varphi(f)} \varphi(Y')]) \end{aligned}$$

The morphisms in $PD_{\bullet}(\mathbf{Wald}(\mathcal{W};\mathcal{V}))$ are generated by $([\varphi],[\alpha : \tau \rightsquigarrow \tau']) : [\varphi] + \partial[\alpha] \to [\varphi]$ and $([\varphi],[\beta : \eta \rightsquigarrow \delta]) : [\varphi] + \partial[\beta] \to [\varphi]$ for α natural weak equivalence and β a natural transformation that is a cofibration in the internal Hom $\mathbf{Wald}(\mathcal{W};\mathcal{V})$. By example 7.6, we can restrict to the generators $([\tau'],[\alpha : \tau \rightsquigarrow \tau']) : [\tau] \to [\tau']$ and $([\delta],[\beta : \eta \rightsquigarrow \delta]) : [\delta/\eta] + [\eta] \to [\delta]$.

Define $\overline{\omega}([\tau'],[\alpha])$ to be the following morphism in $\mathbf{Pic}(PD_{\bullet}\mathcal{W}, PD_{\bullet}\mathcal{V})$:

$$PD_{\bullet}\alpha : PD_{\bullet}\tau \rightsquigarrow PD_{\bullet}\tau'$$

This is the natural transformation defined as follows for all generators $[X] \in PD_{\bullet}\mathcal{W}$

$$PD_{\bullet}\alpha_{[X]} = ([\tau'(X)],[\alpha(X)]) \in \mathrm{Hom}_{PD_{\bullet}\mathcal{V}}([\tau(X)],[\tau'(X)])$$

Define $\overline{\omega}([\delta],[\beta])$ to be the following morphism in $\mathbf{Pic}(PD_{\bullet}\mathcal{W}, PD_{\bullet}\mathcal{V})$:

$$PD_{\bullet}\beta : PD_{\bullet}\delta/\eta + PD_{\bullet}\eta \rightsquigarrow PD_{\bullet}\delta$$

This is the natural transformation defined as follows for all generators $[X] \in PD_{\bullet}\mathcal{W}$

$$PD_{\bullet}\beta_{[X]} = ([\delta(X)],[\beta(X)]) \in \mathrm{Hom}_{PD_{\bullet}\mathcal{V}}([\delta/\eta(X)] + [\eta(X)],[\delta(X)])$$

Normally we would need to check that $PD_{\bullet}\alpha$ and $PD_{\bullet}\beta$ are natural transformation between Picard groupoids. However, it has been shown that the functors

$$D_{\bullet} : \mathbf{Wald} \to \mathbf{squad}$$

and

$$P : \mathbf{squad} \to \mathbf{Pic}$$

are 2-functors. Our definition of $\overline{\omega}$ is really the composition of these two functors on the exact functors of Waldhausen categories and natural transformations between these functors. This means these natural transformations are well defined. Although these computations do not need to be shown, they are in the Appendix for the curious reader.

We do need to check that $\overline{\omega}$ is a well-defined symmetric, monoidal functor between Picard Groupoids. This is a matter of checking simple but tedious computations, which can be found the Appendix. □

9.1 Proof of Main Theorem

Proof. From Propositions 9.2 and 9.3, we have the following maps for all k:

$$PD_{\bullet} : \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_1, \dots, \mathcal{W}_k; \mathcal{V}) \xrightarrow{PD_{\bullet}} \mathrm{Hom}_{\mathbf{Pic}}(PD_{\bullet}(\mathcal{W}_1), \dots, PD_{\bullet}(\mathcal{W}_k); PD_{\bullet}(\mathcal{V}))$$

Notice that the group of objects in $PD_{\bullet}\mathcal{W}$ is generated by $\det_{\mathcal{W}}(X) = [X]_{\mathcal{W}}$ for all objects X of $\mathcal{W}$. So we can define morphisms in **Pic** on the generators alone. We will use the diagram from Proposition 9.2 and $\overline{\omega}$ defined in Proposition 9.3 to determine how $PD_{\bullet}$ maps k-morphisms in **Wald** to k-morphisms in **Pic**. This diagram involves the isomorphisms that exist from the closed structures on **Wald** and **Pic**:

$$\eta : \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_{k+1}; \mathcal{V}) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_2, \dots, \mathcal{W}_{k+1}; \mathbf{Wald}(\mathcal{W}_1, \mathcal{V}))$$

$$\varphi\colon \mathrm{Hom}_{\mathbf{Pic}}(PD_{\bullet}(\mathcal{W}_2),\dots,PD_{\bullet}(\mathcal{W}_{k+1});\mathbf{Pic}(PD_{\bullet}(\mathcal{W}_1),PD_{\bullet}(\mathcal{V}))) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{Pic}}(PD_{\bullet}(\mathcal{W}_1),\dots,PD_{\bullet}(\mathcal{W}_{k+1});PD_{\bullet}(\mathcal{V}))$$

We will describe these isomorphisms point-wise.

For every $f \in \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_{k+1}; \mathcal{V})$ and every $X_i \in \mathcal{W}_i$,

$$\eta(f)(X_2, \dots, X_{k+1}) \in \mathbf{Wald}(\mathcal{W}_1, \mathcal{V})$$

corresponds to the functor that maps X_1 to $f(X_1, X_2, \dots, X_k) \in \mathcal{V}$.

For every $f \in \mathrm{Hom}_{\mathbf{Pic}}(PD_{\bullet}(\mathcal{W}_2), \dots, PD_{\bullet}(\mathcal{W}_{k+1}); \mathbf{Pic}_{\bullet}(PD_{\bullet}(\mathcal{W}_1), PD_{\bullet}(\mathcal{V})))$

$$\varphi(f) = \mathrm{ev}(\mathrm{id}_{PD_{\bullet}\mathcal{W}_1}, f)$$

maps $([X_1]_{\mathcal{W}_1}, \dots, [X_{k+1}]_{\mathcal{W}_{k+1}})$ to $f([X_2]_{\mathcal{W}_1}, \dots, [X_{k+1}]_{\mathcal{W}_{k+1}})([X_1]_{\mathcal{W}_1})$

The diagram also involves the functor $PD_\bullet$ which is defined on 1-morphisms. The map

$$PD_\bullet : \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_2; \mathbf{Wald}(\mathcal{W}_1; \mathcal{V})) \to \mathrm{Hom}_{\mathbf{Pic}}(PD_\bullet(\mathcal{W}_2); PD_\bullet(\mathbf{Wald}(\mathcal{W}_1, \mathcal{V})))$$

maps f to

$$\begin{aligned} PD_\bullet(f) : PD_\bullet(\mathcal{W}_2) &\to PD_\bullet(\mathbf{Wald}(\mathcal{W}_1, \mathcal{V})) \\ [X]_{\mathcal{W}_2} &\mapsto [f(X) : \mathcal{W}_1 \to \mathcal{V}]_{\mathbf{Wald}(\mathcal{W}_1, \mathcal{V})} \end{aligned}$$

$PD_\bullet$, as defined in Proposition 9.2 on the set of 2-morphisms $\mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_1, \mathcal{W}_2; \mathcal{V})$, is given by the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_2; \mathbf{Wald}(\mathcal{W}_1, \mathcal{V})) & \xrightarrow{PD_\bullet} & \mathrm{Hom}_{\mathbf{Pic}}(PD_\bullet(\mathcal{W}_2); PD_\bullet(\mathbf{Wald}(\mathcal{W}_1, \mathcal{V}))) \\ & & \big\downarrow \scriptstyle{\overline{\omega} \circ -} \\ \big\uparrow \scriptstyle{\eta} & & \mathrm{Hom}_{\mathbf{Pic}}(PD_\bullet(\mathcal{W}_2); \mathbf{Pic}(PD_\bullet(\mathcal{W}_1), PD_\bullet(\mathcal{V}))) \\ & & \big\downarrow \scriptstyle{\mathrm{ev}(\mathrm{id}_{PD_\bullet(\mathcal{W}_1)}\ -)} \\ \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_1, \mathcal{W}_2; \mathcal{V}) & \xrightarrow{PD_\bullet} & \mathrm{Hom}_{\mathbf{Pic}}(PD_\bullet(\mathcal{W}_1), PD_\bullet(\mathcal{W}_2); PD_\bullet(\mathcal{V})) \end{array}$$

Therefore, for $f \in \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_1, \mathcal{W}_2; \mathcal{V})$,

$$\begin{aligned} PD_\bullet f(\det_{\mathcal{W}_1}(X), \det_{\mathcal{W}_2}(Y)) &= \mathrm{ev}(\mathrm{id}_{PD_\bullet(\mathcal{W}_1)}, \overline{\omega} \circ PD_\bullet \eta f)(\det_{\mathcal{W}_1}(X), \det_{\mathcal{W}_2}(Y)) \\ &= \mathrm{ev}(\det_{\mathcal{W}_1}(X), \overline{\omega} \circ PD_\bullet \eta f(\det_{\mathcal{W}_2}(Y))) \\ &= \mathrm{ev}(\det_{\mathcal{W}_1}(X), \overline{\omega} \det_{\mathbf{Wald}(\mathcal{W}_1, \mathcal{V})}(\eta f(Y))) \\ &= \mathrm{ev}(\det_{\mathcal{W}_1}(X), PD_\bullet \eta f(Y)) \\ &= PD_\bullet \eta f(Y)(\det_{\mathcal{W}_1}(X)) \\ &= \det_{\mathcal{V}}(\eta f(Y)(X)) \\ &= \det_{\mathcal{V}}(f(X, Y)) \end{aligned}$$

i.e. $(PD_\bullet f)([X]_{\mathcal{W}_1}, [Y]_{\mathcal{W}_2}) = [f(X, Y)]_{\mathcal{V}}$

In fact, this extends and we get:

$$PD_\bullet f(\det_{\mathcal{W}_1}(X_1), \ldots, \det_{\mathcal{W}_{k+1}}(X_{k+1})) = \det_{\mathcal{V}}(f(X_1, \ldots, X_{k+1}))$$

which is the same as $(PD_\bullet f)([X_1], \ldots, [X_{k+1}]) = [f(X_1, \ldots, X_{k+1})])$

We know $PD_\bullet$ is a functor, but to show it is a multifunctor we have to check the composition and symmetric conditions on the set of k-morphisms. Namely, the following two diagrams should commute:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Wald}}(\mathcal{V}_1,\dots,\mathcal{V}_k;\mathcal{U}) \times \prod_{i=1}^{k} \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_{i1},\dots,\mathcal{W}_{il_i};\mathcal{V}_i) & & \\
\Big\downarrow {\scriptstyle PD_\bullet \times \prod PD_\bullet} & & \\
\mathrm{Hom}_{\mathbf{Pic}}(PD_\bullet\mathcal{V}_1,\dots,PD_\bullet\mathcal{V}_k;PD_\bullet\mathcal{U}) \times \prod_{i=1}^{k} \mathrm{Hom}_{\mathbf{Pic}}(PD_\bullet\mathcal{W}_{i1},\dots,PD_\bullet\mathcal{W}_{il_i};PD_\bullet\mathcal{V}_i) & & \Big\downarrow \circ \\
\Big\downarrow \circ & & \\
\mathrm{Hom}_{\mathbf{Pic}}(PD_\bullet\mathcal{W}_{11},\dots,PD_\bullet\mathcal{W}_{kl_k};PD_\bullet\mathcal{U}) & \xleftarrow{PD_\bullet} & \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_{11},\dots,\mathcal{W}_{kl_k};\mathcal{U})
\end{array}$$

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_1,\dots,\mathcal{W}_k;\mathcal{V}) & \xrightarrow{\sigma^*} & \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_{\sigma(1)},\dots,\mathcal{W}_{\sigma(k)};\mathcal{V}) \\
\Big\downarrow {\scriptstyle PD_\bullet} & & \Big\downarrow {\scriptstyle PD_\bullet} \\
\mathrm{Hom}_{\mathbf{Pic}}(PD_\bullet\mathcal{W}_1,\dots,PD_\bullet\mathcal{W}_k;PD_\bullet\mathcal{V}) & \xrightarrow{\sigma^*} & \mathrm{Hom}_{\mathbf{Pic}}(PD_\bullet\mathcal{W}_{\sigma(1)},\dots,PD_\bullet\mathcal{W}_{\sigma(k)};PD_\bullet\mathcal{V})
\end{array}$$

To show $PD_\bullet$ respects composition for $k = 2$, there are two cases to check. Assume $f \in \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_1,\mathcal{W}_2;\mathcal{V})$, $g_i \in \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_i,\mathcal{W}_i)$ for $i = 1,2$, $f' \in \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_2,\mathcal{V})$, and $g \in \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_1,\mathcal{W}_2;\mathcal{V})$. We need the following identities:

$$PD_\bullet(f \circ (g_1,g_2)) = PD_\bullet f \circ (PD_\bullet g_1, PD_\bullet g_2)$$

$$PD_\bullet(f \circ g) = PD_\bullet f \circ PD_\bullet g$$

which we have by defining point-wise on objects $[X]_{\mathcal{W}_1} \in PD_\bullet\mathcal{W}_1$ and $[Y]_{\mathcal{W}_2} \in PD_\bullet\mathcal{W}_2$:

$$\begin{aligned}
PD_\bullet(f \circ (g_1,g_2))([X]_{\mathcal{W}_1},[Y]_{\mathcal{W}_2}) &= [(f(g_1,g_2)(X,Y))]_{\mathcal{V}} \\
&= [f(g_1(X),g_2(Y))]_{\mathcal{V}} \\
&= PD_\bullet f([g_1(X)]_{\mathcal{W}_1},[g_2(Y)]_{\mathcal{W}_2}) \\
&= PD_\bullet f(PD_\bullet g_1([X]_{\mathcal{W}_1}), PD_\bullet g_2([Y]_{\mathcal{W}_2})) \\
&= PD_\bullet f \circ (PD_\bullet g_1, PD_\bullet g_2)([X]_{\mathcal{W}_1},[Y]_{\mathcal{W}_2})
\end{aligned}$$

$$\begin{aligned}
PD_\bullet(f \circ g)([X]_{\mathcal{W}_1},[Y]_{\mathcal{W}_2}) &= [f \circ g(X,Y)]_{\mathcal{V}} \\
&= PD_\bullet f([g(X,Y)]_{\mathcal{W}_2}) \\
&= PD_\bullet f(PD_\bullet g([X]_{\mathcal{W}_1},[Y]_{\mathcal{W}_2})) \\
&= PD_\bullet f \circ PD_\bullet g([X]_{\mathcal{W}_1},[Y]_{\mathcal{W}_2})
\end{aligned}$$

The proof for any k is very similar.

To prove the symmetry condition for $k = 2$, we should note there is only one trivial element in Σ_2, which is the one that flips the two objects. So we need to show:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_1, \mathcal{W}_2; \mathcal{V}) & \xrightarrow{\sigma^*_{\mathbf{Wald}}} & \mathrm{Hom}_{\mathbf{Wald}}(\mathcal{W}_2, \mathcal{W}_1; \mathcal{V}) \\ \Big\downarrow PD_{\bullet} & & \Big\downarrow PD_{\bullet} \\ \mathrm{Hom}_{\mathbf{Pic}}(PD_{\bullet}\mathcal{W}_1, PD_{\bullet}\mathcal{W}_2; PD_{\bullet}\mathcal{V}) & \xrightarrow{\sigma^*_{\mathbf{Pic}}} & \mathrm{Hom}_{\mathbf{Pic}}(PD_{\bullet}\mathcal{W}_2, PD_{\bullet}\mathcal{W}_1; PD_{\bullet}\mathcal{V}) \end{array}$$

$$\begin{aligned} PD_{\bullet}(\sigma^*_{\mathbf{Wald}} f)(\det_{\mathcal{W}_2}(Y), \det_{\mathcal{W}_1}(X)) &= \det_{\mathcal{V}}(\sigma^*_{\mathbf{Wald}} f(Y, X)) \\ &= \det_{\mathcal{V}}(f(X, Y)) \\ &= PD_{\bullet} f(\det_{\mathcal{W}_1}(X), \det_{\mathcal{W}_2}(Y)) \\ &= \sigma^*_{\mathbf{Pic}}(PD_{\bullet} f)(\det_{\mathcal{W}_2}(Y), \det_{\mathcal{W}_1}(X)) \end{aligned}$$

Once again, this extends similarly for any k. $\square$

APPENDIX A

COMPUTATIONS FOR PROOFS

A.1 $\overline{\omega}$ defines a natural transformation on morphisms

The next few computations are to check the natural transformations defined under $\overline{\omega}$ respect the usual commutative diagram for any morphism in $PD_{\bullet}\mathcal{W}$. Once again, we can restrict to generators $([Y'], [f]) : [Y] \to [Y']$ and $([Z], [g]) : [Z/Y] + [Y] \to [Z]$.

First, let's start with $\overline{\omega}([\tau'], [\alpha]) : PD_{\bullet}\tau \rightsquigarrow PD_{\bullet}\tau'$ on the morphism $([Y'], [f : Y \xrightarrow{\sim} Y'])$. We need the following diagram to commute:

$$\begin{array}{ccc} [\tau(Y)] & \xrightarrow{([\tau(Y')], [\tau(f)])} & [\tau(Y')] \\ \Big\downarrow {\scriptstyle ([\tau'(Y)], [\alpha_Y])} & & \Big\downarrow {\scriptstyle ([\tau'(Y')], [\alpha_{Y'}])} \\ [\tau'(Y)] & \xrightarrow{([\tau'(Y')], [\tau'(f)])} & [\tau'(Y')] \end{array}$$

So the composition both ways is:

$$([\tau'(Y')], [\alpha_{Y'}] + [\tau(f)])$$

and

$$([\tau'(Y')], [\tau'(f)] + [\alpha_Y])$$

To prove these are the same morphism, we have to show the second factors are the same. (Since their image under ∂ is the same, any action on their difference is trivial.)

Since α is a natural transformation, we have the following commutative diagram consisting of all weak equivalences for $f : Y \xrightarrow{\sim} Y'$ in $\mathcal{W}$:

$$\begin{array}{ccc} \tau(Y) & \xrightarrow{\tau(f)} & \tau(Y') \\ \Big\downarrow {\scriptstyle \alpha_Y} & & \Big\downarrow {\scriptstyle \alpha_{Y'}} \\ \tau'(Y) & \xrightarrow{\tau'(f)} & \tau'(Y') \end{array}$$

So, by the relations in $D_1\mathcal{V}$,

$$[\alpha_{Y'}] + [\tau(f)] = [\tau'(f)] + [\alpha_Y]$$

which is what we needed to prove the composition both ways agree.

Now, let's work with $\overline{\omega}([\tau'], [\alpha]) : PD_{\bullet}\tau \rightsquigarrow PD_{\bullet}\tau'$ on the morphism $([Z], [g : Y \rightarrowtail Z])$. We need the following diagram to commute:

$$\begin{array}{ccc} [\tau(Z/Y)] + [\tau(Y)] & \xrightarrow{([\tau(Z)], [\tau(g)])} & [\tau(Z)] \\ \Big\downarrow {\scriptstyle ([\tau'(Z/Y)], [\alpha_{Z/Y}]) + ([\tau' Y], [\alpha_Y])} & & \Big\downarrow {\scriptstyle ([\tau'(Z)], [\alpha_Z])} \\ [\tau'(Z/Y)] + [\tau'(Y)] & \xrightarrow[([\tau'(Z)], [\tau'(g)])]{} & [\tau'(Z)] \end{array}$$

So the composition both ways is:

$$([\tau'(Z)], [\alpha_Z] + [\tau(g)])$$

and

$$([\tau'(Z)], [\tau'(g)] + [\alpha_{Z/Y}]^{[\tau'(Y)]} + [\alpha_Y])$$

Therefore, we need

$$[\alpha_Z] + [\tau(g)] = [\tau'(g)] + [\alpha_{Z/Y}]^{[\tau(Y)]} + [\alpha_Y] \tag{A.1}$$

Since α is a natural transformation, we have the following commutative diagram consisting of weak equivalences for the vertical morphisms and cofibration sequences horizontally for $g : Y \rightarrowtail Z$ in $\mathcal{W}$:

$$\begin{array}{ccccc} \tau(Y) & \overset{\tau(f)}{\rightarrowtail} & \tau(Z) & \twoheadrightarrow & \tau(Z/Y) \\ \downarrow {\scriptstyle \alpha_Y} & & \downarrow {\scriptstyle \alpha_Z} & & \downarrow {\scriptstyle \alpha_{Z/Y}} \\ \tau'(Y) & \overset{\tau'(f)}{\rightarrowtail} & \tau'(Z) & \twoheadrightarrow & \tau'(Z/Y) \end{array}$$

So, by the relations in $D_1\mathcal{V}$, we get

$$[\tau'(g)] + [\alpha_Y] + [\alpha_{Z/Y}]^{[\tau(Y)]} = [\alpha_Z] + [\tau(g)]$$

Therefore, equation (A.1) becomes:

$$\begin{aligned}
[\tau'(g)] + [\alpha_Y] + [\alpha_{Z/Y}]^{[\tau(Y)]} &= [\tau'(g)] + [\alpha_{Z/Y}]^{[\tau(Y)]} + [\alpha_Y] \\
\implies [\alpha_Y] + [\alpha_{Z/Y}] + \langle [\tau(Y)], \partial[\alpha_{Z/Y}] \rangle &= [\alpha_{Z/Y}] + [\alpha_Y] + \langle [\tau'(Y)], \partial[\alpha_Y] \rangle \\
\implies [\alpha_Y] + [\alpha_{Z/Y}] &= [\alpha_{Z/Y}] + [\alpha_Y] + \langle \partial[\alpha_{Z/Y}], \partial[\alpha_Y] \rangle \\
&= [\alpha_{Z/Y}] + [\alpha_Y] + [[\alpha_Y], [\alpha_{Z/Y}]] \\
&= [\alpha_Y] + [\alpha_{Z/Y}]
\end{aligned}$$

Now let's move on to checking that $\varpi([\delta], [\beta]) : PD_{\bullet}\delta/\eta + PD_{\bullet}\eta \rightsquigarrow PD_{\bullet}\delta$. We will start on the morphism $([Y'], [f : Y \xrightarrow{\sim} Y'])$ is a natural transformation. We need the following diagram to commute:

$$\begin{tikzcd}
{[\delta/\eta(Y)] + [\eta(Y)]} \arrow[r, "{([\delta/\eta(Y')], [\delta/\eta(f)]) + ([\eta(Y')], [\eta(f)])}"] \arrow[d, "{([\delta(Y)], [\beta_Y])}"'] & {[\delta/\eta(Y')] + [\eta(Y')]} \arrow[d, "{([\delta(Y')], [\beta_{Y'}])}"] \\
{[\delta(Y)]} \arrow[r, "{([\delta(Y')], [\delta(f)])}"'] & {[\delta(Y')]}
\end{tikzcd}$$

So the composition both ways is:

$$([\delta(Y')], [\beta_{Y'}] + [\delta/\eta(f)]^{[\eta(Y')]} + [\eta(f)])$$

and

$$([\delta(Y')], [\delta(f)] + [\beta_Y]$$

We need

$$[\beta_{Y'}] + [\delta/\eta(f)]^{[\eta(Y')]} + [\eta(f)] = [\delta(f)] + [\beta_Y] \tag{A.2}$$

Since β is a natural transformation, we have the following commutative diagram consisting of weak equivalences for the vertical morphisms and cofibration sequences horizontally for $f : Y \xrightarrow{\sim} Y'$ in $\mathcal{W}$:

$$\begin{tikzcd}
\eta(Y) \arrow[r, tail, "\beta_Y"] \arrow[d, "\eta(f)"] & \delta(Y) \arrow[r, two heads] \arrow[d, "\delta(f)"] & \delta/\eta(Y) \arrow[d, "\delta/\eta(f)"] \\
\eta(Y') \arrow[r, tail, "\beta_{Y'}"] & \delta(Y') \arrow[r, two heads] & \delta/\eta(Y')
\end{tikzcd}$$

So, by the relations in $D_1\mathcal{V}$, we get

$$[\delta(f)] + [\beta_Y] = [\beta_{Y'}] + [\eta(f)] + [\delta/\eta(f)]^{[\eta(Y)]}$$

Therefore, equation (A.2) becomes:

$$\begin{aligned}
[\beta_{Y'}] + [\delta/\eta(f)]^{[\eta(Y')]} + [\eta(f)] &= [\beta_{Y'}] + [\eta(f)] + [\delta/\eta(f)]^{[\eta(Y)]} \\
\implies [\delta/\eta(f)] + [\eta(f)] + \langle[\eta(Y')], \partial[\delta/\eta(f)]\rangle &= [\eta(f)] + [\delta/\eta(f)] + \langle[\eta(Y)], \partial[\delta/\eta(f)]\rangle \\
\implies [\delta/\eta(f)] + [\eta(f)] &= [\eta(f)] + [\delta/\eta(f)] + \langle\partial[\eta(f)], \partial[\delta/\eta(f)]\rangle \\
&= [\eta(f)] + [\delta/\eta(f)] + [[\delta/\eta(f)], [\eta(f)]] \\
&= [\delta/\eta(f)] + [\eta(f)]
\end{aligned}$$

Finally, we have to check $\overline{\omega}([\delta], [\beta]) : PD_{\bullet}\delta/\eta + PD_{\bullet}\eta \rightsquigarrow PD_{\bullet}\delta$ on the morphism $([Z], [g : Y \rightarrowtail Z])$. We need the following diagram to commute:

$$\begin{array}{ccc}
[\delta/\eta(Z/Y)] + [\eta(Z/Y)] + [\delta/\eta(Y)] + [\eta(Y)] & \xrightarrow{([\delta/\eta(Z)], [\delta/\eta(g)]) + ([\eta(Z)], [\eta(g)] + \langle[\delta/\eta(Y)], [\eta(Z/Y)]\rangle)} & [\delta/\eta(Z)] + [\eta(Z)] \\
\Big\downarrow {\scriptstyle ([\delta(Z/Y)], [\beta_{Z/Y}]) + ([\delta(Y)], [\beta_Y])} & & \Big\downarrow {\scriptstyle ([\delta(Z)], [\beta_Z])} \\
[\delta(Z/Y)] + [\delta(Y)] & \xrightarrow[([\delta(Z)], [\delta(g)])]{} & [\delta(Z)]
\end{array}$$

Therefore, the composition both ways is:

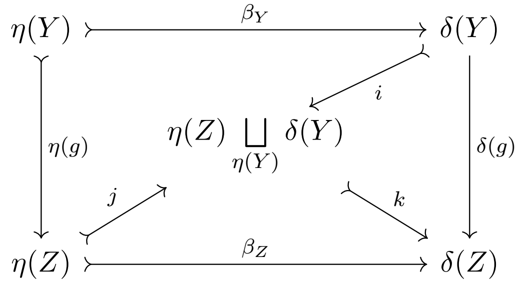
$$([\delta(Z)], [\beta_Z] + [\delta/\eta(g)]^{[\eta(Z)]} + [\eta(g)] + \langle[\delta/\eta(Y)], [\eta(Z/Y)]\rangle)$$

and

$$([\delta(Z)], [\delta(g)] + [\beta_{Z/Y}]^{[\delta(Y)]} + [\beta_Y])$$

$$\implies [\beta_Z] + [\delta/\eta(g)]^{[\eta(Z)]} + [\eta(g)] + \langle[\delta/\eta(Y)], [\eta(Z/Y)]\rangle = [\delta(g)] + [\beta_{Z/Y}]^{[\delta(Y)]} + [\beta_Y] \quad \text{(A.3)}$$

Since β is a natural transformation and is a cofibration, that means the southern arrow condition is satisfied:



And we get the following four diagrams:

$$\begin{array}{ccccc} & & & & \delta/\eta(Z/Y) \\ & & & & \uparrow \\ & & \delta/\eta(Y) & \rightarrowtail & \delta/\eta(Z) \\ & & \uparrow & & \uparrow \\ \eta(Z) & \rightarrowtail & \eta(Z) \coprod_{\eta(Y)} \delta(Y) & \rightarrowtail & \delta(Z) \end{array} \tag{A.4}$$

$$\begin{array}{ccccc} & & & & \delta/\eta(Z/Y) \\ & & & & \uparrow \\ & & \eta(Z/Y) & \rightarrowtail & \delta(Z/Y) \\ & & \uparrow & & \uparrow \\ \delta(Y) & \rightarrowtail & \eta(Z) \coprod_{\eta(Y)} \delta(Y) & \rightarrowtail & \delta(Z) \end{array} \tag{A.5}$$

$$\begin{array}{ccccc} & & & & \eta(Z/Y) \\ & & & & \uparrow \\ & & \delta/\eta(Y) & \rightarrowtail & \delta/\eta(Y) \coprod \eta(Z/Y) \\ & & \uparrow & & \uparrow \\ \eta(Y) & \rightarrowtail & \delta(Y) & \rightarrowtail & \eta(Z) \coprod_{\eta(Y)} \delta(Y) \end{array} \tag{A.6}$$

$$\begin{array}{ccccc} & & & & \delta/\eta(Y) \\ & & & & \uparrow \\ & & \eta(Z/Y) & \rightarrowtail & \delta/\eta(Y) \coprod \eta(Z/Y) \\ & & \uparrow & & \uparrow \\ \eta(Y) & \rightarrowtail & \eta(Z) & \rightarrowtail & \eta(Z) \coprod_{\eta(Y)} \delta(Y) \end{array} \tag{A.7}$$

So, by the relations in $D_1\mathcal{V}$, we get the following four relations:

$$[k] + [j] = [\beta_Z] + [\delta/\eta(g)]^{[\eta(Z)]} \tag{A.8}$$

$$[k] + [i] = [\delta(g)] + [\beta_{Z/Y}]^{[\delta(Y)]} \tag{A.9}$$

$$[i] + [\beta_Y] = [i \circ \beta_Y] + [\delta/\eta(Y) \rightarrowtail \delta/\eta(Y) \coprod \eta(Z/Y)]^{[\eta(Y)]} \tag{A.10}$$

$$[j] + [\eta(g)] = [j \circ \eta(g)] + [\eta(Z/Y) \rightarrowtail \delta/\eta(Y) \bigsqcup \eta(Z/Y)]^{[\eta(Y)]} \tag{A.11}$$

Using (A.6) and (A.7) as well as the fact that $i \circ \beta_Y = j \circ \eta(g)$, we get:

$$\begin{aligned}[i] + [\beta_Y] &= [j] + [\eta(g)] - [\eta(Z/Y) \rightarrowtail \delta/\eta(Y) \bigsqcup \eta(Z/Y)]^{[\eta(Y)]} + [\delta/\eta(Y) \rightarrowtail \delta/\eta(Y) \bigsqcup \eta(Z/Y)]^{[\eta(Y)]} \\ &= [j] + [\eta(g)] + \langle [\delta/\eta(Y)], [\eta(Z/Y)] \rangle^{[\eta(Y)]}\end{aligned}$$

Any action on the image of $\langle \cdot, \cdot \rangle$ is trivial. Using (A.4) the right hand side further simplifies to

$$-[k] + [\beta_Z] + [\delta/\eta(g)]^{[\eta(Z)]} + [\eta(g)] + \langle [\delta/\eta(Y)], [\eta(Z/Y)] \rangle$$

Using (A.5), the left hand side simplifies to:

$$-[k] + [\delta(g)] + [\beta_{Z/Y}]^{[\delta(Y)]} + [\beta_Y]$$

Therefore, we get

$$[\delta(g)] + [\beta_{Z/Y}]^{[\delta(Y)]} + [\beta_Y] = [\beta_Z] + [\delta/\eta(g)]^{[\eta(Z)]} + [\eta(g)] + \langle [\delta/\eta(Y)], [\eta(Z/Y)] \rangle$$

which is exactly what (A.3) needed.

Technically, we must also check these are *symmetric monoidal* natural transformations. This means that for $\overline{\omega}([\tau'], [\alpha])$, we need

$$\begin{array}{ccc} PD_{\bullet}\tau([X]) + PD_{\bullet}\tau([Y]) & \longrightarrow & PD_{\bullet}\tau'([X]) + PD_{\bullet}\tau'([Y]) \\ \downarrow & & \downarrow \\ PD_{\bullet}\tau([X] + [Y]) & \longrightarrow & PD_{\bullet}\tau'([X] + [Y]) \end{array}$$

and

$$\begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ PD_{\bullet}\tau(0) & \longrightarrow & PD_{\bullet}\tau'(0) \end{array}$$

However, these commutativity diagrams are trivially satisfied since

$PD_{\bullet}\tau([X] + [Y]) := PD_{\bullet}\tau([X]) + PD_{\bullet}\tau([Y])$ and $PD_{\bullet}\tau(0) = 0$. Similarly for $\overline{\omega}([\delta], [\beta])$.

A.2 $\overline{\omega}$ is a well-defined symmetric, monoidal functor

Let $[\varphi],\ [\varphi_0],\ [\varphi_1],\ [\varphi_2] \in PD_{\bullet}(\mathbf{Wald}(\mathcal{W};\mathcal{V}))$.

I) Functor Requirements:

a) Composition Preserved:

$$\begin{aligned}\overline{\omega}(([\varphi_0],[\alpha_0]) \circ ([\varphi_0]+\partial[\alpha_0],[\alpha_1]))[X] &= \overline{\omega}(([\varphi_0],[\alpha_0]+[\alpha_1]))[X] \\ &= ([\varphi_0(X)],[\alpha_0(X)]+[\alpha_1(X)]) \\ &= ([\varphi_0(X)],[\alpha_0(X)]) \circ ([\varphi_0(X)]+\partial[\alpha_0(X)],[\alpha_1(X)]) \\ &= \overline{\omega}(([\varphi_0],[\alpha_0]))[X] \circ \overline{\omega}(([\varphi_0]+\partial[\alpha_0],[\alpha_1]))[X]\end{aligned}$$

b) Identity Morphism Preserved: $\overline{\omega}(([\varphi],0))[X] = ([\varphi(X)],0)$

II) Monoidal Requirements:

a) Identity Object Preserved: $\overline{\omega}(*) = \overline{\omega}([0_{\mathcal{W}}]) = [\varphi(0_{\mathcal{W}})] = [0_{\mathcal{V}}] = *$

b) Monoidal Structure on Objects Preserved: True by definition.

c) Monoidal Structure on Morphisms Preserved:

$$\begin{aligned}\overline{\omega}(([\varphi_0],[\alpha_0])+([\varphi_1],[\alpha_1]))[X] &= \overline{\omega}(([\varphi_0]+[\varphi_1],[\alpha_0]^{[\varphi_1]}+[\alpha_1]))[X] \\ &= ([\varphi_0(X)]+[\varphi_1(X)],[\alpha_0(X)]^{[\varphi_1(X)]}+[\alpha_1(X)]) \\ &= ([\varphi_0(X)],[\alpha_0(X)])+([\varphi_1(X)],[\alpha_1(X)]) \\ &= \overline{\omega}(([\varphi_0],[\alpha_0]))[X]+\overline{\omega}(([\varphi_1],[\alpha_1]))[X]\end{aligned}$$

d) Associativity Respected: We need the following diagram to commute:

$$\begin{array}{ccc} (\overline{\omega}([\varphi_0])+\overline{\omega}([\varphi_1]))+\overline{\omega}([\varphi_2]) & \rightsquigarrow & \overline{\omega}([\varphi_0])+(\overline{\omega}([\varphi_1])+\overline{\omega}(\varphi_2)) \\ \wr\downarrow & & \wr\downarrow \\ \overline{\omega}([\varphi_0]+[\varphi_1])+\overline{\omega}([\varphi_2]) & & \overline{\omega}([\varphi_0])+\overline{\omega}([\varphi_1]+[\varphi_2]) \\ \wr\downarrow & & \wr\downarrow \\ \overline{\omega}(([\varphi_0]+[\varphi_1])+[\varphi_2]) & \rightsquigarrow & \overline{\omega}([\varphi_0]+([\varphi_1]+[\varphi_2])) \end{array}$$

which by definition really means:

$$\begin{array}{ccc} (PD_{\bullet}\varphi_0+PD_{\bullet}\varphi_1)+PD_{\bullet}\varphi_2 & \rightsquigarrow & PD_{\bullet}\varphi_0+(PD_{\bullet}\varphi_1+PD_{\bullet}\varphi_2) \\ \wr\downarrow id & & \wr\downarrow id \\ (PD_{\bullet}\varphi_0+PD_{\bullet}\varphi_1)+PD_{\bullet}\varphi_2 & & PD_{\bullet}\varphi_0+(PD_{\bullet}\varphi_1+PD_{\bullet}\varphi_2) \\ \wr\downarrow id & & \wr\downarrow id \\ (PD_{\bullet}\varphi_0+PD_{\bullet}\varphi_1)+PD_{\bullet}\varphi_2 & \rightsquigarrow & PD_{\bullet}\varphi_0+(PD_{\bullet}\varphi_1+PD_{\bullet}\varphi_2) \end{array}$$

which is true since all vertical natural transformations are identities.

e) Unitality Respected: We need the following diagrams to commute:

$$\begin{array}{ccc} * + \overline{\omega}([\varphi]) & \overset{id}{\rightsquigarrow} & \overline{\omega}(*) + \overline{\omega}([\varphi]) \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\ \overline{\omega}([\varphi]) & \underset{id}{\leftsquigarrow} & \overline{\omega}(* + [\varphi]) \end{array}$$

$$\begin{array}{ccc} \overline{\omega}([\varphi]) + * & \overset{id}{\rightsquigarrow} & \overline{\omega}([\varphi]) + \overline{\omega}(*) \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\ \overline{\omega}([\varphi]) & \underset{id}{\leftsquigarrow} & \overline{\omega}([\varphi] + *) \end{array}$$

which is true since all natural transformations are identities.

III) Symmetry Requirements: We need the following diagram to commute:

$$\begin{array}{ccc} \overline{\omega}([\varphi_0]) + \overline{\omega}([\varphi_1]) & \rightsquigarrow & \overline{\omega}([\varphi_1]) + \overline{\omega}([\varphi_0]) \\ \downarrow & & \downarrow \\ \overline{\omega}([\varphi_0] + [\varphi_1]) & \rightsquigarrow & \overline{\omega}([\varphi_1] + [\varphi_0]) \end{array}$$

which by definition really means:

$$\begin{array}{ccc} [\varphi_0(X)] + [\varphi_1(X)] & \longrightarrow & [\varphi_1(X)] + [\varphi_0(X)] \\ \downarrow & & \downarrow \\ (PD_\bullet\varphi_0 + PD_\bullet\varphi_1)([X]) & \longrightarrow & (PD_\bullet\varphi_1 + PD_\bullet\varphi_0)([X]) \end{array}$$

This means does

$$\overline{\omega}([\varphi_1] + [\varphi_0], \langle[\varphi_1], [\varphi_0]\rangle)_{[X]} = ([\varphi_1(X)] + [\varphi_0(X)], \langle[\varphi_1(X)], [\varphi_0(X)]\rangle)$$

Well, by definition,

$$\begin{aligned} \overline{\omega}([\varphi_1] + [\varphi_0], \langle[\varphi_1], [\varphi_0]\rangle)_{[X]} &= (([\varphi_1] + [\varphi_0]([X])), \langle[\varphi_1], [\varphi_0]\rangle([X])) \\ &= ([\varphi_1(X)] + [\varphi_0(X)], \langle[\varphi_1(X)], [\varphi_0(X)]\rangle) \end{aligned}$$

Together, this shows that $\overline{\omega}$ is a well-defined symmetric, monoidal functor between $PD_\bullet(\mathbf{Wald}(\mathcal{W};\mathcal{V}))$ and $\mathbf{Pic}(PD_\bullet\mathcal{W};PD_\bullet\mathcal{V})$.

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BIOGRAPHICAL SKETCH

I was born in Miami, FL by Cuban immigrants. My father was a math professor in Cuba who taught me at the age of three how to read and write, as well as some basic mathematics such as geometry and arithmetic. This is when I fell in love with the subject.

I am the first in my family to attend college (in the United States), let alone graduate with a doctorate degree. I graduated from my local high school as Valedictorian and was a recipient of the Gates Millennium Scholarship which covered my cost of attendance for my entire undergraduate and graduate career. I attended Florida State University as an undergraduate Pure Mathematics major and continued at FSU as a graduate student, where I worked as a Teaching Assistant. I have had the pleasure of doing research for the last several years with my PhD advisor, Dr. Ettore Aldrovandi.

Apart from several talks at FSU's algebra seminars, I was invited to speak at two conferences: UF/FSU Topology conference in Gainesville, FL in Spring of 2017 and the 2017 Lloyd Roeling UL Lafayette Mathematics Conference in Fall of 2017. I enjoyed these conferences so much as I was able to meet so many peers from around the country and learn about their research interests. These experiences will stay with me forever.