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THE SUMMABILITY OF INFINITE SERIES

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PREFACE

The purpose of this paper is to study methods by which a value can be assigned to an infinite series. The reason for studying about these methods lies in the fact that infinite series often appear as the end result of a calculation or computation. A desire to obtain a usable end result leads us to the investigation of methods for evaluating infinite series.

The first chapter of this paper deals with the method of ordinary convergence. If a series converges to a number S, we give the series the value S. Several tests are given by which we can determine whether a series converges or diverges.

The class of divergent series, presents a problem, since there is no way of evaluating a divergent series. The second chapter shows how early mathematicians struggled with the problem of assigning a value to a divergent series.

Several methods (processes) of summability or limitability are developed in the third chapter. As these methods are developed, they are required to satisfy two conditions, namely, the permanence condition and the extension condition. We also compare one process of summability with another and determine whether or not the two processes are compatible.

Many processes of summability other than the ones mentioned in this paper are known and used. We give only a few of the most easily applied methods of summability.

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CHAPTER I

BASIC DEFINITIONS AND THEOREMS

Definition 1-1. Sequence. If by some process of construction a succession of real numbers can be formed, whereby to each positive integer n there is assigned one and only one real number, u_n , then the real numbers

$$u_1 , u_2 , u_3 , \cdots , u_n , \cdots$$

are said to form a sequence.

Each real number u_i , where $i = 1,2,\cdots$, is called a term of the sequence.

The general term, u_n , is a rule for determining the number corresponding to the integer n. The notation for a sequence is $\{u_n\}$.

Definition 1-2. Convergence of a Sequence. A sequence $\{u_n\}$ is said to be convergent if there exists a finite number L such that for any $\epsilon > 0$, one can find a positive integer N, such that

$$|L - u_n| < \epsilon \text{ for all } n > N.$$

The terms u_i of the sequence are said to approach the limit L.

$$\lim_{n\to\infty} u_n = L .$$

Remark: A sequence which converges to zero is called a null sequence.

Definition 1-3. Infinite Series. Let u_1 , u_2 , u_3 , $\cdots$, u_n , $\cdots$ be a sequence of real numbers, $\{u_n\}$.

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

is called an infinite series.

Consider the sequence $\{s_n\}$ where s_n , the nth term of the sequence, is the sum of the first n terms of $\{u_n\}$.

$$s_n = \sum_{i=1}^{n} u_i$$

The numbers s_n are nth partial sums. $\{s_n\}$ is the sequence of partial sums.

Definition 1-4. Convergence of Infinite Series. An infinite series $\sum_{n=1}^{\infty} u_n$ is said to converge if and only if the sequence of partial sums, $\{s_n\}$, is convergent. A series which does not converge is called divergent.

Thus the criterion for the convergence of $\sum_{n=1}^{\infty} u_n$ is as follows: Given a finite number S , if for any $\epsilon > 0$, $\exists$ a positive integer N $\ni$

$$|S - s_n| < \epsilon \text{ for all } n > N,$$

then we say the series $\sum_{n=1}^{\infty} u_n$ converges to S. Symbolically, $\lim_{n\to\infty} s_n = S$.

The number S is called the sum of the series $\sum_{n=1}^{\infty} u_n$. Obviously, we cannot add infinitely many numbers together in the arithmetic sense and obtain a sum. We write $\sum_{n=1}^{\infty} u_n = S$, by which we mean $\lim_{n\to\infty} s_n = S$.