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Character Varieties of Knots and Links with Symmetries

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CHARACTER VARIETIES OF KNOTS AND LINKS WITH SYMMETRIES

By

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ABSTRACT

Abstract : Let M be a hyperbolic manifold. The $SL_2(\mathbb{C})$ character variety of M is essentially the set of all representations $\rho : \pi_1(M) \to SL_2(\mathbb{C})$ up to trace equivalence. This algebraic set is connected to many geometric properties of the manifold M. We examine the effect of symmetries of M on its character variety. We compute the $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$ character varieties for an infinite family of two-bridge hyperbolic knots with symmetry. We explore the effect the symmetry has on the character variety and exploit this symmetry to factor the character variety. We then find the geometric genus of both components of the character variety. We compute the $SL_2(\mathbb{C})$ character variety for the Borromean ring complement in S^3. Further, we explore how the symmetries effect this character variety. Finally, we prove some general results about the structure of character varieties of links with symmetries.

CHAPTER 1

INTRODUCTION

Let M be a hyperbolic 3-manifold of finite volume. An intriguing question is how the fundamental group $\pi_1(M)$ relates to the topological properties of the manifold M. In this paper, we will be interested in the connection between $\pi_1(M)$ and the metric structures that can be given to M. By Mostow-Prasad rigidity, we know that there is only one complete hyperbolic metric that can be given to M. However, there are many incomplete metrics that can be given to M as well. Many of these give insight into the complete metric.

The theory of character varieties has been developed by Culler, Shalan, and Thurston as explained in [9] and [22].

In this chapter, we will define the $SL_2(\mathbb{C})$ character variety of a finitely generated group G, and then discuss the connection of the character variety of $G = \pi_1(M)$ to the geometry of a finite-volume hyperbolic 3-manifold M.

1.1 The $SL_2(\mathbb{C})$ Representation Variety

We will now define the $SL_2(\mathbb{C})$ representation variety of a finitely generated group G.

Let G be a finitely generated group. Let $R(G)$ be the set of all representations of G into $SL_2(\mathbb{C})$. This is the set of all group homomorphisms

$$\rho : G \to SL_2(\mathbb{C}).$$

We can define G by a finite set of generators $g_1, g_2, \ldots, g_n$ and a set of relations $\{r_\alpha\}_{\alpha \in A}$, where each r_α is a word in $g_1, g_2, \ldots, g_n$. Any homomorphism as above is determined by the set of $\rho(g_i)$. For each i, we have

$$\rho(g_i) = \begin{pmatrix} w_i & x_i \\ y_i & z_i \end{pmatrix} = G_i$$

with $w_i z_i - x_i y_i = 1$. Identifying each representation ρ with the point

$$(w_1, x_1, y_1, z_1, \ldots, w_n, x_n, y_n, z_n) \in \mathbb{C}^{4n},$$

and letting V be the set of all such points, it is easy to show that V is an algebraic set.

We will now sketch this proof. Since each ρ is a homomorphism into $SL_2(\mathbb{C})$, each point corresponds to a homomorphism into $SL_2(\mathbb{C})$ if and only if the determinant of each G_i is 1, and $\rho(r_\alpha)$ is the identity matrix. For each word $r_\alpha = r_\alpha(g_1, \ldots, g_n)$, let $r_\alpha(G_1, \ldots, G_n)$ be the matrix obtained by substituting the matrix G_i in for the element g_i. Of course some words may contain powers of ${g_i}^{-1}$. Since each $G_i \in SL_2(\mathbb{C})$, we have

$$G_i^{-1} = \begin{pmatrix} z_i & -x_i \\ -y_i & w_i \end{pmatrix}.$$

Therefore for each word r_α we have

$$r_\alpha(G_1, \ldots, G_n) = \begin{pmatrix} P_\alpha & Q_\alpha \\ R_\alpha & S_\alpha \end{pmatrix}$$

where the $P_\alpha, Q_\alpha, R_\alpha$, and S_α are polynomials in the variables

$$w_1, x_1, y_1, z_1, \ldots, w_n, x_n, y_n, z_n.$$

Therefore $\rho \in R(G)$ if and only if for all α and all i we have

$$w_i z_i - x_i y_i - 1 = 0$$

and

$$P_\alpha - 1 = 0,\ S_\alpha - 1 = 0,\ Q_\alpha = 0,\ R_\alpha = 0.$$

This shows that V is the vanishing set of a finite set of polynomials, and therefore is an algebraic set. Of course, this set depends on the set of generators chosen. A different set of generators will result in a different algebraic set, but as shown in [9] these sets are isomorphic. Therefore, for a fixed group G, up to isomorphism, the set V is unique. We will call this set $R(G)$.

Definition 1.1.1. Let G be a finitely generated group. If $g_1, \ldots, g_n$ is a set of generators of G, the $SL_2(\mathbb{C})-$ *representation variety* of G is the set of points $R(G)$ in $\mathbb{C}^{4n}$ as defined above.

Note that in general, $R(G)$ is not an irreducible algebraic set.

1.2 The $SL_2(\mathbb{C})$ Character Variety

Definition 1.2.1. Let $\rho : G \to SL_2(\mathbb{C})$ be a representation. The *character* of ρ is the function $\chi_\rho : G \to \mathbb{C}$ defined by $\chi_\rho(g) = \text{trace}(\rho(g))$. We denote the set of all characters by $C(G) = \{\chi_\rho \mid \rho \in R(G)\}$. The *trace map* is the function $\tau : R(G) \to C(G)$ defined by $\tau(\rho) = \chi_\rho$.

$C(G)$ is often called the *character variety* of G.

Note that the group $SL_2(\mathbb{C})$ acts on $R(G)$ by conjugation : if ρ is a representation of G into $SL_2(\mathbb{C})$ and $A \in SL_2(\mathbb{C})$, we can define the representation $A \cdot \rho$ as

$$A \cdot \rho(g) = A\rho(g)A^{-1}.$$

Let $\hat{R}(G)$ be the set of orbits. Two homomorphisms ρ and ρ' are called *equivalent* if they are in the same orbit. It is clear that if ρ and ρ' are equivalent representations, they must have the same character. Therefore the trace map given in the previous definition induces a well-defined map

$$\hat{R}(G) \to C(G).$$

This is not an injective map. Even though it is clear that conjugate representations have the same character, Shalen demonstrates in [22] that the converse is not true. Note that any homomorphism of G into the group of matrices of the form

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \quad \lambda \in \mathbb{C}$$

has the same character as the trivial representation that sends each $g \in G$ to I_2, the 2×2 identity matrix.

A representation $\rho : G \to SL_2(\mathbb{C})$ is *reducible* if it is equivalent to a representation by upper triangular matrices. Culler and Shalen prove the following proposition in [9]:

Proposition 1.2.2. *Let G be a finitely generated group. If two representations $\rho, \rho' : G \to SL_2(\mathbb{C})$ have the same character, then either ρ and ρ' are equivalent or they are both reducible.*

We will be mainly interested in *irreducible* representations, that is, representations that are not reducible. First, note that we want to avoid considering abelian representations. If M is a knot complement and $\rho : \pi_1(M) \to SL_2(\mathbb{C})$ is an abelian representation, we have $\rho(g) = I_2$ for every element g in the commutator subgroup of $\pi_1(M)$. Therefore $\rho(\ell) = I_2$ for every longitude of M.

Since all meridians of M are conjugate, if m is any meridian of M we have $\rho(m) = A^{\pm 1}$ for some $A \in SL_2(\mathbb{C})$. Therefore if m is any chosen meridian of M, $\rho(m)$ generates the image of ρ, and $\rho(\pi_1(G)) \cong \mathbb{Z}$. So ρ is a also a representation of the trivial knot. These representations don't give any useful information about the topology of the knot. Avoiding reducible representations ensures that we avoid the abelian ones. Not every reducible representation is abelian. However, any discrete faithful representation ρ_0 must be irreducible. Avoiding reducible representations ensures that $\chi_\rho = \chi_{\rho'}$ if and only if ρ and ρ' are conjugate.

Note that an equivalent definition of an irreducible representation is a representation such that the only subspaces of $\mathbb{C}^2$ that are invariant under the action of $\rho(G)$ are the trivial subspace and $\mathbb{C}^2$. For example, it is clear that any representation that maps G into the subspace of upper triangular matrices is indeed reducible because it fixes the subspace $(a, 0)$ of $\mathbb{C}^2$.

Culler and Shalen give a detailed proof of the following theorem in [9]:

Theorem 1. The set of characters, $C(G)$, is an algebraic set.

We will now sketch this proof. For details see [9].

The first step in the proof is to set up a one-to-one correspondence between the characters in $C(G)$ and a set of points in $\mathbb{C}^m$. For each $g \in G$ we define a function $\tau_g : R(G) \to \mathbb{C}$ by $\tau_g(\rho) = \chi_\rho(g)$. Let T denote the ring generated by all functions $\tau_g, g \in G$. If $g_1, \ldots, g_n$ is a set of generators for G, the set of functions of the form

$$\tau_{g_{i_1} \cdots g_{i_k}} \text{ with } 1 \leq i_1 < \cdots < i_k \leq n$$

forms a set of generators for T. The proof is based on the matrix identity

$$\text{trace}(XY) = \text{trace}(X)\text{trace}(Y) + \text{trace}(XY^{-1}).$$

This shows that T is finitely generated. So we can fix a finite set of elements $\gamma_1, \ldots, \gamma_m \in T$ which completely determine T. Now we can define a map

$$t : R(G) \to \mathbb{C}^m \text{ by } t(\rho) = (\tau_{\gamma_1}(\rho), \ldots, \tau_{\gamma_m}(\rho)).$$

We claim that $C(G)$ is isomorphic to $t(R(G))$. Each point of $t(R(G))$ corresponds to a unique character $\rho \in R(G)$.

The goal is to show that $t(R(G))$ is an algebraic set, or in other words, to show that $t(R(G))$ is closed under the Zariski topology. Recall that the *Zariski topology* on $\mathbb{C}^m$ is the topology where the closed sets are the algebraic sets. The Zariski topology is coarser than the usual topology, as any set which is closed under the Zariski topology is also closed under the usual topology.

Note that $t(R(G))$ is the image of an algebraic set under a regular map (a map defined by polynomials). The image of an algebraic set under a polynomial map need not be an algebraic set, but its closure (under the usual topology on $\mathbb{C}^m$) is an algebraic set. Therefore, to prove that $t(R(G))$ is closed under the Zariski topology, it is enough to show that it is a closed set under the usual topology.

Since $R(G)$ is an algebraic set, we can write $R(G) = V_1 \cup \cdots \cup V_j$, where each V_i is an irreducible algebraic set. Write $R(G) = A \cup B$, where A is the union of all V_i which contain an irreducible representation, and B is the set of all reducible representations in $R(G)$. Then $t(R(G)) = t(A) \cup t(B)$, and it suffices to show that both $t(A)$ and $t(B)$ are closed under the usual topology. Culler and Shalen first show that $t(A)$ is closed. It is shown that if V_i contains an irreducible representation, then $t(V_i)$ is an affine variety (and therefore closed under the usual topology). Therefore $t(A)$ is an algebraic set.

Next it is shown that $t(B)$ is closed. Every representation $\rho \in B$ is equivalent to a representation by upper triangular matrices so we can assume that for the generators $g_1, \ldots, g_n$ we have

$$\rho(g_i) = \begin{pmatrix} w_i & x_i \\ 0 & w_i^{-1} \end{pmatrix}$$

which has the same character as the representation given by

$$\rho'(g_i) = \begin{pmatrix} w_i & 0 \\ 0 & w_i^{-1} \end{pmatrix}.$$

Given an upper triangular $\rho \in R(G)$, it follows that $\rho' \in R(G)$. Recall from the previous section that each word r_α corresponds to polynomials $P_\alpha, Q_\alpha, R_\alpha, S_\alpha$ in the variables

$$w_1, x_1, y_1, z_1, \ldots, w_n, x_n, y_n, z_n,$$

and $\rho \in R(G)$ if and only if ρ can be identified with a point in $\mathbb{C}^{4n}$ such that $w_i z_i - x_i y_i = 1$ for all i and $P_\alpha = S_\alpha = 1$, $Q_\alpha = R_\alpha = 0$ for all α. Since ρ is upper triangular, this representation corresponds to a point

$$(w_1, x_1, 0, z_1, \ldots, w_n, x_n, 0, z_n),$$

with $w_i z_i = 1$ for all i. The representation ρ' corresponds to the point obtained by taking the above point and setting each $x_i = 0$. Setting each $y_i = 0$ results in the polynomials P_α and S_α being expressed only in the variables w_i, z_i. Therefore if the point corresponding to ρ satisfies $P_\alpha = S_\alpha = 1$, the point corresponding to ρ' must as well. Since each $x_i = y_i = 0$ in the point corresponding to ρ' we automatically have $Q_\alpha = R_\alpha = 0$ for this second point.

Therefore, if D is the set of all diagonal representations in $R(G)$, we have $t(B) = t(D)$. It follows that $t(B)$ is the image of the map $t : D \to \mathbb{C}^m$. Recall that a map $f : X \to Y$ is *proper* if $f^{-1}(S) \subset X$ is compact whenever $S \subset Y$ is compact. It it easy to show that $t : D \to \mathbb{C}^m$ is a proper map. Each representation $\rho \in D$ corresponds to a point

$$(w_1, 0, 0, {w_1}^{-1}, \ldots, w_n, 0, 0, {w_n}^{-1}) \in \mathbb{C}^{4n}.$$

Suppose we have a sequence of points

$$p_j = (w_{j_1}, 0, 0, {w_{j_1}}^{-1}, \ldots, w_{j_n}, 0, 0, {w_{j_n}}^{-1}) \in \mathbb{C}^{4n}$$

such that $\{p_j\} \to \infty$. This can only happen if either the set $\{w_{j_k}\}$ or the set $\{{w_{j_k}}^{-1}\}$ is unbounded. But this implies that the sequence $\{t(p_j)\} \to \infty$, because n of the coordinates of $t(p_j)$ are of the form $w_{j_k} + {w_{j_k}}^{-1}$. Therefore $t : D \to \mathbb{C}^m$ is indeed a proper map, and $t(D)$ is a closed set. It follows that $C(G)$ is an algebraic set.

Here we will assume that $G = \pi_1(M)$, where M is a complete, finite volume hyperbolic 3-manifold. Since we are interested in the connection between the character variety and hyperbolic metrics of M, we will be interested only in irreducible representations. By 1.2.2 we know that for irreducible representations, ρ and ρ' have the same character if and only if they are conjugate. So for our purposes, the $SL_2(\mathbb{C})$ character variety of G is the set of all representations $\rho : G \to SL_2(\mathbb{C})$ up to conjugation. Since $R(G)$ is an algebraic set, we can't "mod out" by conjugation, so the character variety is used.

Definition 1.2.3. Let M be a complete, finite volume hyperbolic manifold. The $SL_2(\mathbb{C})$-*character variety* of M is the Zariski closure of the set of characters of irreducible representations $\rho : \pi_1(M) \to SL_2(\mathbb{C})$. We denote this set by $X(M)$.

Recall that $X(M)$ is well-defined up to algebraic isomorphism. If $\Gamma \cong \pi_1(M)$ is an explicit group presentation we use $X(\Gamma)$ to denote the concrete algebraic set we obtain, which is well-defined.

1.3 The $PSL_2(\mathbb{C})$ Character Variety

For any representation $\rho : G \to SL_2(\mathbb{C})$, we can obtain a representation $\bar{\rho} : G \to PSL_2(\mathbb{C})$ by composing with the quotient map $q : SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$ defined by $A \mapsto \{A, -A\}$. The set of all characters of irreducible representations $\bar{\rho} : G \to PSL_2(\mathbb{C})$ is also an affine algebraic set which we call $Y(G)$. The set $X(G)$ is a double cover of $Y(G)$.

Let M be a hyperbolic 3-manifold and let $G = \pi_1(M)$. Since G is a finitely generated group we can choose a set of generators for G and define the $SL_2(\mathbb{C})$ character variety of G as above. We are interested in the connection between this algebraic set and the geometric structures that we can give to M. The geometric structures of M derive from the geometry of hyperbolic 3-space. There are many models of hyperbolic 3-space. Here we will use the *upper half-space model*. This consists of the set of points in upper half-space

$$\mathbb{H}^3 = \{(x, y, z) | x, y, z \in \mathbb{R} \text{ and } z > 0\}$$

together with the *hyperbolic metric*. If γ is a piecewise differentiable path in $\mathbb{H}^3$, we define the hyperbolic length of γ using the hyperbolic length formula:

$$\text{length}(\gamma) = \int_\gamma \frac{1}{\text{Im}(z)}.$$

Since M is hyperbolic, there exists a covering map $p : \mathbb{H}^3 \to M$. Recall that a *covering map* is a continuous, surjective map $p : \mathbb{H}^3 \to M$ such that every point $q \in M$ has a neighborhood that is evenly covered by p. The particular model of hyperbolic 3-space we choose to work with does not matter, although of course the specific covering map will depend upon the model chosen.

Since $\text{Isom}^+(\mathbb{H}^3) \cong PSL_2(\mathbb{C})$, where $\text{Isom}^+(\mathbb{H}^3)$ is the group of orientation preserving isometries of $\mathbb{H}^3$, M is a isomorphic to a quotient of $\mathbb{H}^3$ by a discrete subgroup of $PSL_2(\mathbb{C})$, and there exists a discrete faithful representation

$$\bar{\rho}_0 : \pi_1(M) \hookrightarrow PSL_2(\mathbb{C})$$

which is unique up to conjugation (up to orientation reversal). Then $M \cong \mathbb{H}^3/\bar{\rho}_0(\pi_1(M))$. This gives us the unique complete metric on M (up to orientation reversal). Any loop in $\mathbb{H}^3/\bar{\rho}_0(\pi_1(M))$ lifts to a path in the covering space $\mathbb{H}^3$, and we can find the length of the path using the hyperbolic metric.

In [7] Culler shows that if the image of $\bar{\rho}_0$ is a discrete subgroup of $PSL_2(\mathbb{C})$ with no elements of order 2, then the representation above can be lifted to a discrete faithful representation

$$\rho_0 : \pi_1(M) \hookrightarrow SL_2(\mathbb{C}).$$

Since M is orientable, $\pi_1(M)$ is torsion free, so a lift ρ_0 exists. There is more that one lift. Fix a lift ρ_0. Thurston [23] showed that the character of ρ_0 is contained in a unique component of $X(G)$. We will call this component $X_0(M)$.

Definition 1.3.1. Let $X(M)$ be the character variety of a hyperbolic manifold M. A *canonical component* $X_0(M)$ is any irreducible component of $X(M)$ which contains a point corresponding to a discrete faithful representation of $\pi_1(M)$.

There can be more than one canonical component $X_0(M)$ due to the lifting from $PSL_2(\mathbb{C})$ to $SL_2(\mathbb{C})$, and due to the fact that there are two discrete faithful characters in the $PSL_2(\mathbb{C})-$character variety due to orientation reversing.

If we have an arbitrary irreducible representation $\bar{\rho} : \pi_1(M) \to PSL_2(\mathbb{C})$, that representation can be lifted to a representation $\rho : \pi_1(M) \to SL_2(\mathbb{C})$ so that the images under ρ and $\bar{\rho}$ are isomorphic.

Some of the connections between the character variety and the topology of the manifold M are well known. Thurston [23] has showed that if M is hyperbolic and has n cusps, the canonical component $X_0(M)$ has complex dimension n. Both Boyer, Luft, and Zhang [2] and Ohtsuki, Riley, and Sakuma [17] have shown that the number of irreducible components of $X(M)$ can be arbitrarily large. Boyer, Luft, and Zhang have done this by using injections while Ohtsuki, Riley, and Sakuma have done this by using epimorphisms between 2-bridge link groups.

Computations of character varieties have been done for individual knot complements, but there are only a few examples of computations for infinite families of one-cusped manifolds (see [1], [15], [24], and [25]) or for manifolds with more than one cusp (see [14] and [20]).

The applications of character varieties are very extensive. Culler, Gordon, Luecke, and Shalen used character varieties to prove the cyclic surgery theorem [8]. Boyer and Zhang also used character varieties in their proof of the finite filling theorem [4]. Character varieties have been used to study actions on trees and detecting surfaces, as in [3]. This can also be seen using the A-polynomial, which is related to character varieties and essentially birational to it for knots. The A-polynomial was originally defined in [5]. The A-polynomial has many conjectural relationships with quantum physics (see [11], for example).

For a given hyperbolic manifold M, many natural questions arise regarding the properties of $X(M)$. First, we want to know how many irreducible components $X(M)$ has. We also want to identify the canonical component, which is the component corresponding to the unique complete metric structure on M. When M has exactly one cusp, any canonical component $X_0(M)$ is a complex curve. Two measures of the complexity of a complex curve are geometric genus and gonality. The *gonality* of an algebraic curve C is the minimal degree of a dense map from C to $\mathbb{C}$. In [19], Petersen and Reid studied the connection between the gonality of a family of curves to the underlying manifold M. They looked at how the gonality of these curves behaves in families of Dehn fillings on 2-cusped hyperbolic manifolds. They studied families of 1-cusped manifolds obtained by Dehn fillings of a single cusp of a fixed 2-cusped hyperbolic 3-manifold M. They showed that if $M(-, p/q)$ is the 1-cusped manifold obtained by Dehn filling one of the cusps of M, then the gonality $\gamma(X_0(M(-, p/q))$ of the canonical component is bounded by a constant, and the constant depends on the underlying manifold M.

1.4 The A-Polynomial and the Eigenvalue Variety

In this section we define the A-polynomial for a knot and then generalize the definition to an n-component link. The definition of the A-polynomial is given in [6].

Recall that the *torus* T is the 2-space $S^1 \times S^1$. We denote the *solid torus* $S^1 \times D^2$ by S. Note that S is a 3-manifold with boundary T.

Definition 1.4.1. Let S be a solid torus with $\partial S = T$. A *meridian* of T is a simple closed curve on T which bounds a disk in S. A *longitude* of T is a simple closed curve on T which represents a generator of $\pi_1(S)$.

If $M = S^3 - K$ is a knot complement, the boundary of M is homeomorphic to a torus T. If we fix the knot K, the meridian of T is well-defined up to sign. That is, up to homeomorphism, there are exactly 2 meridians of T (one for each choice of orientation).

Let M be a knot complement in S^3. If T is the torus boundary of M, we can fix a meridian μ and a longitude λ of T. If $G = \pi_1(M)$, we have shown in Chapter 1 that $R(G) = \text{Hom}(\pi_1(M), SL_2(\mathbb{C}))$

can be identified with an affine algebraic set. We define R_U to be the set of all $\rho \in R(G)$ such that both $\rho(\mu)$ and $\rho(\lambda)$ are upper triangular. This subset of $R(G)$ is also an algebraic set. Let $\rho \in R_U$. We have seen that if we choose a set of n generators of G, the set $R(G)$ can be identified with an algebraic subset of $\mathbb{C}^{4n}$. Writing μ and λ in terms of the generators $g_1, \ldots, g_N$ allows us to express

$$\rho(\mu) = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \text{ and } \rho(\lambda) = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

where each P_{ij}, Q_{ij} is a polynomial in the $4n$ variables $w_1, x_1, y_1, z_1, \ldots, w_n, x_n, y_n, z_n$ as before. A representation $\rho \in R(G)$ is also in R_U if and only the polynomial relations $P_{21} = 0$ and $Q_{21} = 0$ are also satisfied. This shows that R_U is also an algebraic set.

If $\rho \in R_U$ we have

$$\rho(\mu) = \begin{pmatrix} M & * \\ 0 & M^{-1} \end{pmatrix} \text{ and } \rho(\lambda) = \begin{pmatrix} L & * \\ 0 & L^{-1} \end{pmatrix}$$

Note that every representation ρ is equivalent to a representation of this form. Every matrix in $SL_2(\mathbb{C})$ is conjugate to an upper triangular matrix. Therefore we can assume without loss of generality that $\rho(\mu)$ is upper triangular. However, since $\rho(\mu)$ and $\rho(\lambda)$ must commute, either $\rho(\lambda)$ must be upper triangular as well, or $\rho(\mu)$ is diagonal and $\rho(\lambda)$ is lower triangular. If the latter is the case it is possible to conjugate so that both $\rho(\mu)$ and $\rho(\lambda)$ are upper triangular.

The advantage of both of these matrices being upper triangular is that we can see that the eigenvalues of the first matrix are M and M^{-1}, and the eigenvalues of the second are L and L^{-1}. We define the *eigenvalue map*

$$(\xi_\mu \times \xi_\lambda) : R_U \to \mathbb{C}^2 \text{ by } \rho \mapsto (M, L).$$

A point (M, L) is in the image of the eigenvalue map if there exists a representation $\rho \in R_U$ with $\rho(\mu)$ and $\rho(\lambda)$ as above. Since R_U is an algebraic set we can write $R_U = V_1 \cup \cdots \cup V_m$, with each V_i an algebraic variety. Let $\overline{\xi(V_i)}$ be the Zariski closure of $\xi(V_i)$. This is an algebraic subset of $\mathbb{C}^2$. When $\overline{\xi(V_i)}$ has dimension 1, there is a polynomial $F_i(M, L)$ which defines the curve. The A-polynomial is the product of all such $F_i(M, L)$. This polynomial is unique up to scaling, regardless of the chosen μ and λ. Note that R_U contains the set of all abelian representations. These are the set of all representations of the "untangled" knot, which is the trivial knot. Since any longitude of the trivial knot is mapped to the identity matrix, the polynomial representing the reducible representations is

$L-1$. This is always a factor of the A-polynomial and is often ignored because it doesn't give any useful information about the topology of the knot.

It is easy to see that in the A-polynomial the abelian representations and reducible representations both coincide with the polynomial $L-1$. Any longitude $\ell \in \pi_1(M)$ is in the commutator subgroup, so if ρ is any abelian representation, we must have $\rho(\ell) = I_2$ and $L = 1$. If ρ is a reducible representation, we can assume without loss of generality that it is upper triangular. For each meridian $\mu_j \in \pi_1(M)$ we have

$$\rho(\mu_j) = \begin{pmatrix} M_j & * \\ 0 & M_j{}^{-1} \end{pmatrix} \quad \text{and} \quad \rho(\mu_j^{-1}) = \begin{pmatrix} M_j{}^{-1} & * \\ 0 & M_j \end{pmatrix}$$

Since each longitude ℓ can be expressed in terms of meridians, and is in the commutator subgroup, it is clear that we must have $L = 1$.

A generalization of the definition of the A-polynomial can be made for an n-component link M. This has been done by Klaff and Tillmann in [12].

Let M be an orientable, n-cusped hyperbolic 3-manifold of finite volume, and let $Y_0(M)$ be a canonical component in the $PSL_2(\mathbb{C})$-character variety of M. We define the *restriction map* $r : Y_0(M) \to Y(\partial M)$ as follows : for each cusp C_i of M, choose a meridian μ_i and longitude λ_i of the torus boundary T_i of C_i. For any character $\chi_\rho \in Y_0(M)$, we restrict the map $\chi_\rho : \pi_1(M) \to \mathbb{C}$ to the set of μ_i, λ_i. The *eigenvalue variety* is the Zariski closure of the set of points $(m_1, \ell_1, \ldots, m_n, \ell_n)$ in $\mathbb{C}^{2n}$ with the property that there is a representation $\rho : \pi_1(M) \to SL_2(\mathbb{C})$ such that $\rho(M_i)$ and $\rho(L_i)$ have eigenvalues m_i and ℓ_i respectively with respect to a common eigenvector. The image of the canonical component $Y_0(M)$ under the eigenvalue map is birational to $Y_0(M)$.

1.5 Two-Bridge Knots

A *two-bridge knot* is a knot which admits a projection with exactly two maxima and two minima.

Let (p, q) be a pair of coprime odd integers with $-p < q \leq p$. This pair corresponds to a two-bridge knot $K(p, q)$ obtained using an algorithm described in [15]. The algorithm is as follows. First choose $\epsilon \in \{0, 1\}$ such that $0 < q/p + \epsilon \leq 1$. Then write $q/p + \epsilon$ as a continued fraction :

$$\frac{q}{p} + \epsilon = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots + a_s}}}}$$

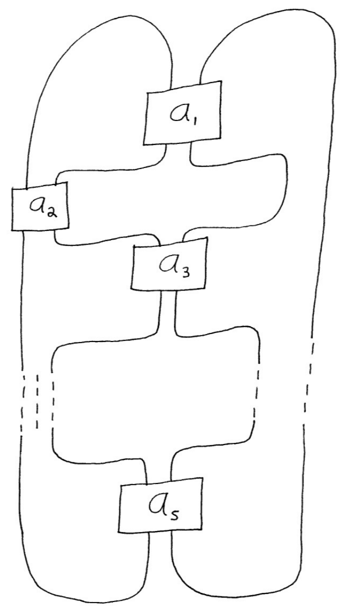

Figure 1.1: The $[a_1, \ldots, a_s]$ knot, a 2-bridge knot

where each $a_i \geq 1$. This gives us a sequence of positive integers $[a_1, \ldots, a_s]$. We can always obtain a sequence where s is odd. If s is even in the continued fraction above, we can rewrite the fraction as follows. If $a_s = 1$, replace a_{s-1} by $a_{s-1} + 1$. This will be the final term in the new sequence. If $a_s = s > 1$, replace the final entry a_s by $a_s = a - 1$ and let $a_{s+1} = 1$.

We use the sequence obtained to create the following knot, where the number in the block corresponds to a number of half-twists. When j is odd, the block j-th block contains a_{2j-1} left-handed half twists. When j is even, the j-th block contains a_{2j} right-handed half-twists. This is called the *4-plat presentation* of the 2-bridge knot $K(p, q)$.

Every two-bridge knot can be presented in this form. The knots $K(p, q)$ and $K(p', q')$ are ambient isotopic (the same knot) if and only if $p = p'$ and either $q = q'$ or $qq' \equiv 1 \pmod{p}$. This follows from the following fact. Let q/p be the fraction equal to the continued fraction associated to the sequence $a_1, \ldots, a_s$ as defined above. Let q'/p' be the fraction equal to the continued fraction associated to the reverse sequence $a_s, \ldots, a_1$. Then $p = p'$ and $qq' \equiv 1 \pmod{p}$. The knot $K(p', q')$ is the knot obtained by turning the knot $K(p, q)$ upside down. So the knots $K(p, q)$ and $K(p', q')$ are ambient isotopic if and only if either $q/p = q'/p'$, or the sequences obtained in the continued fraction expansion of q'/p' is the reverse of the sequence obtained in the expansion of q/p.

If we have both $q = q'$ and $qq' \equiv 1 \pmod{p}$, then the sequences $a_1, \ldots, a_s$ and $a_s, \ldots, a_1$ are the same, and the knot $K(p, q)$ contains a flip symmetry obtained by turning the knot upside down. If G is the symmetry group of a two-bridge knot, then G either has order 4 or order 8. If $|G| = 8$ then G contains this extra flip symmetry.

Not all two-bridge knots are hyperbolic, but when $K(p, q)$ is a hyperbolic knot this extra symmetry can be used to factor the character variety. In [15] Macasieb, Petersen, and van Luijk computed the character variety of an infinite family of double twist knots and used the symmetry of these knots to factor the character variety.

1.6 Dehn Fillings

Suppose that M is a hyperbolic knot complement in S^3. We can think of M as being either open or closed. If we want to think of M as a closed manifold, we can remove a tubular neighborhood $N(K)$ of the knot K. Then $M = S^3 - \mathring{N}(K)$ is a closed manifold with torus boundary. If we want to think of M as open, consider a closed neighborhood $\overline{N(K)}$ of the knot. $\overline{N(K)}$ is homeomorphic to $T \times [0, \infty)$, where T is a torus. Think of $\overline{N(K)}$ as an infinite number of tori wrapped around the knot K. When we think of M as open, we refer to the neighborhood $\overline{N(K)}$ as the *cusp* of M, and think of it as $T \times [0, \infty)$. The reason for this is that it reflects how geodesics behave under the complete hyperbolic metric. Any geodesic which approaches the knot must extend indefinitely. We will sometimes talk of M as open and sometimes talk of the manifold M as having a torus boundary.

Suppose we identify M as an open manifold with cusp C. Since we think of the cusp as $T \times [0, \infty)$, M is homeomorphic to a manifold with end $T \times [0, 1)$. A *Dehn filling* of the cusp results from removing a neighborhood of C and gluing in a solid torus S by a diffeomorphism of the boundary of S with the boundary of C. To do this, we identify M with the interior of the manifold whose end is $T \times [0, 1]$ which has torus boundary $T \times \{1\}$. The gluing results from a diffeomorphism of the boundary of S with $T \times \{1\}$.

If we think of M as having a boundary, then M is homeomorphic to a manifold with end $T \times [0, 1]$ which has torus boundary $T \times \{1\}$. The Dehn filling results from gluing a solid torus S to M by an identification of the boundary of S with the torus boundary of M.

There are infinitely many ways to do this. We need a bijection between the 2 boundaries. Let m_S be any chosen meridian of the solid torus S, and let m, ℓ be a chosen meridian and longitude of the cusp boundary T. The diffeomorphism is determined solely by where m_S is mapped to. In order for the identification to be a bijection, we need $m_S \mapsto m^p \ell^q$, where $(p, q) = 1$. The ratio p/q is called the *filling slope*, and we refer to the resulting manifold as $M(p/q)$. Note that there are two possible orientations for each m and each ℓ. Since $p/q = -p/-q$, there are really two possibilities for $M(p/q)$. The choice of orientation for m, ℓ is called the *framing*.

The character variety $X(M)$ includes the character variety of the filled manifold (which we denote by $X(M(p/q))$) in a natural way. Fix a meridian m and a longitude ℓ of the cusp C to be filled. If Γ is a presentation for $\pi_1(M)$, a presentation $\Gamma(p/q)$ for $\pi_1(M(p/q))$ results from adding the relation $m^p \ell^q$ to the relations in Γ. This extra relation induces the quotient map $\varpi : \pi_1(M) \to \pi_1(M(p/q))$. The inclusion is then given by $i : X(M(p/q)) \to X(M)$ where $i(\chi_\rho) = \chi_{\rho'}$ and $\rho' = \rho \circ \varpi$. In particular, $X_0(M(p/q)) \subset X_0(M)$.

As long as $(p, q) = 1$, a Dehn filling as described above will result in a manifold. It is possible for different filling slopes to give the same manifold. When $p/q \neq p'/q'$ but $M(p/q)$ and $M(p'/q')$ are the same manifold, we refer to p/q and p'/q' as *cosmetic fillings*. However, as p and q range over all possibilities, an infinite number of distinct manifolds will result. This is due to a result from Thurston in [23]. We define the *height* of the filling slope by $|p| + |q|$. If M is a one-cusped hyperbolic 3-manifold of finite volume, then the volume of the filled manifold $M(p/q)$ limits to the volume of M as $|p| + |q| \to \infty$. Therefore there are an infinite number of distinct volumes. Recall from Mostow-Prasad rigidity that if M and N are finite volume hyperbolic 3-manifolds, then $\pi_1(M)$ and $\pi_1(N)$ are isomorphic if and only if there exists an isometry $\phi : M \to N$. Therefore if $M(p/q)$ and $M(p'/q')$ have distinct volumes, then $\pi_1(M(p/q))$ and $\pi_1(M(p'/q'))$ are not isomorphic, and as natural subvarieties of $X_0(M)$, the points $X_0(M(p/q))$ and $X_0(M(p'/q'))$ are distinct. So any time you do an infinite number of Dehn fillings, you will obtain an infinite number of distinct points inside $X_0(M)$.

We will be interested in the Dehn fillings which result in a *hyperbolic* manifold. Thurston also showed that $M(p/q)$ is hyperbolic when the height of the slope is sufficiently large. Therefore, all but finitely many of the $M(p/q)$ are hyperbolic.

We will be most interested in the canonical component $X_0(M)$. In the case where M has one cusp, $X_0(M)$ is a one-dimensional variety, and all but finitely many p/q Dehn fillings correspond to a single point $X_0(M(p/q))$ on this curve. Each of these points corresponds to a metric on the original M that is "almost" complete. In other words, with this metric M is incomplete, but we can obtain a complete metric by performing a p/q Dehn filling on the cusp of M.

This can be generalized to an n-cusped manifold : if $C_1, \ldots, C_n$ are the cusps of M, all but finitely many Dehn fillings of C_1 result in an $(n-1)$-cusped hyperbolic manifold. We refer to the manifold resulting from a p_1/q_1 Dehn filling of C_1 by $M(p_1/q_1, -, \ldots, -)$. We can continue filling the cusps in this way, restricting the fillings to those which result in a hyperbolic manifold. By Thurston we know that there are infinitely many ways to Dehn fill all n cusps to give a hyperbolic manifold $M(p_1/q_1, p_2/q_2, \ldots, p_n/q_n)$. By Thurston infinitely many of the points $X(M(p_1/q_1, p_2/q_2, \ldots, p_n/q_n))$ must lie on the canonical component $X(M)$. For this reason the canonical component is often referred to as the *Dehn surgery* component.

CHAPTER 2

SYMMETRIES OF CHARACTER VARIETIES

2.1 General Lemmas and 1-Cusped Manifolds

We are interested in how symmetries act on character varieties. We will be most interested in non-trivial symmetries of M.

Definition 2.1.1. A symmetry σ is *non-trivial* if there exists an element $\alpha \in \pi_1(M)$ such that $\sigma(\alpha)$ is not freely homotopic to either α or α^{-1}.

The effect of the symmetry group of a hyperbolic manifold on its character variety has been explored for many families of manifolds. In [1], Baker and Petersen looked at the symmetry groups of once-punctured torus bundles M_n, where M_n is the knot complement obtained by $-(n+2)$-Dehn filling one boundary component of the Whitehead link complement. M_n is hyperbolic when $|n| > 2$. When $|n| > 3$, the symmetry group of M_n is generated by two involutions, spin and flip. The flip involution is an orientation reversing involution of order 2. It acts trivially on all unoriented free homotopy classes of loops in M_n. That is, when σ is a flip involution, we have $\sigma(\alpha) = \alpha^{\pm 1}$ for all $\alpha \in M_n$. The flip involution induces a trivial action on the character variety. However, the spin involution induces a non-trivial action on the character variety.

The following lemma shows that if a symmetry σ has the property that $\sigma(\alpha) = \alpha^{\pm 1}$ for all $\alpha \in \pi_1(M)$, σ must act trivially on the character variety $X(M)$.

Lemma 2.1.2. *Let M be a 3-manifold and $\sigma : M \to M$ a symmetry. If σ fixes all free, unoriented homotopy classes of loops, then σ acts trivially on $X(M)$.*

Proof. Let $\rho : \pi_1(M) \to SL_2(\mathbb{C})$ be an irreducible representation. For each $g \in G$ we have

$$\rho(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $ad - bc = 1$. If σ is a symmetry of M as above, we have $\sigma(g) = g^{\pm 1}$. If $\sigma(g) = g$ then $\rho(\sigma(g)) = \rho(g)$. If $\sigma(g) = g^{-1}$ we have

$$\rho(\sigma(g)) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

In both cases we have $\chi_{\sigma(g)}(\rho) = a + d = \chi_g(\rho)$. Since ρ is irreducible, it follows that $\rho(\sigma(g))$ is conjugate to $\rho(g)^{\pm 1}$. □

We now consider the case where M is a hyperbolic knot complement with a non-trivial symmetry σ. Let $G = \pi_1(M)$, and suppose that $\alpha \in G$.

For any $\alpha \in G$, we can define the *length* of α as follows. Consider the set of all loops ℓ homotopic to α. Each loop ℓ lifts to a path γ in the covering space $\mathbb{H}^3$. If γ is a piecewise differentiable path, we can find the hyperbolic length of γ using the hyperbolic length formula:

$$\text{length}(\gamma) = \int_\gamma \frac{1}{\text{Im}(z)}.$$

Then the length of α is equal to $\inf\{\text{length}(\gamma)\}$, where γ ranges over the set of all piecewise differentiable lifts. We know that under the complete metric, α and $\sigma(\alpha)$ must have the same length. If this were not the case, there would be two complete metrics on M: one in which $\text{length}(\alpha) < \text{length}(\sigma(\alpha))$, and another in which $\text{length}(\alpha) > \text{length}(\sigma(\alpha))$. By the Mostow-Prasad rigidity theorem this is impossible. The complete metric on M corresponds to a discrete faithful representation χ_0. For arbitrary $\alpha \in G$, there is not a perfect correspondence between the length of α and $\chi_0(\alpha)$. To see this, note that in general we do not have $\chi_0(\alpha^n) = n\chi_0(\alpha)$. However, the following lemma shows that we must have $\chi_0(g) = \pm\chi_0(\sigma(g))$.

Lemma 2.1.3. *Let σ be a symmetry on M. Let $g \in \pi_1(M)$. If χ_0 is a discrete faithful representation, $\chi_0(g) = \pm\chi_0(\sigma(g))$.*

Proof. The character χ_0 corresponds to a discrete faithful representation

$$\rho_0 : \pi_1(M) \to SL_2(\mathbb{C})$$

which is a lift of a discrete faithful representation

$$\bar{\rho_0} : \pi_1(M) \to PSL_2(\mathbb{C}).$$

Elements of $PSL_2(\mathbb{C})$ are equivalence classes of the form $[A] = \{\pm A\}$, where $A \in SL_2(\mathbb{C})$. Let $\rho_0(g) = A$, and let α be the geodesic in M homeomorphic to g. Since ρ_0 is a discrete faithful representation, A is either *parabolic* ($\text{trace}(A) = \pm 2$) or *hyperbolic* ($\text{trace}(A) > 2$). A is parabolic if and only if α is a trivial loop or a peripheral curve. If α is a trivial loop, it is homotopic to a point, so its length is 0. In the case where α is a peripheral curve, length is not well defined because we can slide α along the cusp. We first consider the case where A is parabolic. Clearly if α is either a trivial loop or a peripheral curve, $\sigma(\alpha)$ must be as well, and therefore the lemma holds. If A is hyperbolic, then we can express trace(A) in terms of the *translation length* ℓ_0 of A:

$$|\text{trace}(A)| = 2\cosh(\ell_0(A)/2).$$

Since α and $\sigma(\alpha)$ must have the same length, the lemma follows. □

Note that if we move to the $PSL_2(\mathbb{C})-$character variety we will have $\chi_0(g) = \chi_0(\sigma(g))$. Also we must have $\chi_0(g) = \chi_0(\sigma(g))$ whenever $\rho(g)$ and $\rho(\sigma(g))$ are conjugate. In this paper, all of the symmetries we consider are orientation-preserving symmetries which are rotations. Therefore, here we will always have $\chi_0(g) = \chi_0(\sigma(g))$, even when working with the $SL_2(\mathbb{C})-$character variety.

The symmetry group of a hyperbolic knot must be finite and either cyclic or dihedral. This has been shown by Riley [21] and Kodama and Sakuma [13]. However, the situation is different for arbitrary links. Paoluzzi and Porti prove the following theorem in [18].

Theorem 2.1.4. *For any finite group H acting on S^3, there exists a hyperbolic link with symmetry group H.*

Definition 2.1.5. Let σ be a symmetry of the $n-$cusped manifold M. Let $M(p_1/q_1, p_2/q_2, \ldots, p_n/q_n)$ be the manifold resulting from a p_i/q_i Dehn filling of C_i, $1 \leq i \leq n$. We say that

$$(p_1/q_1, p_2/q_2, \ldots, p_n/q_n)$$

respects the symmetry σ of M if σ is also a symmetry of $M(p_1/q_1, p_2/q_2, \ldots, p_n/q_n)$.

We are interested in the effect of a non-trivial symmetry of M on $X(M)$.

Question 2.1.6. What effect does a Dehn filling of the cusps of M have on the symmetry group of M?

The following lemma states that the original symmetry of M is preserved as long as we Dehn fill all of the cusps in the same orbit with the same filling slope.

Lemma 2.1.7. *Let M be a manifold with n torus boundary components whose interior is homeomorphic to a link complement in S^3 and $\sigma : M \to M$ an orientation preserving symmetry. Suppose that $(p_1/q_1, \ldots, p_n/q_n)$ is a hyperbolic Dehn filling of the cusps of M such that $p_i/q_i = p_j/q_j$ when C_i and C_j are in the same orbit. Then there exists a symmetry*

$$\tau : M(p_1/q_1, \ldots, p_n/q_n) \to M(p_1/q_1, \ldots, p_n/q_n)$$

such that τ restricted to M is σ.

Proof. Let G be a finite group of orientation-preserving isometries of the hyperbolic structure. Let $\sigma \in G$. Then σ restricts to a symmetry on ∂M. Let m_i, ℓ_i be a meridian and longitude of C_i. We first suppose σ takes C_i to C_j where $i \neq j$. By the Gordon-Luecke Theorem, elements of G must take meridians to meridians. Therefore we must have $\sigma(m_i) = m_j$ where m_j is a meridian of C_j. Since any homeomorphism must take generators of $\pi_1(M)$ to generators of $\pi_1(M)$, we must have $\sigma(\ell_i) = {m_j}^t \ell_j^e$ where ℓ_j is a longitude of C_j and e is either 1 or -1. The linking number of m_j and ℓ_j is 1 so we must have $t = 0$. Suppose $e = -1$. The only way this could happen is if the homeomorphism changes orientation of one of the original loops m_j and ℓ_j, but not the other. For this to happen, the homeomorphism would change the orientation of C_j. The only way this could happen is if the homeomorphism changes the orientation on M, but σ is orientation preserving. So we must have $e = 1$. In the case where $\sigma(C_i) = C_i$, the same argument shows that we have $\sigma(m_i) = m_{i0}$ and $\sigma(\ell_i) = \ell_{i0}$, where m_{i0}, ℓ_{i0} are also a meridian and longitude of C_i.

A p/q Dehn filling of C_i takes a meridian of a solid torus T_i to ${m_i}^p {\ell_i}^q$ on C_i. Since C_i and C_j are in the same orbit, we perform a p/q Dehn filling on C_j as well, taking a meridian of a solid torus T_j to ${m_j}^p {\ell_j}^q$ on C_j. We must have either $\sigma(m_i) = m_j$ and $\sigma(\ell_i) = \ell_j$, or $\sigma(m_i) = {m_j}^{-1}$ and $\sigma(\ell_i) = {\ell_j}^{-1}$. In the first case we have $\sigma({m_i}^p {\ell_i}^q) = {m_j}^p {\ell_j}^q$, and in the second case we have $\sigma({m_i}^p {\ell_i}^q) = {m_j}^{-p} {\ell_j}^{-q}$. In both cases, the symmetry of T_i, T_j agrees with the p/q gluing. Therefore, if the original manifold glues the meridian of the solid torus T_i to $m^p l^q$ on C_i, $\sigma(M \cup T_i)$ is obtained by gluing the meridian of T_i to either ${m_j}^p {\ell_j}^q$ or ${m_j}^{-p} {\ell_j}^{-q}$. This is exactly where the meridian of T_j is glued to in the original Dehn filled manifold. Since ∂M is the set of boundaries of an ϵ-neighborhood of each cusp C_k with a symmetry, ∂M has the same symmetry as M does. ∂M has a symmetry obtained by restriction of the symmetry on M and $\sigma(\partial M) = \partial M$. We can extend σ to a homeomorphism τ of $M(p_1/q_1, \ldots, p_n/q_n)$ which takes M to M and ∂M to ∂M. Since $\partial M = \partial T$,

τ fixes the framing of each solid torus for $T_k, 1 \leq k \leq n$. Therefore $\tau(T_k) = T_k$ is a symmetry of each T_k. As a result we can extend σ to $M(p_1/q_1, \ldots, p_n/q_n)$ by the symmetry τ on each T_k, as needed. □

Note that this lemma says if M is a knot complement then any Dehn filling of the cusp respects the symmetry σ of M. If M is a link complement with at least two components this may not be the case. For example, if M has exactly two cusps and σ fixes each cusp, then all Dehn fillings respect the symmetry σ. However, if σ permutes the cusps, if we Dehn fill the first cusp with a p/q filling then we need to Dehn fill the second cusp with a p/q filling if we want to preserve the symmetry. Of course it is possible that there exists a way perform a p/q filling on C_1 and a p'/q' filling on C_2 where $p/q \neq p'/q'$ which will respect this second type of symmetry, but only if p'/q' is one of the finite number of cosmetic fillings. Since these fillings are rare, we will ignore these. In general, if M has n cusps then if we fill one cusp with p/q filling then we have to fill all the cusps in the orbit with a p/q filling.

This gives rise to a question. Suppose that M is an n-cusped hyperbolic manifold with symmetry σ. Do there always exist an infinite number of fillings $(p_1/q_1, \ldots, p_n/q_n)$ which respect the symmetry σ? For example, suppose that M is a 2 component link and σ is a symmetry which permutes the cusps. Are there infinitely many values of p/q such that $M(p/q, p/q)$ is hyperbolic?

From Thurston we know that all but finitely many $M(p/q, -)$ are hyperbolic, and for each of these manifolds all but finitely many of the double-filled $M(p/q, p'/q')$ are hyperbolic. However it is possible that for a given $M(p/q, -)$, one of the non-hyperbolic fillings is $M(p/q, p/q)$. Does this indeed happen, and if so, how often?

To answer this question we consider the quotient orbifold. The quotient orbifold $\mathcal{O} = M/\sigma$ is a quotient space. This orbifold is defined as follows: points p and q in M are identified in the quotient when $q = \sigma^n(p)$ for some integer n. This resulting quotient $\mathcal{O}$ is a hyperbolic manifold everywhere except for the singular locus which is a graph. We can do a Dehn filling on a torus boundary of $\mathcal{O}$ (or on a Euclidean pillowcase boundary) in the same way we can Dehn fill a cusp of M. Fix a cusp of $\mathcal{O}$, and let $\mathcal{O}(p/q)$ be the orbifold resulting from a p/q Dehn filling of the cusp. The orbifold surgery theorem states that all but finitely many slopes p/q will result in a hyperbolic orbifold.

In the case where M is a 2-cusped orbifold as above, we have a hyperbolic orbifold $O(p/q)$ for all but finitely many slopes p/q. For each p/q hyperbolic orbifold filling, we have a hyperbolic manifold

$M(p/q, p/q)$ which results from "unfolding" the orbifold. It follows that all but finitely many of the double fillings are hyperbolic.

At this point we define two algebraic sets.

Definition 2.1.8. Let σ be a symmetry of the manifold M. If $f, g \in \mathbb{C}[x_1, \ldots, x_m]$, let $\mathrm{Van}\{f = g\}$ be the set of points $p \in \mathbb{C}^m$ such that $f(p) = g(p)$. Then we define the set

$$V_\sigma = X_0(M) \cap \mathrm{Van}\{\chi_\rho(\alpha_i) = \chi_\rho(\sigma(\alpha_i))\},$$

where α_i ranges over the set of all elements of $\pi_1(M)$.

Let Z_σ be the Zariski closure of $X_0(M) \cap \{X_0(M(p_1/q_1, p_2/q_2, \ldots, p_n/q_n)\}$, where

$$(p_1/q_1, p_2/q_2, \ldots, p_n/q_n)$$

ranges over the set of all hyperbolic $n-$cusped fillings of the n cusps of M which respect the symmetry σ of M.

Remark 2.1.9. Recall that we know there are infinitely many distinct points of this type. Since Thurston showed that all but finitely many of the points in $\{X_0(M(p_1/q_1, p_2/q_2, \ldots, p_n/q_n)\}$ lie on $X_0(M)$, the dimension of Z_σ is at least 1.

At this point it is clear why, when M is a knot complement with a non-trivial symmetry σ, the symmetry can often be used to identify the canonical component $X_0(M)$. A symmetry σ gives rise to an algebraic set W_σ with the property that if $\chi_0 \in X_0(M)$ is a point corresponding to a discrete faithful representation of M we must have $\chi_0 \in W_\sigma$. Lemma 2.1.7 states that σ is also a symmetry of $M(p/q)$, and infinitely many of the Dehn filled $M(p/q)$ are hyperbolic. We know from Thurston that infinitely many of the points $X_0(M(p/q))$ are on $X_0(M)$. Each of these points corresponds to a discrete faithful representation on $M(p/q)$, and therefore these points must be in W_σ as well. Since W_σ is an algebraic set the Zariski closure of the set of all such $X_0(M(p/q))$ must be in W_σ as well. The Zariski closure of an infinite number of points has dimension at least 1. Since they all lie on $X_0(M)$, a complex curve, the Zariski closure is $X_0(M)$. Therefore if f is the polynomial which defines W_σ and g is the polynomial which defines $X_0(M)$, we must have $g|f$. Therefore computing f may enable us to identify $X_0(M)$.

As an example of this, we will see how this can be used to factor the character variety of the $(3, 2k + 1, 3)$ family of knots. This is a family of two-bridge knots that will be defined in a later section.

Lemma 2.1.10. *Let M be an n-cusped hyperbolic manifold. Fix a subset $C_1, \ldots, C_j$ of cusps of M. Let $M(p_1/q_1, \ldots, p_j/q_j, -)$ and $M(p_1'/q_1', \ldots, p_j'/q_j', -)$ be the manifolds resulting from Dehn fillings of the corresponding cusps of M. If these fillings result in non-homeomorphic hyperbolic manifolds, then the varieties $X_0(M(p_1/q_1, \ldots, p_j/q_j, -))$ and $X_0(M(p_1'/q_1', \ldots, p_j'/q_j', -))$ are distinct varieties in their natural inclusion in $X_0(M)$.*

Proof. Let $\Gamma = \pi_1(M)$, and let $G = \pi_1(M(p_1/q_1, \ldots, p_j/q_j, -))$, $G' = \pi_1(M(p_1'/q_1', \ldots, p_j'/q_j', -))$. Suppose that $(p_1/q_1, \ldots, p_j/q_j, -)$ and $(p_1'/q_1', \ldots, p_j'/q_j', -)$ are two distinct sets of Dehn fillings of $C_1, \ldots, C_j$ as above such that $M(p_1/q_1, \ldots, p_j/q_j, -)$ and $M(p_1'/q_1', \ldots, p_j'/q_j', -)$ are both hyperbolic, and such that $X_0(M(p_1/q_1, \ldots, p_j/q_j, -))$ and $X_0(M(p_1'/q_1', \ldots, p_j'/q_j', -))$ are both equal to the same irreducible $(n-j)$-dimensional algebraic set. Call this variety W. Since these 2 manifolds are distinct, there must be 2 distinct points Q, Q' on W, where Q is the point on W corresponding to the discrete faithful representation ρ_0 of $\pi_1(M(p_1/q_1, \ldots, p_j/q_j, -))$, and Q' is the point on W corresponding to the discrete faithful representation ρ_0' of $\pi_1(M(p_1'/q_1', \ldots, p_j'/q_j', -))$.

Let m_i, ℓ_i be a meridian and longitude of C_i for $1 \leq i \leq j$. Then we have $G \cong \Gamma/K$, where K is the normal closure of the subgroup of Γ trivialized by the extra relations ${m_i}^{p_i} = {\ell_i}^{q_i}$. We similarly write $G' \cong \Gamma/K'$.

Recall that the variety W is equal to $X_0(M(p_1'/q_1', \ldots, p_j'/q_j', -))$. Therefore the point Q is on $X_0(M(p_1'/q_1', \ldots, p_j'/q_j', -))$. Therefore this point corresponds not only to the discrete faithful representation ρ_0 for $X_0(M(p_1/q_1, \ldots, p_j/q_j, -))$, but also to a character for $X_0(M(p_1'/q_1', \ldots, p_j'/q_j', -))$. This means that ρ_0 is a representation of Γ which factors through both of the projections

$$p : \Gamma \to \Gamma/K \quad \text{and} \quad p' : \Gamma \to \Gamma/K'.$$

Since ρ_0 is a discrete faithful character for $X_0(M(p_1/q_1, \ldots, p_j/q_j, -))$, $\rho_0 : G \to SL_2(\mathbb{C})$ is an injective homomorphism and therefore an isomorphism between G and $\rho_0(G)$. If $p : \Gamma \to \Gamma/K$ is the projection map, then

$$\rho_0 \circ p : \Gamma \to SL_2(\mathbb{C})$$

is a representation with kernel K.

We can also express this representation as

$$\rho \circ p' : \Gamma \to SL_2(\mathbb{C})$$

which also has kernel K. Since K' is the kernel of the projection map p' we must have $K' \subset K$. Reversing the roles of G and G', we also have $K \subset K'$. Therefore $K = K'$ and $G = G'$. But then $M(p_1/q_1, \ldots, p_j/q_j, -)$ and $M(p_1'/q_1', \ldots, p_j'/q_j', -)$ have the same fundamental group, and are therefore the same manifold, which is a contradiction. $\square$

Lemma 2.1.11. *Suppose that M is an n-cusped hyperbolic manifold with cusps $C_i, 1 \leq i \leq n$. Let $M(p_1/q_1, p_2/q_2, \ldots, p_n/q_n)$ be an n-cusped hyperbolic filling of M which results in a hyperbolic manifold. Let Z be the Zariski closure of $\{X_0(M(p_1/q_1, p_2/q_2, \ldots, p_n/q_n)\}$, where*

$$(p_1/q_1, p_2/q_2, \ldots, p_n/q_n)$$

ranges over the set of all such possible hyperbolic n-cusped fillings of M which are contained in $X_0(M)$. Then $Z = X_0(M)$.

Proof. First consider the case when $n = 1$. An infinite number of distinct p/q hyperbolic Dehn filling of the single cusp of M will result in a different manifold. Therefore we have infinitely many distinct points $X_0(M(p/q))$ on the curve $X_0(M)$. The Zariski closure of an infinite number of distinct points must have dimension of at least 1, and since all of the points lie on the one-dimensional $X_0(M)$, we must have $Z = X_0(M)$.

Now suppose the statement is true for $k \leq n$. We first consider the set of all hyperbolic fillings of the first cusp of M. If $(p/q, -, \ldots, -)$ is such a filling, then $X_0(M(p/q, -, \ldots, -))$ is an irreducible set of dimension $n-1$. Fix p/q, and consider the set of all fillings $(p/q, p_2/q_2, \ldots, p_n/q_n)$ where p_i/q_i (for $2 \leq i \leq n$) ranges over the set of fillings of the remaining cusps of M which result in a hyperbolic manifold. By the inductive hypothesis the set of all points $X_0(M(p/q, p_2/q_2, \ldots, p_n/q_n))$ is Zariski dense in $X_0(M(p/q, -, \ldots, -))$. Each $X_0(M(p/q, -, \ldots, -))$ has dimension $n - 1$. Therefore the dimension of Z must be either $n - 1$ or n. If the dimension is $n - 1$, all of the points must lie in a finite number of the varieties of the form $X_0(M(p/q, -, \ldots, -))$. However, by Lemma 2.1.10, an infinite number of the distinct p/q Dehn fillings of C_1 will result in an infinite number of distinct $(n-1)$-dimensional varieties $X_0(M(p/q, -, \ldots, -))$, so the Zariski closure of all of the $X_0(M(p/q, -, \ldots, -))$ is all of $X_0(M)$. Therefore $Z = X_0(M)$. $\square$

This lemma states that for any n-cusped hyperbolic manifold, under a natural inclusion of character varieties, the points corresponding to the closed manifolds which are Dehn fillings of all n

cusps is Zariski dense in $X_0(M)$, which means these points are 'spread out' in the algebraic sense. That is, the Zariski closure has dimension n, which is as large as it can possibly be, as all of these metrics correspond to points lying in the $n-$dimensional variety $X_0(M)$. This leads to the natural question : what if M has a non-trivial symmetry group, and we want to consider the dimension of the set of all metric completions which respect a given symmetry of M?

We already know the answer for the $n = 1$ case, as every p/q Dehn filling of the cusp of M will result in a closed manifold $M(p/q)$ which has all of the symmetries of the original M. There may be additional symmetries of $M(p/q)$ that the original M does not have, but the important point here is that the symmetry group of M is a subgroup of the symmetry group of $M(p/q)$. When M is a knot complement, the set of all hyperbolic p/q Dehn fillings correspond to an infinite number of distinct points $X_0(M(p/q))$ on the irreducible curve $X_0(M)$. Therefore the Zariski closure of the set of all of these points must be equal to the entire canonical component. When $n > 1$, this question is much more interesting.

The following proposition shows that in the case where M is an $n-$cusped manifold and σ is a symmetry which fixes each cusp, the dimension of this set is n.

Proposition 2.1.12. *Suppose that M is an n-cusped hyperbolic manifold with cusps $C_1, \ldots, C_n$. Let σ be a symmetry on M such that $\sigma(C_i) = C_i$ for all i. Then $Z_\sigma = V_\sigma = X_0(M)$.*

Proof. Since each C_i is in a different orbit under σ, by Lemma 2.1.7, all hyperbolic Dehn fillings respect the symmetry σ. Furthermore, by Lemma 2.1.11, $Z_\sigma = X_0(M)$. Since $Z_\sigma \subseteq V_\sigma \subseteq X_0(M)$, it follows that $V_\sigma = X_0(M)$. □

We first consider all possible permutations of the cusps induced by the symmetry for $n = 2$ and $n = 3$, and then some special cases for general n.

2.2 Symmetries of a Two-Cusped Manifold

Now we consider the case where M is a 2-cusped hyperbolic manifold. Denote the cusps of M by C_1 and C_2. Here we will use the notation $M(p/q, -)$ to refer to the manifold resulting from a p/q Dehn filling of C_1, and the notation $M(-, p/q)$ for the manifold resulting from a p/q Dehn filling of C_2. Let σ be a symmetry of M.

There are two possible scenarios that can happen for two cusped manifolds. Either the symmetry can permute the cusps, or it can take each cusp to itself. We will consider each case separately.

Case 1 : The symmetry fixes each cusp (set-wise)

We consider the case when $\sigma(C_1) = C_1$ and $\sigma(C_2) = C_2$.

In this case σ is also a symmetry of $M(p/q, -)$ and $M(-, p'/q')$ for any p/q and any p'/q'. Recall that by Thurston, since M is hyperbolic, $X_0(M)$ has dimension equal to the number of cusps of M, so $\dim(X_0(M))$ is 2 here.

Two motivating questions are:

1. If $S = \{X_0(M(p/q, -)\}$ is the set of all curves $X_0(M(p/q, -))$ as p/q ranges over the set of all hyperbolic Dehn fillings, what is the dimension of the Zariski closure of S?

2. Is it true that $X_0(M(p/q, -)) \subset V_\sigma$ for each (p/q)?

To answer (1), note that since $M(p/q, -)$ is a one-cusped manifold, each $X_0(M(p/q, -))$ is a one-dimensional curve contained in $X_0(M)$, a 2-dimensional algebraic variety. Therefore the Zariski closure is $X_0(M)$ unless the family of curves is just a finite number, that is, infinitely many of the curves are the same. The following proposition shows that this is not possible, and therefore the Zariski closure is indeed $X_0(M)$. It is clear that the answer to (2) is yes. Since $X_0(M(p/q, -))$ is the Zariski closure of the set of all $X_0(M(p/q, p'/q'))$ such that $M(p/q, p'/q')$ is hyperbolic, and $X_0(M(p/q, p'/q')) \in V_\sigma$ for each such pair $(p/q, p'/q')$, the Zariski closure of this set of points must be contained in V_σ as well.

Recall from Definition 2.1.8 that Z_σ corresponds to the representations of fundamental groups of fillings that respect the symmetry σ. In this case, the corresponding filled manifolds are all $M(p/q, p'/q')$. That is, Z_σ is the Zariski closure of all $X(M(p/q, p'/q'))$ as a subset of $X_0(M)$.

Proposition 2.2.1. *Suppose that M is a 2-cusped hyperbolic manifold with cusps C_1 and C_2. Let σ be a symmetry on M such that $\sigma(C_1) = C_1$ and $\sigma(C_2) = C_2$. Then $Z_\sigma = V_\sigma = X_0(M)$, and Z_σ and V_σ both have dimension 2.*

Proof. This is a special case of Proposition 2.1.12. □

The projection of the Whitehead link shown in Figure 2.1 shows that this is a 2-component link with this property. One symmetry of this link is a reflection about the vertical line passing through the middle of the link.

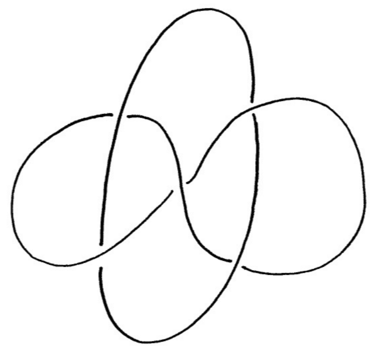

Figure 2.1: The Whitehead Link, a hyperbolic link

Case 2 : The symmetry permutes the cusps

We consider the case when $\sigma(C_1) = C_2$ and $\sigma(C_2) = C_1$.

In this case the symmetry σ flips the 2 cusps. Note that σ is a symmetry of the double-filled manifold $M(p/q, p'/q')$ if and only if $p/q = p'/q'$ up to framing.

When the symmetry σ permutes the cusps, Z_σ is the Zariski closure of the set $\{X_0(M(p/q, p/q))\}$, as p/q ranges over the set of all filling slopes resulting in a hyperbolic double filling of M.

Proposition 2.2.2. *Suppose that M is a two-cusped hyperbolic manifold with cusps C_1 and C_2. Let σ be a symmetry on M such that $\sigma(C_1) = C_2$ and $\sigma(C_2) = C_1$. Then Z_σ has dimension 1.*

Proof. Since Z_σ contains an infinite number of distinct points, Z_σ must have dimension 1 or 2. Suppose Z_σ is of dimension 2. Since $X_0(M)$ is an irreducible set of dimension 2 and $Z_\sigma \subset X_0(M)$, $Z_\sigma = X_0(M)$.

Consider the curve $X_0(M(-, p/q))$ for any p/q such that $M(p/q, p/q)$ is hyperbolic. Let ρ be a discrete faithful representation for $M(-, p/q)$. If α is any peripheral curve along C_1, we have $\chi_\rho(\alpha) = \pm 2$. Therefore we must have $\chi_\rho(\sigma(\alpha)) = \pm 2$. Therefore $\sigma(\alpha)$ must either be a trivial curve or a peripheral curve of C_2. If $\alpha = {m_1}^a {\ell_1}^b$ for a meridian m_1 and a longitude ℓ_1 of C_1, then $\sigma(\alpha) = {m_2}^a {\ell_2}^b$ for a meridian m_2 and a longitude ℓ_2 of C_2. We can assume without loss of generality that a p/q Dehn filling of C_2 results from adding the relation $m^p = \ell^q$ to the group presentation for $\pi_1(M)$. Since this filling removes the second cusp, ${m_2}^a {\ell_2}^b$ is not peripheral. The only way this can be a trivial curve is if it is a power of ${m_2}^p {\ell_2}^{-q}$, which is the case only when $a = pm$ and $b = -qm$ for some integer m. However this is not the case for every peripheral curve α of C_1. Therefore it is not always the case that $\chi_\rho(\sigma(\alpha)) = \pm 2$, so it is not possible for $Z_\sigma = X_0(M)$. Therefore Z_σ must have dimension 1. □

2.3 Symmetries of a Three-Cusped Manifold

Now consider the case where M is a 3-cusped hyperbolic manifold and σ is a symmetry on M. Denote the cusps of M by C_1, C_2, C_3. Up to renaming of the cusps, there are three possibilities depending on how the symmetry permutes the cusps. We will look at each possibility in turn.

Case 1 : The symmetry fixes all of the cusps, so

$$\sigma(C_1) = C_1, \sigma(C_2) = C_2 \text{ and } \sigma(C_3) = C_3.$$

Proposition 2.3.1. *If σ fixes all three cusps of a three-cusped manifold M, then $Z_\sigma = V_\sigma = X_0(M)$, and Z_σ and V_σ both have dimension 3.*

Proof. This is a special case of Proposition 2.1.12. □

Case 2 : The symmetry fixes one cusp and permutes the other two cusps, so

$$\sigma(C_1) = C_1, \sigma(C_2) = C_3 \text{ and } \sigma(C_3) = C_2.$$

Theorem 2.3.2. *If σ permutes exactly two cusps of a three cusped manifold M, then the dimension of V_σ is 2 and the dimension of Z_σ is 2.*

Proof. Let σ be a symmetry of M as above. Z_σ is the Zariski closure of $\{X_0(M(p_0/q_0, p/q, p/q)\}$, where $(p_0/q_0, p/q, p/q)$ ranges over the set of all hyperbolic fillings of the three cusps of M such that C_2 and C_3 are filled the same. Consider the quotient orbifold $\mathcal{O} = M/\sigma$. This quotient map sends C_2 and C_3 to a cusp C of $\mathcal{O}$. There are infinitely many hyperbolic Dehn fillings of C, and all but finitely many of these will result in non-homeomorphic orbifolds. Let $\mathcal{O}(p/q)$ be the orbifold resulting from a p/q Dehn filling of the cusp C. If $\mathcal{O}(p/q)$ and $\mathcal{O}(p'/q')$ are non-homeomorphic orbifolds, then $M(-, p/q, p/q)$ and $M(-, p'/q', p'/q')$ are non-homeomorphic manifolds. Therefore there are infinitely many non-homeomorphic manifolds $M(-, p/q, p/q)$. Denote this manifold by $M_{p/q}$. Each $X_0(M_{p/q})$ is a curve in $X_0(M)$. By Lemma 2.1.10, infinitely many of the curves $X_0(M_{p/q})$ must be distinct, and therefore the Zariski closure of the set of all of these curves has dimension 2 or 3. Therefore Z_σ has dimension 2 or 3.

However, we see that the dimension of V_σ cannot be 3 by considering a hyperbolic Dehn filling of C_3 only. If α is a peripheral curve along C_2 and ρ a discrete faithful representation for the resulting two-cusped manifold, we must have $\chi_\rho(\alpha) = \pm 2$. However, we see that we do not have $\chi_\rho(\sigma(\alpha)) = \pm 2$ for all possible p/q fillings of C_3, by an argument similar to the one we used in the two-dimensional case. Therefore, since $Z_\sigma \subseteq V_\sigma$ and the dimension of Z_σ is at least 2, both V_σ and Z_σ have dimension 2. □

Case 3 : The symmetry cyclically permutes all three cusps, so

$$\sigma(C_1) = C_2, \sigma(C_2) = C_3 \text{ and } \sigma(C_3) = C_1.$$

Theorem 2. If σ permutes all three cusps of a three cusped manifold M, then the dimension of V_σ is 1 or 2, and the dimension of Z_σ is 1 or 2.

Proof. Let σ be a symmetry of M as above. Z_σ is the Zariski closure of $\{X_0(M(p/q, p/q, p/q)\}$, where $(p/q, p/q, p/q)$ ranges over the set of all hyperbolic triple p/q fillings of the cusps of M. By considering the quotient orbifold $\mathcal{O} = M/\sigma$ we see that Z_σ contains an infinite number of distinct points, so $\dim(Z_\sigma \geq 1)$. We see that the dimension of V_σ must be less than 3 by considering a hyperbolic Dehn filling of C_1 only and considering a peripheral curve along C_3 as in the previous cases. Therefore $1 \leq \dim(Z_\sigma) \leq \dim(V_\sigma) \leq 3$, and the lemma follows. □

2.4 Symmetries of an n-cusped manifold

We now consider the case where M is an n-cusped hyperbolic manifold. For the general case there are many possible permutations of the cusps of M under a symmetry. We will consider the case where M has a symmetry of order n that cyclically permutes all n cusps of M. In this case, Z_σ is the Zariski closure of the set of points $X_0(M(p/q, p/q, ..., p/q))$ that lie on $X_0(M)$, where the manifolds $M(p/q, p/q, ..., p/q)$ are hyperbolic.

Here we will use the eigenvalue variety, which is birational to $Y_0(M)$ when restricted to $Y_0(M)$. Choose a meridian m_1 and a longitude ℓ_1 of C_1. For each C_j with $2 \leq n$ we define m_j to be the meridian of C_j such that $\sigma^j(m_1) = m_j$. We define ℓ_j similarly. Recall that the *eigenvalue variety* is the Zariski closure of the set of points $(M_1, L_1, \ldots, M_n, L_n)$ in $\mathbb{C}^{2n}$ with the property that there exists a representation $\rho : \pi_1(M) \to SL_2(C)$ such that for each i, $\rho(m_i)$ and $\rho(\ell_i)$ have eigenvalues M_i, L_i with respect to a common eigenvector $\vec{v_i}$.

We can conjugate $\rho(m_1)$ and $\rho(\ell_1)$ so that both matrices are upper triangular:

$$\rho(m_1) = \begin{pmatrix} M & * \\ 0 & M^{-1} \end{pmatrix} \text{ and } \rho(\ell_1) = \begin{pmatrix} L & * \\ 0 & L^{-1} \end{pmatrix}$$

which gives the pair $(M_1, L_1) = (M, L)$.

Now suppose that ρ is a representation which respects σ. Then $\chi_\rho(m_i)$ must be the same for all i, and the same is true for the set of $\chi_\rho(\ell_i)$. So we must have

$$\chi_\rho(m_i) = M + M^{-1} \text{ and } \chi_\rho(\ell_i) = L + L^{-1} \text{ for all } i.$$

For each i, there exists a representation equivalent to ρ such that

$$\rho(m_i) = \begin{pmatrix} M_i & * \\ 0 & M_i{}^{-1} \end{pmatrix} \text{ and } \rho(\ell_i) = \begin{pmatrix} L_i & * \\ 0 & L_i{}^{-1} \end{pmatrix}$$

Since we must have $(M_i, L_i) = (M^{\pm 1}, L^{\pm 1})$ for each i, there are only a finite number of possible (M_i, L_i) for $i \geq 2$ for each (M_1, L_1). Therefore the Zariski closure of the set of all (M_1, L_1) in $\mathbb{C}^2$ has the same dimension as the eigenvalue variety.

We have shown the following.

Proposition 2.4.1. *Suppose that M is an $n-$component hyperbolic link with cusps $C_1, ..., C_n$. Let σ be a symmetry on M such that $\sigma(C_i) = C_{i+1}$ for $1 \leq i \leq i - 1$ and $\sigma(C_n) = C_1$. Then Z_σ and V_σ have dimension 1 or 2.*

For Z_σ we can do even better. Recall that in this case, we know there are infinitely hyperbolic manifolds of the form $M(p/q, \ldots, p/q)$ by looking at the quotient orbifold $\mathcal{O} = M/\sigma$. We have an inclusion map

$$i : \pi_1(M) \hookrightarrow \pi_1(\mathcal{O})$$

which follows from the fact that any deck transformation of M is a deck transformation of $\mathcal{O}$. It is easy to see that the converse is not the case. Any non-trivial rotation of M induces a deck transformation of $\mathcal{O}$ which is not a deck transformation of M as M is not fixed.

Given a representation $\rho : \pi_1(\mathcal{O}) \to SL_2(\mathbb{C})$, we define a representation $\rho_M : \pi_1(M) \to SL_2(\mathbb{C})$ as follows:

$$\rho_M = \rho \circ i : \pi_1(M) \to SL_2(\mathbb{C}).$$

The map $\rho \mapsto \rho \circ i$ gives a map $R(\mathcal{O}) \to R(M)$ inducing a map

$$i^* : Y_0(\mathcal{O}) \to Y_0(M).$$

The following theorem is proved in [10].

Theorem 2.4.2. *Let Q be a hyperbolic orbifold of finite volume. Then $dim_{\mathbb{C}}(Y_0(Q)) = m$, where m is the number of flexible cusps of Q.*

Therefore $i^*(Y_0(\mathcal{O}))$ has dimension 0 or 1. However we know that the dimension is at least 1 because of the infinite number of Dehn surgery points. Therefore $\dim(i^*(Y_0(\mathcal{O}))) = 1$.

Proposition 2.4.3. *Suppose that M is an $n-$component hyperbolic link with cusps $C_1, ..., C_n$. Let σ be a symmetry on M such that $\sigma(C_i) = C_{i+1}$ for $1 \leq i \leq i-1$ and $\sigma(C_n) = C_1$. Then Z_σ has dimension 1.*

Proof. Under the map i^* the point $Y_0(M(p/q, \ldots, p/q)/\sigma)$ is mapped to $Y_0(M(p/q, \ldots, p/q))$ due to the inclusion $i : \pi_1(M(p/q, \ldots, p/q)) \to \pi_1(M(p/q, \ldots, p/q)/\sigma)$. We wish to show that $\mathcal{O}(p/q)$ and $M(p/q, \ldots, p/q)/\sigma$ are homeomorphic. Let $D \subset M(p/q, \ldots, p/q)$ be a fundamental domain for $M(p/q, \ldots, p/q)/\sigma$. Then since $M(p/q, \ldots, p/q)/\sigma$ and $\mathcal{O}(p/q)$ are both homeomorphic to D they are homeomorphic to each other. Therefore if ρ is a discrete faithful representation for $M(p/q, \ldots, p/q)/\sigma$ it is also a discrete faithful representation for $\mathcal{O}(p/q)$. It follows that $i^*(Y_0(\mathcal{O}(p/q))) = Y_0(M(p/q, \ldots, p/q))$. Since the points $Y_0(\mathcal{O}(p/q))$ are Zariski dense in $Y_0(\mathcal{O})$, it follows that $i^*(Y_0(\mathcal{O}))$ has dimension 1. Therefore the set of all $Y_0(M(p/q, \ldots, p/q))$ has dimension 1. □

CHAPTER 3

THE $(3, 2K + 1, 3)$ FAMILY OF KNOTS

In this section we consider an infinite family of hyperbolic 2-bridge knots. These knots consist of a series of three "stacked" tangles - at the top and bottom a set of 3 right-handed half-twists, and in the middle a set of $2k + 1$ right-handed half-twists. The odd number of half-twists in each tangle ensures that we always have a knot and not a 2-component link. The $(3, 5, 3)$ knot is shown below. This family of knots presents an example of a family of 2-bridge knots with a non-trivial symmetry which we will show factors the character variety. Note that all of these knots contain the extra "flip" symmetry.

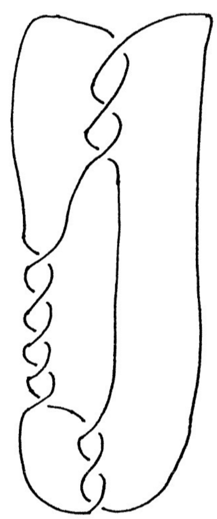

Figure 3.1: The $(3, 5, 3)$ knot

Recall from the discussion in section 1.5 that in general 2-bridge knots either have symmetry group of order 4 or 8. Since this family of knots contains the extra flip symmetry of order 2, the symmetry group of each of these knots has order 8. We will be interested only in this extra symmetry. Recall that the other symmetries have no effect on the character variety because they act trivially on the base point free unoriented loops.

In this chapter, we will find a group presentation for this family of knots. We will use this presentation to compute a natural model for the $SL_2(\mathbb{C})$-character variety and a smooth model for the $PSL_2(\mathbb{C})$-character variety. We will exploit the symmetry of these knots to identify the canonical component. We will then compute the geometric genus of each component.

3.1 Group Presentation of the $(3, 2k+1, 3)$ Knot

Here we denote the $(3, 2k+1, 3)$ knot by K_k. Let $M_k = S^3 - K_k$ and $\Gamma_k = \pi_1(M_k)$. We first determine a presentation for Γ_k.

Before considering the knot as a whole, we will first consider an individual $p-$tangle. A $p-$tangle is a series of p right-handed half twists as shown in Figure 3.2.

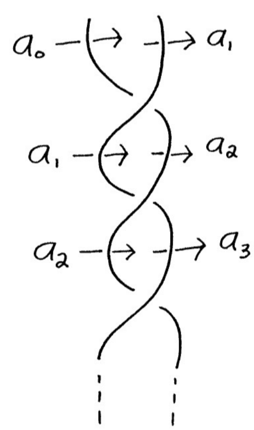

Figure 3.2: A $p-$tangle

In [15] it is shown that all of the loops in this tangle can be written in terms of the loops a_0 and a_1. Let $\alpha = a_0 a_1$, which is the loop that travels around the whole tangle. Note that $\alpha = a_i a_{i+1}$ for all i. The following lemma is used.

Lemma 3.1.1. *For all* $j \geq 0, a_j = \alpha^{-d}\ a_{j-2d}\ \alpha^d$, *where* $d = \lfloor \frac{j}{2} \rfloor$.

Proof. We prove this by induction on j. When $j = 0$ or 1, we have $d = 0$ so the statement holds. Assume the statement is true for a_j. Since $\alpha = a_j a_{j+1}$, we have $a_{j+1} = (a_j)^{-1}\alpha$. First consider the case where $j+1$ is even. If $j+1 = 2m$, then $j = 2m-1$ and $d = m-1$. Therefore $a_j = \alpha^{-m+1}a_1\ \alpha^{m-1}$. By the inductive hypothesis we can write

$$a_{j+1} = \alpha^{-m+1}{a_1}^{-1}\alpha^{m-1}\alpha$$

$$a_{j+1} = \alpha^{-m}(a_0a_1{a_1}^{-1})\alpha^m$$

$$a_{j+1} = \alpha^{-m}a_0\ \alpha^m$$

Now we consider the case where $j+1$ is odd. If $j+1 = 2m+1$, then $j = 2m$ and $d = m$. In this case $a_j = \alpha^{-m}a_0\ \alpha^m$ and by the inductive hypothesis

$$a_{j+1} = \alpha^{-m}\ {a_0}^{-1}\alpha^m\alpha$$

$$a_{j+1} = \alpha^{-m}({a_0}^{-1}a_0a_1)\alpha^m$$

$$a_{j+1} = \alpha^{-m}a_1\ \alpha^m$$

Therefore the statement holds for $j+1$. □

The $(3, 2k+1, 3)$ knot is created by the gluing together of 3 tangles. The first tangle is a series of 3 right-handed half twists, where all loops can be written in terms of a_0 and a_1 as described above. Let b_0 and b_1 be the analogous loops at the top of the center tangle, where $\beta = b_0b_1$. Then each b_i can be expressed in terms of b_0 and b_1 in the same way as described for the first tangle, where b_0, b_1, β are substituted for a_0, a_1, α. Note that if this tangle consists of $2k+1$ right-handed half-twists, the two loops at the bottom of the tangle are b_{2k+1} and b_{2k+2}. Let c_0 and c_1 be the analogous loops at the top of the third tangle, where $\gamma = c_0c_1$. Then each c_i can be expressed in terms of c_0 and c_1 using the same formula, substituting c_0, c_1, γ for a_0, a_1, α.

The gluing consists of attaching the three tangles using the following identifications:

$$b_0 = {a_0}^{-1}$$

$$b_1 = a_3$$

$$c_0 = b_{2k+2}$$

$$c_1 = a_4$$

$$c_3 = {b_{2k+1}}^{-1}$$

$$c_4 = a_1.$$

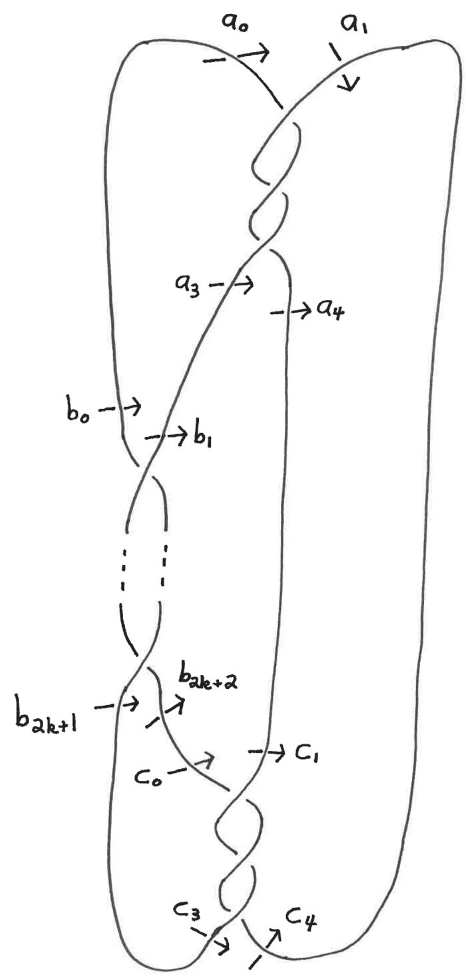

Figure 3.3: The (3, 2k+1, 3) Knot

The resulting knot is shown in Figure 3.3.

We let $x_0 = a_0$ and $x_1 = a_1$. Note that every loop can be expressed in terms of x_0 and x_1. This is clear for the loops along the first tangle. We move to the second tangle. Since by Lemma 3.1.1 $a_3 = (x_0x_1)^{-1}x_1(x_0x_1)$ we have

$$b_0 = {x_0}^{-1}$$

$$b_1 = (x_0x_1)^{-1}x_1(x_0x_1).$$

Then $\beta = {x_0}^{-1}(x_0x_1)^{-1}x_1(x_0x_1)$ and using Lemma 3.1.1 we write

$$b_{2k+2} = \beta^{-k-1}\ {x_0}^{-1}\beta^{k+1}.$$

Now we have

$$c_0 = \beta^{-k-1} \; {x_0}^{-1} \beta^{k+1}$$
$$c_1 = (x_0 x_1)^{-1} {x_1}^{-1} (x_0 x_1)^2 = \beta^{-1} x_1.$$

This shows that x_0 and x_1 are generators for $\pi_1(M_k)$.

Note that the relation $c_4 = a_1$ follows from the previous relations. To see this note that if $c_3 = {b_{2k+1}}^{-1}$ then $b_{2k+1}c_3$ is the trivial loop, so $c_4 = b_{2k+1}c_3c_4$. We have $c_3c_4 = \gamma$, therefore $c_4 = b_{2k+1}c_0c_1$, and $c_0 = b_{2k+2}, c_1 = a_4$ which implies $c_4 = b_{2k+1}b_{2k+2}c_1$. Since $b_0b_1 = b_{2k+1}b_{2k+2}$ this gives $c_4 = b_0b_1a_4 = {a_0}^{-1}a_3a_4 = {a_0}^{-1}(a_0a_1) = a_1$.

Since the first four identifications are used to express the loops along the second and third tangles in terms of x_0 and x_1, the relation needed for the group presentation is $c_3 = {b_{2k+1}}^{-1}$.

We have shown the following.

Lemma 3.1.2. *The group presentation for $\Gamma_k = \pi_1(M_k)$, where $M_k = S^3 - K_k$, is given by*

$$\Gamma = \langle g, h | g\beta^k h \beta^{-k-1} g^{-1} \beta^k = \beta^k h \beta^{-k-1} g^{-1} \beta^k h^{-1} \rangle$$

where $\beta = {x_0}^{-1}(x_0x_1)^{-1}x_1(x_0x_1)$.

3.2 Character Variety of the $(3, 2k+1, 3)$ Knot

We use the above group presentation to compute the character variety. Recall from the discussion in the introduction that if ρ is a representation $\pi_1(M) \to SL_2(\mathbb{C})$ where $\pi_1(M)$ is generated by x_0 and x_1, all $\chi_\rho(\alpha)$ can be written in terms of $\chi_\rho(x_0), \chi_\rho(x_1)$, and $\chi_\rho(x_0x_1)$. Note that in our case x_0 and x_1 are conjugate loops and therefore $\chi_\rho(x_0) = \chi_\rho(x_1)$. Therefore the character variety will be a polynomial in two variables.

By [1] since $\pi_1(M_k)$ is generated by x_0 and x_1, up to conjugation

$$\rho(x_0) = \begin{pmatrix} c & 0 \\ s & c^{-1} \end{pmatrix} \text{ and } \rho(x_1) = \begin{pmatrix} d & 1 \\ 0 & d^{-1} \end{pmatrix}.$$

Here we must have $\chi_\rho(x_0) = \chi_\rho(x_1)$, so it follows from [15] that we can write

$$\rho(x_0) = X_0 = \begin{pmatrix} d & 0 \\ 2-u & d^{-1} \end{pmatrix} \text{ and } \rho(x_1) = X_1 = \begin{pmatrix} d^{-1} & -1 \\ 0 & d \end{pmatrix}$$

The character variety is determined by the traces of the matrices

$$X_0 = \begin{pmatrix} d & 0 \\ 2-u & d^{-1} \end{pmatrix}$$
$$X_1 = \begin{pmatrix} d^{-1} & -1 \\ 0 & d \end{pmatrix}$$
$$X_0X_1 = \begin{pmatrix} 1 & -d \\ 2d^{-1}+ud^{-1} & -1+u \end{pmatrix}.$$

Note that $\rho(\alpha) = X_0X_1$, and this matrix corresponds to the loop around the first tangle.

The $SL_2(\mathbb{C})$ character variety is a polynomial in terms of $\chi_\rho(x_0) = \text{trace}(X_0)$ and $\chi_\rho(x_0) = \text{trace}(X_0X_1)$. Here we set $x = d + d^{-1}$. The $SL_2(\mathbb{C})$ character variety is therefore a polynomial in terms of the variables

$$x = \chi_\rho(x_0) = \chi_\rho(x_1) \text{ and}$$
$$u = \chi_\rho(x_0x_1).$$

Since $\beta = {x_0}^{-1}(x_0x_1)^{-1}x_1(x_0x_1)$ we have $\rho(\beta) = B = {X_0}^{-1}{X_1}^{-1}{X_0}^{-1}X_1X_0X_1$. This corresponds to the loop around the second tangle. An elementary computation shows

$$B = \begin{pmatrix} 2-u+d^{-2}(u-1)^2 & (1-u)(d^{-1}u-d) \\ -2d+3du-du^2+d^{-1}(2u-3u^2+u^3) & d^2(u-1)^2+2u^2-u^3 \end{pmatrix}.$$

Computing the character variety for this family of knots requires computing B^k for arbitrary k. To do this we follow [15], which uses the Fibonacci polynomials along with the Cayley-Hamilton Theorem.

Definition 3.2.1. Let n be an integer. The *Fibonacci polynomials* are defined recursively as follows:

$$f_0(t) = 0, \quad f_1(t) = 1, \quad f_{n+1}(t) = tf_n(t) - f_{n-1}(t).$$

Remark 3.2.2. If t is the trace of a matrix M, we can use Fibonacci polynomials to calculate the j th power of M:

$$M^j = f_j(t)M - f_{j-1}(t)I.$$

This follows from the Cayley-Hamilton Theorem.

Since the trace of the matrix B is equal to $t = (u-1)^2x^2 + 3u - u^3$, we have

$$B^k = f_k(t)B - f_{k-1}(t)I,$$

where the $f_k(t)$ are obtained recursively using Definition 3.2.1.

Let $G = \rho(\gamma)$. Since $\gamma = \beta^{-k-1}g^{-1}\beta^k h$ we have $G = B^{-k-1}{X_0}^{-1}B^k X_1$. Any character χ_ρ on $X_0(M)$ must respect the flip symmetry which sends α to γ. Therefore for any χ_ρ on $X_0(M)$ we must have $\chi_\rho(\alpha) = \chi_\rho(\gamma)$. It follows that the polynomial which determines $X_0(M)$ must divide $V_k(x,u) = \text{trace}(G) - u$. Note that $V_k(x,u)$ is not necessarily irreducible, as there may be representations respecting the flip symmetry which do not lie on the canonical component.

To find the character variety we use the group presentation given in Lemma 3.1.2. We must have

$$\rho({x_0}^{-1}\beta^k x_1 \beta^{-k-1}{x_0}^{-1}\beta^k) = \rho(\beta^k x_1 \beta^{-k-1}{x_0}^{-1}\beta^k x_1)$$

which corresponds to the matrix relation

$${X_0}^{-1}B^k X_1 B^{-k-1}{X_0}^{-1}B^k = B^k X_1 B^{-k-1}{X_0}^{-1}B^k X_1.$$

The character variety is determined by the set of points (x,u) in $\mathbb{C}^2$ such that

$${X_0}^{-1}B^k X_1 B^{-k-1}{X_0}^{-1}B^k - B^k X_1 B^{-k-1}{X_0}^{-1}B^k X_1$$

is the zero matrix. Using Maple, we see that the above matrix is equal to

$$\begin{pmatrix} 0 & Z_k(x,u) \\ (u-2)Z_k(x,u) & 0 \end{pmatrix}.$$

This matrix is the zero matrix if and only if $Z_k(x,u) = 0$, and $Z_k(x,u)$ is the polynomial which defines the $SL_2(\mathbb{C})-$character variety of the $(3, 2k+1, 3)$ knot. Using the relation $x = d + d^{-1}$, we can write the $(1,2)$ entry as a function of x and not d.

The polynomial $Z_k(x,u)$ factors as $Z_{1,k}(x,u)Z_{2,k}(x,u)$, where

$$\begin{aligned} Z_{1,k}(x,u) &= f_k(t)u^4 + f_k(t)(-2x^2+2)u^3 + f_k(t)(x^4-1)u^2 \\ &\quad + (f_k(t)(-2x^4+4x^2-3) + f_{k-1}(t))u \\ &\quad + f_k(t)(x^4-x^2-1) + f_{k-1}(t)(-x^2+1) \end{aligned}$$

and

$$\begin{aligned}Z_{2,k}(x,u) &= {f_k}^2(t)u^6 - {f_k}^2(t)(2x^2+1)u^5 + f_k(t)(x^4 f_k(t) - 6f_k(t) + 6x^2)u^4 \\ &+ f_k(t)(-5x^4 f_k(t) + x^2 f_k(t) + 6f_k(t) + 2f_{k-1}(t))u^3 \\ &+ f_k(t)(9x^4 f_k(t) - 17x^2 f_k(t) + 8f_k(t) - 2x^2 f_{k-1} - f_{k-1}(t))u^2 \\ &+ f_k(t)(-7x^4 f_k(t) + 16x^2 f_k(t) - 8f_k(t) + 5x^2 f_{k-1}(t) - 7f_{k-1}(t))u \\ &+ 2x^4 {f_k}^2(t) - 4x^2 {f_k}^2(t) + {f_k}^2(t) - 3x^2 f_k(t) f_{k-1}(t) + 4f_k(t) f_{k-1}(t) - {f_{k-1}}^2(t)\end{aligned}$$

where $t = (u-1)^2 x^2 + 3u - u^3$.

Theorem 3.2.3. *A natural model for the $SL_2(\mathbb{C})$ character variety of M_k, where M_k is the two-bridge knot defined above, is given by the set of points (x,u) in $\mathbb{C}^2$ such that $Z_k(x,u) = 0$, where $Z_k(x,u) = Z_{1,k}(x,u)Z_{2,k}(x,u)$ and $t = (u-1)^2x^2 + 3u - u^3$.*

3.2.1 Analyzing the $(3, 2k+1, 3)$ Character Variety

We are interested in 2 polynomials : $Z_k(x,u) = Z_{1,k}(x,u)Z_{2,k}(x,u)$, which defines the $\mathrm{SL}_2(\mathbb{C})$-character variety, and $V_k(x,u)$, which is the polynomial obtained by setting the trace of the matrix G equal to u.

At this point we will perform a series of birational transformations which will replace $Z_k(x,u)$ and $V_k(x,u)$ with birationally equivalent polynomials which are easier to work with. Our goals are to identify the canonical component, identify the number of irreducible components, and compute the geometric genus of each irreducible component. At this point we know we have at least two irreducible components since $Z_{1,k}(x,u) \neq Z_{2,k}(x,u)$.

The trace of the matrix B is $t = (u-1)^2x^2 + 3u - u^3$. The $\mathrm{PSL}_2(\mathbb{C})$-character variety is obtained by making the substitution $x^2 = y$. Since in this case we have

$$y = \frac{t - 3u + u^3}{(u-1)^2}$$

we can do an additional birational map to get everything in terms of the variables t and u. Note that when $u = 1$, $t = 2$ and therefore $f_m(t) = m$ for all m. It follows that we have $Z_{1,k}(t,1) = x^2 - 2k$ and $Z_{2,k}(t,1) = (5k^2 - 5k)x^4 + (-6k^2 + 6k)x^2 + (-2k^2 + 4k - 1)$, and neither polynomial vanishes except at finitely many points. Therefore, this is a valid birational transformation.

This will work because when $u = 1$, the polynomial defining the $\mathrm{PSL}_2(\mathbb{C})$-character variety, as well as the polynomial $T_k(t,u)$, the image of $V_k(x,u)$ under the birational map, reduces to $x^2 - 2$, so it is not identically 0. These birational transformations will replace $Z_{1,k}(x,u)$ with $W_{1,k}(t,u)$ and $Z_{2,k}(x,u)$ with $W_{2,k}(t,u)$, where

$$\begin{aligned}
W_{1,k}(t,u) &= (f_k(t) - f_{k-1}(t))u^2 + 2(f_k(t) - f_{k+1}(t))u + (f_{k+2}(t) - f_{k+1}(t)) \\
W_{2,k}(t,u) &= (2-t){f_k}^2(t)u^2 \\
&\quad + ({f_{k-1}}^2(t) + 2f_k(t)f_{k-1}(t) - 3{f_k}^2(t) - 2tf_k(t)f_{k-1}(t) + t^2{f_k}^2(t))u \\
&\quad + f_{k-1}(t)f_{k+1}(t) + 4f_k(t)f_{k+1}(t) - 2tf_k(t)f_{k+1}(t) - {f_k}^2(t).
\end{aligned}$$

For all k, $Z_k(x,u)$ factors as $Z_{1,k}(x,u)Z_{2,k}(x,u)$, so the polynomial $W_{1,k}(t,u)W_{2,k}(t,u)$ is birational to the polynomial which defines the $PSL_2(\mathbb{C})$-character variety of the $(3, 2k+1, 3)$ knot.

We have shown the following.

Theorem 3.2.4. *The $PSL_2(\mathbb{C})$ character variety of the knot M_k is birational to the vanishing set of $W_{1,k}(t,u)W_{2,k}(t,u)$, where $W_{1,k}(t,u)$ and $W_{2,k}(t,u)$ are the polynomials defined above.*

We use the polynomial $T_k(t,u)$ to identify the canonical component.

Lemma 3.2.5. *For all k, $T_k(t,u)$ factors as $(2-u)(f_{k+1}(t) - f_k(t))W_{1,k}(t,u)$. Moreover, $W_{1,k}(t,u)$ is irreducible.*

Proof. Set $R_k(t,u) = T_k(t,u) + (u-2)(f_{k+1}(t) - f_k(t))W_{1,k}(t,u)$. We want to show that $R_k(t,u)$ is identically 0. In Maple $R_k(t,u)$ factors as

$$u(u-1)^2(-1 - tf_k(t)f_{k-1}(t) + {f_k}^2(t) + {f_{k-1}}^2(t))$$

It is enough to show that

$$h_k(t) = (-1 - tf_k(t)f_{k-1}(t) + {f_k}^2(t) + {f_{k-1}}^2(t))$$

is identically zero. When $k = 1$ we have $f_{k-1}(t) = 0$ and $f_k(t) = 1$, so $h_1(t) = 0$. Substituting $k+1$ in for k and using the recursive formula $f_{k+1}(t) = tf_k(t) - f_{k-1}(t)$, we find that $h_{k+1}(t) = h_k(t)$, so it follows by induction that $h_k(t)$ is identically 0 for all k.

We will now show that $W_{1,k}(t,u)$ is irreducible for all k. We have $W_{1,k}(t,u) = a_k(t)u^2 + b_k(t)u + c_k(t)$, where

$$a_k(t) = f_k(t) - f_{k-1}(t)$$
$$b_k(t) = 2(f_k(t) - f_{k+1}(t))$$
$$c_k(t) = f_{k+2}(t) - f_{k+1}(t).$$

Since $W_{1,k}(t,u)$ is quadratic in u, we must show that it cannot split into 2 linear factors of u, and we must also show that $a_k(t), b_k(t)$, and $c_k(t)$ do not share a common factor. If we write $t = z + z^{-1}$ we can write $f_k(t) = \frac{z^k - z^{-k}}{z - z^{-1}}$. Then we can express the discriminant in terms of z. But the discriminant is $\frac{-4(z-1)^2}{z}$, which is not a square in $\mathbb{C}[t]$. Therefore $W_{1,k}(t,u)$ does not split into linear factors of u.

An elementary calculations shows that for all $k \geq 1$ we have:

$$a_k - c_k = b_{k+1} + ta_{k+1}$$
$$a_{k+1} = -\frac{1}{2}b_k$$
$$b_{k+1} = -2c_k.$$

Now suppose that for some k, a_k, b_k, and c_k all share a common factor in $\mathbb{C}[t]$. Let n be the smallest such integer for which this is the case. Since $a_1 = 1$ it must be the case that $n > 1$. We then have

$$a_{n-1} = \frac{1}{2}b_n + ta_n$$
$$b_{n-1} = -2a_n$$
$$c_{n-1} = -\frac{1}{2}b_n.$$

Therefore a_{n-1}, b_{n-1} and c_{n-1} must share the same common factor. But this contradicts the fact that n was the smallest such integer chosen. This shows that $W_{1,k}(t,u)$ is irreducible for all k. □

Lemma 3.2.6. *For all k, $W_{2,k}(t,u)$ is irreducible.*

Proof. We write $W_{2,k}(t,u) = a_k u^2 + b(t)u + c(t)$, where

$$a_k = (2-t){f_k}^2(t)$$
$$b_k = {f_{k-1}}^2(t) + 2f_k(t)f_{k-1}(t) - 3{f_k}^2(t) - 2tf_k(t)f_{k-1}(t) + t^2{f_k}^2(t)$$
$$c_k = f_{k-1}(t)f_{k+1}(t) + 4f_k(t)f_{k+1}(t) - 2tf_k(t)f_{k+1}(t) - {f_k}^2(t).$$

We first show that a_k, b_k, and c_k have no common factor. We showed earlier that

$$-1 - tf_k(t)f_{k-1}(t) + {f_k}^2(t) + {f_{k-1}}^2(t) = 0$$

for all k. Substituting

$$tf_k(t)f_{k-1}(t) - {f_k}^2(t) + 1$$

in for ${f_{k-1}}^2(t)$ in the formula for b_k gives

$$b_k = (t^2 - 4){f_k}^2(t) + (2 - t)f_k(t)f_{k-1}(t) + 1$$

so no factor of $(2 - t)$ or of $f_k(t)$ can divide b_k. Therefore a_k and b_k share no common factor.

To show that $W_{2,k}(t, u)$ cannot split into 2 linear factors of u it is enough to show that the discriminant is not a square in $\mathbb{C}(t)$. Let $D_k(t) = [b_k(t)]^2 - 4a_k(t)c_k(t)$, with $a_k(t), b_k(t), c_k(t)$ as above.

At this point we make the substitutions

$$f_j(t) = \frac{z^j - z^{-j}}{z - z^{-1}}$$
$$t = z + z^{-1}$$

and multiply by $z^{4k}(z - 1)^4(z + 1)^4$ to clear the denominator. We obtain the polynomial:

$$\begin{aligned} P_k(z) = {} & z^{8k+8} - 8z^{8k+7} + 26z^{8k+6} - 44z^{8k+5} + 41z^{8k+4} - 20z^{8k+3} + 4z^{8k+2} \\ & + 16z^{6k+7} - 80z^{6k+6} + 164z^{6k+5} - 176z^{6k+4} + 104z^{6k+3} - 32z^{6k+2} \\ & + 4z^{6k+1} - 12z^{4k+7} + 78z^{4k+6} - z^{4k+5} + 276z^{4k+4} - 204z^{4k+3} + 78z^{4k+2} \\ & - 12z^{4k+1} + 4z^{2k+7} - 32z^{2k+6} + 104z^{2k+5} - 176z^{2k+4} + 164z^{2k+3} \\ & - 80z^{2k+2} + 16z^{2k+1} + 4z^6 - 20z^5 + 41z^4 - 44z^3 + 26z^2 - 8z + 1. \end{aligned}$$

We will show that $D_k(t)$ is separable by showing that $P_k(z)$ has no multiple roots other than $z = 1$ and $z = -1$. These multiple roots of $P_k(z)$ do not correspond to multiple roots of $D_k(t)$. When $z = 1, t = 2$ and we can verify that $t = 2$ is not a root of $D_k(t)$ by direct substitution. We only need to use that fact that $f_k(2) = k$ for all k to obtain $D_k(2) = 1$. We obtain $D_k(-2) = 144k^4 + 96k^3 + 24k^2 + 8k + 1$, which is clearly odd for all k and therefore never 0.

The polynomial $P_k(z)$ is divisible by $z^{2k+1} + 1$: we can write

$$P(z) = (z^{2k+1} + 1)Q_k(z),$$

where

$$\begin{aligned}Q_k(z) = {} & z^{6k+7} - 8z^{6k+6} + 26z^{6k+5} - 44z^{6k+4} + 41z^{6k+3} - 20z^{6k+2} + 4z^{6k+1} \\ & + 15z^{4k+6} - 72z^{4k+5} + 138z^{4k+4} - 132z^{4k+3} + 63z^{4k+2} - 12z^{4k+1} - 12z^{2k+6} \\ & + 63z^{2k+5} - 132z^{2k+4} + 138z^{2k+3} - 72z^{2k+2} + 15z^{2k+1} + 4z^6 - 20z^5 + 41z^4 \\ & - 44z^3 + 26z^2 - 8z + 1.\end{aligned}$$

The polynomial $z^{2k+1} + 1$ is separable over $\mathbb{Q}$. Suppose that $Q_k(z)$ and $z^{2k+1} + 1$ share a root. If z is such a root, we can write $z^{2k+1} = -1$ and can rewrite $Q_k(z)$ as:

$$\begin{aligned}Q_k(z) = {} & -z^4 + 8z^3 - 26z^2 + 44z - 41 + 20z^{-1} - 4z^{-2} + 15z^4 - 72z^3 + 138z^2 \\ & - 132z + 63 - 12z^{-1} + 12z^5 - 63z^4 + 132z^3 - 138z^2 + 72z - 15 + 4z^6 \\ & - 20z^5 + 41z^4 - 44z^3 + 26z^2 - 8z + 1.\end{aligned}$$

Multiplying through by z^2 and factoring, we obtain

$$4(z+1)^3(z-1)^5.$$

Since $z = 1$ is clearly not a root of $z^{2k+1} + 1$, this shows us that the only common root of $Q_k(z)$ and $z^{2k+1} + 1$ is $z = -1$.

Consider the polynomial $Q_k(z)$ modulo 3. We obtain

$$\begin{aligned}\overline{Q_k}(z) = {} & z^{6k+7} + z^{6k+6} - z^{6k+5} + z^{6k+4} - z^{6k+3} + z^{6k+2} \\ & + z^{6k+1} + z^6 + z^5 - z^4 + z^3 - z^2 + z + 1.\end{aligned}$$

This factors over $\mathbb{F}_3$ as:

$$\overline{Q_k}(z) = (z^{6k+1} + 1)(z-1)^4(z+1)^2.$$

It can be shown that $z = 1$ is a root of multiplicity 4 of the original polynomial $Q_k(z)$. This can be shown by taking derivatives:

$$Q_k(1) = 0,\ Q_k'(1) = 0,\ Q_k''(1) = 0,\ Q_k'''(1) = 0,\ Q_k^{(4)}(1) = 192.$$

In the same way it can be shown that $z = -1$ is a root of multiplicity 3 of $Q_k(z)$:

$$Q_k(-1) = 0,\ Q_k'(-1) = 0,\ Q_k''(-1) = 0,\ Q_k'''(-1) = 6912k^3 + 1152k^2 + 576k + 76.$$

Since

$$Q_k'''(-1) = 3(2304k^3 + 384k^2 + 192k + 25) + 1$$

this is never 0. It follows that the only multiple roots of $P_k(z)$ are $z = 1$ and $z = -1$. Therefore $D_k(t)$ is separable. □

We have shown the following.

Theorem 3.2.7. *The $PSL_2(\mathbb{C})$ character variety of M_k is birational to the vanishing set of $W_k(t,u) = W_{1,k}(t,u)W_{2,k}(t,u)$, where $W_{1,k}(t,u)$ and $W_{2,k}(t,u)$ are both irreducible. The canonical component is given by the vanishing set of $W_{1,k}(t,u)$.*

3.2.2 The Geometric Genus of the Canonical Component

Recall that the geometric genus of the hyperelliptic curve $y^2 = f(x)$, where $f(x)$ is a separable polynomial, is equal to $\lfloor \frac{d-1}{2} \rfloor$, where d is the degree of $f(x)$.

We now find the geometric genus of the canonical component. Recall that the canonical component is the vanishing set of $W_{1,k}(t,u) = a_k(t)u^2 + b_k(t)u + c_k(t)$, where

$$a_k(t) = f_k(t) - f_{k-1}(t)$$
$$b_k(t) = 2(f_k(t) - f_{k+1}(t))$$
$$c_k(t) = f_{k+2}(t) - f_{k+1}(t).$$

If $a_k(t)u^2 + b_k(t)u + c_k(t) = 0$ we can write

$$\left(a_k(t)u + \frac{b_k(t)}{2}\right)^2 = \frac{{b_k}^2(t) - 4a_k(t)c_k(t)}{4}$$

We now make the birational transformation $w_k(t,u) = a_k(t)u + b_k(t)/2$ to obtain the hyperelliptic curve

$$[w_k(t,u)]^2 = \frac{{b_k}^2(t) - 4a_k(t)c_k(t)}{4}.$$

Note that the inverse map corresponds to the substitution $u = (2w_k(t,u) - b_k(t))/a_k(t)$. Since $a_k(t)$ is a polynomial of degree $k-1$, it has at most $k-1$ vanishing points. Therefore this is a valid birational transformation.

We now wish to show that the polynomial on the right is separable. We make the substitutions

$$f_j(t) = \frac{z^j - z^{-j}}{z - z^{-1}}$$
$$t = z + z^{-1}.$$

The image of $\frac{1}{4}b^2(t) - a(t)c(t)$ under this substitution is $g_k(z)/z^{2k}(z^2-1)^2$ where

$$g_k(z) = 2z^{4k+2} + z^{2k+3} + 2z^{2k+1} + z^{2k-1} + 2.$$

Our goal is to show that the only multiple root of $g_k(z)$ is $z = -1$. This multiple root does not correspond to any multiple roots of the polynomial $b^2(t) - 4a(t)c(t)$, because the substitution formula is invalid when $t = \pm 2$, and it can be verified by direct substitution that these are not multiple roots of ± 2 of $b^2(t) - 4a(t)c(t)$.

Lemma 3.2.8. *The polynomial $g_k(z) = 2z^{4k+2} + z^{2k+3} + 2z^{2k+1} + z^{2k-1} + 2$ has no multiple roots other than $z = -1$ (which is a root of multiplicity 2) for all $k \geq 1$.*

Proof. First we show that the root $z = -1$ has multiplicity 2. An elementary calculation shows that -1 is a root of $g_k(z)$ and $g_k'(z)$. When $k = 1$ we have

$$g_k''(z) = 60z^4 + 20z^3 + 12z,$$

and when $k > 1$ we have

$$g_k''(z) = (32k^2 + 24k + 4)z^{4k} + (4k^2 + 10k + 6)z^{2k+1}$$
$$+ (8k^2 + 4k)^{2k-1} + (4k^2 - 6k + 2)z^{2k-3}.$$

$z = -1$ is not a root of $g_k''(z)$ in either case. Therefore -1 is a root of multiplicity 2.

Let

$$h_k(z) = \frac{g_k'(z)}{z^{2k-2}} = (8k+4)z^{2k+3} + (2k+3)z^4 + (4k+2)z^2 + (2k-1).$$

We will now show that the only root that $g_k(z)$ and $h_k(z)$ have in common is $z = -1$.

We first show that $z = -1$ is the only real root of $g_k(z)$. This can be shown by using $h_k(z)$, since $h_k(z)$ and $g_k'(z)$ have the same sign for nonzero real z. Clearly $g_k(z) > 0$ for nonnegative real z. Therefore we only need to consider values of z in the interval $(-\infty, 0)$. If $z < -1$,

$$h_k(z) < (8k+4)z^{2k+3} + (8k+4) < 0,$$

so $g_k(z)$ is decreasing on $(-\infty, -1)$. Since $g_k(-1) = 0$, we have $g_k(z) > 0$ for all z in $(-\infty, -1)$.

If $-1 < z < 0$,

$$h_k(z) > (8k+4)z^{2k+3} + (8k+4)z^4 > 0$$

so $g_k(x)$ is increasing on $(-1, 0)$. Therefore $g_k(z) > 0$ for all z in $(-1, 0)$. Therefore $z \neq -1$ cannot be a root of both g_k and h_k unless z is imaginary.

Now suppose that z is an imaginary root of both g_k and h_k. Then from the formula for h_k,

$$z^{2k+3} = \frac{-(2k+3)z^4 - (4k+2)z^2 - (2k-1)}{8k+4} \tag{$*$}$$

Since z is a root of g_k, it is also a root of $z^4 g_k$:

$$z^4 g_k(z) = 2z^{4k+6} + z^{2k+7} + 2z^{2k+5} + z^{2k+3} + 2z^4$$

which we can rewrite as

$$z^4 g_k(z) = 2\left(z^{2k+3}\right)^2 + z^{2k+3}\left(z^2+1\right)^2 + 2z^4. \tag{$**$}$$

Substituting the expression for z^{2k+3} from $(*)$ into $(**)$, we obtain

$$-\frac{1}{8}(z-1)^2(z+1)^2 R_k(z) = 0,$$

where

$$R_k(z) = (4k^2+4k-3)z^4 + (24k^2+24k-2)z^2 + (4k^2+4k-3).$$

The roots of R_k give all possibilities for z. We can use the quadratic formula to solve for z^2. Using Maple, we see that the discriminant is $32(4k^2+4k-1)(2k+1)^2$, which is positive for $k \geq 1$. Therefore z^2 is a real number. But in the formula $(*)$, we see that the value on the left would be imaginary, while the value on the right would be real. Therefore, there is no such z. $\square$

It follows that the polynomial $\frac{1}{4}{b_k}^2(t) - a_k(t)c_k(t)$ is separable. Recall that the degree of the Fibonacci polynomial $f_j(t)$ is $j-1$, so the degree of $\frac{1}{4}{b_k}^2(t) - a_k(t)c_k(t)$ is $2k-2$. We have shown the following.

Theorem 3.2.9. *The geometric genus of $X_0(M_k)$ is $k-2$.*

3.2.3 The Geometric Genus of the Non-Canonical Component

Recall that the polynomial corresponding to the non-canonical component is $a_k(t)u^2 + b_k(t)u + c_k(t)$, where

$$a_k(t) = (2-t){f_k}^2(t)$$
$$b_k(t) = {f_{k-1}}^2(t) + 2f_k(t)f_{k-1}(t) - 3{f_k}^2(t) - 2tf_k(t)f_{k-1}(t) + t^2{f_k}^2(t)$$
$$c_k(t) = f_{k-1}(t)f_{k+1}(t) + 4f_k(t)f_{k+1}(t) - 2tf_k(t)f_{k+1}(t) - {f_k}^2(t)$$

Using the birational transformation $v = 2a_k(t)u - b_k(t)$ we can write this as the hyperelliptic curve $v^2 = b_k(t)^2 - 4a_k(t)c_k(t)$. We just need to verify that the right-hand side does not vanish when $t = 2$ since $v = 2a_k(t)(u - b_k(t)/2a_k(t))$. Direct substitution of $t = 2$ into $D_k(t) = b_k(t)^2 - 4a_k(t)c_k(t)$ gives $D_k(2) = 1$.

Our goal is to show that the polynomial $D_k(t) = b_k(t)^2 - 4a_k(t)c_k(t)$ has no multiple roots. If this is the case the geometric genus is given by $\lfloor \frac{d-1}{2} \rfloor$, where d is the degree of the polynomial $D_k(t)$.

At this point we make the substitutions

$$f_j = \frac{z^j - z^{-j}}{z - z^{-1}}$$
$$t = z + z^{-1}$$

and multiply by $z^{4k}(z-1)^4(z+1)^4$ to clear the denominator. We obtain the polynomial:

$$\begin{aligned} P_k(z) = {} & z^{8k+8} - 8z^{8k+7} + 26z^{8k+6} - 44z^{8k+5} + 41z^{8k+4} - 20z^{8k+3} + 4z^{8k+2} \\ & + 16z^{6k+7} - 80z^{6k+6} + 164z^{6k+5} - 176z^{6k+4} + 104z^{6k+3} - 32z^{6k+2} \\ & + 4z^{6k+1} - 12z^{4k+7} + 78z^{4k+6} - z^{4k+5} + 276z^{4k+4} - 204z^{4k+3} + 78z^{4k+2} \\ & - 12z^{4k+1} + 4z^{2k+7} - 32z^{2k+6} + 104z^{2k+5} - 176z^{2k+4} + 164z^{2k+3} \\ & - 80z^{2k+2} + 16z^{2k+1} + 4z^6 - 20z^5 + 41z^4 - 44z^3 + 26z^2 - 8z + 1. \end{aligned}$$

We will show that $D_k(t)$ is separable by showing that $P_k(z)$ has no multiple roots other than $z = 1$ and $z = -1$. These multiple roots of $P_k(z)$ do not correspond to multiple roots of $D_k(t)$. When $z = 1, t = 2$ and we can verify that $t = 2$ is not a root of $D_k(t)$ by direct substitution. We only need to use that fact that $f_k(2) = k$ for all k to obtain $D_k(2) = 1$. We obtain $D_k(-2) = 144k^4 + 96k^3 + 24k^2 + 8k + 1$, which is clearly odd for all k and therefore never 0.

Lemma 3.2.10. *Let p be a prime. Suppose that $f(x) = \sum_{j=0}^{n} a_j x^j$, where each $a_j \in \mathbb{Z}$ and $p \nmid a_n$. Let $\overline{f}(x) = \sum_{j=0}^{n} \overline{a_j} x^j$, where $\overline{a_j} = a_j$ modulo p. Suppose that $\overline{f}(x)$ is irreducible in $\mathbb{F}_p$. Then $f(x)$ is irreducible in $\mathbb{Q}$.*

Proof. Suppose that $f(r) = 0$ where $r \in \mathbb{Q}$. Then we can write $f(x) = (ax - b)h(x)$, for some $a, b \in \mathbb{Z}$ and $h(x) \in \mathbb{Z}[x]$. Then $\overline{f}(x) = (\overline{a}x - \overline{b})\overline{h}(x)$. Since p does not divide the leading coefficient of $f(x)$, $\overline{a} \neq 0$ and $\overline{b}\overline{a}^{-1} \in \mathbb{F}_p$ is a root of $\overline{f}(x)$. Therefore $\overline{f}(x)$ is reducible in $\mathbb{F}_p$. □

The polynomial $P_k(z)$ is divisible by $z^{2k+1} + 1$: we can write

$$P_k(z) = (z^{2k+1} + 1)Q_k(z),$$

where

$$\begin{aligned} Q_k(z) = {} & z^{6k+7} - 8z^{6k+6} + 26z^{6k+5} - 44z^{6k+4} + 41z^{6k+3} - 20z^{6k+2} + 4z^{6k+1} \\ & + 15z^{4k+6} - 72z^{4k+5} + 138z^{4k+4} - 132z^{4k+3} + 63z^{4k+2} - 12z^{4k+1} - 12z^{2k+6} \\ & + 63z^{2k+5} - 132z^{2k+4} + 138z^{2k+3} - 72z^{2k+2} + 15z^{2k+1} + 4z^6 - 20z^5 + 41z^4 \\ & - 44z^3 + 26z^2 - 8z + 1. \end{aligned}$$

The polynomial $z^{2k+1} + 1$ is separable over $\mathbb{Q}$. Suppose that $Q_k(z)$ and $z^{2k+1} + 1$ share a root. If z is such a root, we can write $z^{2k+1} = -1$ and can rewrite $Q_k(z)$ as:

$$\begin{aligned} Q_k(z) = {} & -z^4 + 8z^3 - 26z^2 + 44z - 41 + 20z^{-1} - 4z^{-2} + 15z^4 - 72z^3 + 138z^2 \\ & - 132z + 63 - 12z^{-1} + 12z^5 - 63z^4 + 132z^3 - 138z^2 + 72z - 15 + 4z^6 \\ & - 20z^5 + 41z^4 - 44z^3 + 26z^2 - 8z + 1. \end{aligned}$$

Multiplying through by z^2 and factoring, we obtain

$$4(z + 1)^3(z - 1)^5.$$

Since $z = 1$ is clearly not a root of $z^{2k+1} + 1$, this shows us that $z = -1$ is the only common root of $Q_k(z)$ and $z^{2k+1} + 1$ is $z = -1$. Consider the polynomial $Q_k(z)$ modulo 3. We obtain

$$\begin{aligned} \overline{Q_k}(z) = {} & z^{6k+7} + z^{6k+6} - z^{6k+5} + z^{6k+4} - z^{6k+3} + z^{6k+2} \\ & + z^{6k+1} + z^6 + z^5 - z^4 + z^3 - z^2 + z + 1. \end{aligned}$$

This factors over $\mathbb{F}_3$ as:

$$\overline{Q_k}(z) = (z^{6k+1} + 1)(z-1)^4(z+1)^2.$$

It can be shown that $z = 1$ is a root of multiplicity 4 of the original polynomial $Q_k(z)$. This can be shown by taking derivatives:

$$Q_k(1) = 0, Q_k'(1) = 0, Q_k''(1) = 0, Q_k'''(1) = 0, Q_k^{(4)}(1) = 192.$$

In the same way it can be shown that $z = -1$ is a root of multiplicity 3 of $Q_k(z)$:

$$Q_k(-1) = 0, Q_k'(-1) = 0, Q_k''(-1) = 0, Q_k'''(-1) = 6912k^3 + 1152k^2 + 576k + 76.$$

Since

$$Q'''(-1) = 3(2304k^3 + 384k^2 + 192k + 25) + 1$$

this is never 0. It follows from Lemma 3.2.10 that $Q(z) = R(z)(z-1)^4(z+1)^2$, where $R(z)$ does not have multiple roots. Therefore, $P_k(z)$ has no multiple roots other than $1, -1$, and $D_k(t)$ is separable. An elementary calculation shows that $D_k(t)$ has degree $4k - 1$. We have shown the following.

Theorem 3.2.11. *Let $X_1(M_k)$ be the non-canonical component of $X(M_k)$. The geometric genus of $X_1(M_k)$ is $2k - 1$.*

CHAPTER 4

THE BORROMEAN RINGS

In this chapter we will find the $SL_2(\mathbb{C})$-character variety of the Borromean ring complement and show that it is irreducible. We will then look at Z_σ and V_σ for the symmetry group σ of the Borromean rings.

4.1 Group Presentation of the Borromean Ring Complement

We will use the group presentation given by Montesinos-Amilibia in [16]. When M is the complement of the Borromean rings in S^3, $\pi_1(M) = \Gamma$ is generated by the three meridians shown in the figure below.

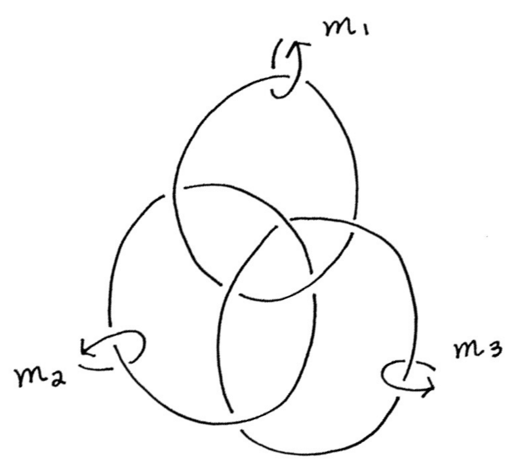

Figure 4.1: The Borromean rings

The relations derive from this fact : when considering any individual ring, its meridian must commute with its longitude. Therefore the three relations are $[m_1, [{m_3}^{-1}, m_2]] = [m_2, [{m_1}^{-1}, m_3]] = [m_3, [{m_2}^{-1}, m_1]] = 1$. Any one of these relations follows from the other two. We use the last two relations as these will make the matrices involved more manageable. Note that the relation $[m_3, [{m_2}^{-1}, m_1]] = 1$ is equivalent to $m_1 m_2 {m_1}^{-1} m_3 m_1 = m_2 m_3 {m_2}^{-1} m_1 m_2$. Writing the relation in this form will make it easier to compute the polynomials that determine the character variety. Therefore we have the following group presentation for the fundamental group of the link complement.

$$\Gamma = \langle m_1, m_2, m_3 | m_1 m_2 {m_1}^{-1} m_3 m_1 = m_2 m_3 {m_2}^{-1} m_1 m_2,$$
$$m_2 {m_1}^{-1} m_3 m_1 {m_3}^{-1} = {m_1}^{-1} m_3 m_1 {m_3}^{-1} m_2 \rangle.$$

Let ρ be an irreducible representation of the fundamental group of the Borromean ring complement Γ into $SL_2(\mathbb{C})$.

$$\rho(m_1) = A,\ \rho(m_2) = B, \text{ and } \rho(m_1) = C.$$

We now use the relations from the group presentation to obtain the matrices whose entries will determine the character variety. Let $R1$ be the matrix corresponding to the first relation and $R2$ be the matrix corresponding to the second relation.

4.2 A Natural Model

In this section we determine a natural model for the character variety of the complement of the Borromean rings, given the presentation for the fundamental group as above.

Proposition 4.2.1. *With Γ as above, $\rho : \Gamma \to SL_2(\mathbb{C})$ is a representation exactly when $R1 = R2 = 0$ with*

$$R1 = ABA^{-1}CA - BCB^{-1}AB$$
$$R2 = BA^{-1}CAC^{-1} - A^{-1}CAC^{-1}B.$$

where

$$A = \rho(m_1), B = \rho(m_2), \text{ and } C = \rho(m_1)$$

are all matrices in $SL_2(\mathbb{C})$.

Definition 4.2.2. For the character variety of the Borromean rings, we define the following variables

$$x = \text{trace}(A)$$
$$y = \text{trace}(B)$$
$$z = \text{trace}(C)$$
$$t = \text{trace}(AB)$$
$$u = \text{trace}(AC)$$
$$v = \text{trace}(BC)$$
$$w = \text{trace}(ABC).$$

Recall from the discussion of the $SL_2(\mathbb{C})$ character variety that using trace relations, all other traces can be written in terms of these traces, so these traces determine the whole character variety. The character variety $X(\Gamma)$ is the set of points (x, y, z, t, u, v, w) in $\mathbb{C}^7$ such that $R1$ and $R2$ are both equal to the zero matrix.

Here we have 3 generators. Therefore we have an additional polynomial relation. The traces of the seven matrices above are related by a polynomial which is quadratic in $w = \text{trace}(ABC)$: we have $\mu(w, x, y, z, t, u, v) = 0$ where

$$\mu(w, x, y, z, t, u, v) = w^2-(tz+vx+uy-xyz)w+tuv+(t^2+u^2+v^2)-(xyt+yzv+xzu)+(x^2+y^2+z^2)-4.$$

At this point we use a lemma by Baker and Petersen in [1] which will make $R1$ and $R2$ more manageable. If we have a representation $\rho : \Gamma \to SL_2(\mathbb{C})$, for $\gamma_1, \gamma_2 \in \Gamma$, up to conjugation we have

$$\rho(\gamma_1) = \begin{pmatrix} a & d \\ 0 & a^{-1} \end{pmatrix} \text{ and } \rho(\gamma_2) = \begin{pmatrix} b & u \\ c & b^{-1} \end{pmatrix}$$

where $cu = 0$. If $u = 0$, then we can conjugate so that $d = 0$ or $d = 1$.

We are interested in irreducible representations. If $\rho : \Gamma \to SL_2(\mathbb{C})$ is an irreducible representation of the Borromean ring complement, then among the matrices A, B, C, there must exist two which are not simultaneously diagonalizable.

Note that in our case Γ is dihedral of order 6, and Γ is generated by 3 elements. Any permutation of the generators m_1, m_2, m_3 induces an isomorphism of Γ corresponding to one of the 6 symmetries of the Borromean rings. Therefore, we can assume without loss of generality that for any irreducible representation ρ, A and B are not simultaneously diagonalizable. Therefore we can assume $d = 1$. Up to conjugation we have

$$A = \begin{pmatrix} a & 1 \\ 0 & a^{-1} \end{pmatrix}, B = \begin{pmatrix} b & 0 \\ c & b^{-1} \end{pmatrix}, \text{ and } C = \begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

where $ps - qr = 1$.

We consider two cases.

The case where $c \neq 0$: We have

$$\begin{aligned}
\text{trace}(A) &= a + a^{-1} = x \\
\text{trace}(B) &= b + b^{-1} = y \\
\text{trace}(C) &= p + s = z \\
\text{trace}(AB) &= c + ab + \frac{1}{ab} = t \\
\text{trace}(AC) &= r + ap + \frac{s}{a} = u \\
\text{trace}(BC) &= pb + cq + \frac{s}{ab} = v \\
\text{trace}(ABC) &= pab + pc + \frac{r}{b} + \frac{cq}{a} + \frac{s}{ab} = w.
\end{aligned}$$

We can set $c = t - ab - \frac{1}{ab}$ and $r = u - ap - \frac{s}{a}$. Note that since c is nonzero we can write $q = (v - pb - \frac{s}{b})/c$. We then have

$$\begin{aligned}
\text{trace}(A) &= x \\
\text{trace}(B) &= y \\
\text{trace}(C) &= z \\
\text{trace}(AB) &= t \\
\text{trace}(AC) &= u \\
\text{trace}(BC) &= v \\
\text{trace}(ABC) &= pt - \frac{z}{ab} + \frac{u}{b} + \frac{v}{a} - p\left(\frac{a}{b} + \frac{b}{a}\right) = w.
\end{aligned}$$

In our computations, we use the fact that $ps - qr = 1$ to write

$$C^{-1} = \begin{pmatrix} s & -q \\ -r & p \end{pmatrix}.$$

Using Maple we see that

$$R1[2,2] = \frac{(b-1)(b+1)(ua - a^2p - s)}{ab^3}.$$

But $ua - a^2p - s = ar$. The entry a is always nonzero and in this special case we must have r nonzero. Therefore $c = 0$ implies that $b = 1$ or $b = -1$. Substituting either of these values in for b results in both $R1$ and $R2$ equal to the zero matrix.

Because c is nonzero we can make the substitution $q = (v - pb - \frac{s}{b})/c$.

Lemma 4.2.3. *The matrix $R1$ is equal to the zero matrix if and only if $tu - xw, tv - yw, vx - uy$ are all zero.*

Proof. The trace of the $R1$ matrix is equal to the polynomial

$$\phi(x, y, w, t, u, v) = (x^2 - y^2)w + ytv - tux - 2xv + 2uy.$$

We now consider the entry $R1[1,2]$. This entry is equal to

$$\frac{-a^2p}{b} + pta + \frac{au}{b} - pb - tu + v - \frac{z}{b} - \frac{p}{b} + \frac{pt}{a} + \frac{u}{ab} - \frac{z}{a^2b} - \frac{pb}{a^2} + \frac{v}{a^2}.$$

Since $x = a + \frac{1}{a}$ we can write this as:

$$ptx - \frac{zx}{ab} + \frac{ux}{b} + \frac{vx}{a} - px\left(\frac{a}{b} + \frac{b}{a}\right) - tu.$$

Using the identity for w, this simplifies as $xw - tu$.

The entry $R1[2,1]$ is equal to:

$$\frac{(tab - a^2b^2 - 1) \times \psi}{a^2b^3}$$

where

$$\psi = pa^2b^2 + a^2p - tab^3p - ab^2u + ab^2tv - ptab - au + s + b^4p - vb + p + 2b^2p - b^3v + sb^2.$$

Since $tab - a^2b^2 - 1 = abc$ and c is nonzero here, we must have $\psi = 0$. When we divide ψ by ab^2 we obtain

$$tv - y\left(pt - \frac{z}{ab} + \frac{u}{b} + \frac{v}{a} - p\left(\frac{a}{b} + \frac{b}{a}\right)\right).$$

This is equal to $tv - yw$.

The entry $-R1[1,1]$ is

$$\frac{1}{ab}(-vtab^2 + ab^3pt + a^2btu - a^3bpt - b^2p + a^2s + b^3v - b^4p - a^3u + a^4p + bv - b^2s - au + a^2p).$$

Since

$$(b^2 - a^2)\left(\frac{v}{a} + \frac{u}{b}\right) - uy + vx = \frac{b^2 v}{a} - \frac{a^2 u}{b} - \frac{u}{b} + \frac{v}{a}$$

we can rewrite this entry as

$$pt(b^2 - a^2) - p\left(\frac{a}{b} + \frac{b}{a}\right)(b^2 - a^2) - \frac{z(b^2 - a^2)}{ab} + (b^2 - a^2)\left(\frac{v}{a} + \frac{u}{b}\right) - uy + vx + atu - btv.$$

We have $tu = xw$ and $tv = yw$. Substituting xw in for tu and yw in for tv and factoring out the $(b^2 - a^2)$ from the other terms gives

$$(b^2 - a^2)w - uy + vx + axw - byw.$$

Since $x = a + \frac{1}{a}$ and $y = b + \frac{1}{b}$, this is equal to $vx - uy$. We have shown that the three equations are satisfied when $R1 = 0$.

Suppose we have $tu = xw, tv = yw, vx = uy$. In this case we can write $\phi(x, y, w, t, u, v)$ as

$$x(xw) - y(yw) + ytv - tux + 2(uy - xv) = x(xw - tu) + y(tv - yw) = 0.$$

Therefore $\text{trace}(R1) = 0$. □

The case where $c = 0$:

Lemma 4.2.4. *The representations where $c = 0$ are given by*
$y = 2, t = x, w = u, v = z$ *and* $y = -2, t = -x, w = -u, v = -z$

Proof. When $c = 0$ we have

$$\text{trace}(A) = a + a^{-1} = x$$
$$\text{trace}(B) = b + b^{-1} = y$$
$$\text{trace}(C) = p + s = z$$
$$\text{trace}(AB) = ab + \frac{1}{ab} = t$$
$$\text{trace}(AC) = r + ap + \frac{s}{a} = u$$
$$\text{trace}(BC) = pb + \frac{s}{ab} = v$$
$$\text{trace}(ABC) = pab + \frac{u}{b} - \frac{pa}{b} = w.$$

Note that in this case A and B are both upper triangular, and we must have r nonzero for ρ to be an irreducible representation.

We now use the relations from the group presentation to obtain the matrices whose entries will determine the character variety. Let $R1$ be the matrix corresponding to the first relation and $R2$ be the matrix corresponding to the second relation. We have

$$R1 = ABA^{-1}CA - BCB^{-1}AB$$
$$R2 = BA^{-1}CAC^{-1} - A^{-1}CAC^{-1}B$$

Here we use the fact that $ps - qr = 1$ to write $C^{-1} = \begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$. Using Maple we see that

$$R1[2,2] = \frac{(b-1)(b+1)(ua - a^2p - s)}{ab^3}.$$

But $ua - a^2p - s = ar$. a is always nonzero and in this special case we must have r nonzero. Therefore $c = 0$ implies that $b = 1$ or $b = -1$. Substituting either of these values in for b results in both $R1$ and $R2$ equal to the zero matrix.

Note that in the general case we have $t = c + ab + \frac{1}{ab}$, so if $b = 1$, $c = 0$ iff $t = x$. If $b = -1$, $c = 0$ iff $t = -x$. Putting everything together we see that $c = 0$ if and only if we have $b = 2, t = x$, or we have $b = -2, t = -x$. The first solution gives the algebraic set $b = 2, t = x, w = u, v = z$. The second solution gives the algebraic set $b = -2, t = -x, w = -u, v = -z$. □

Lemma 4.2.5. *The matrix $R2$ is equal to the zero matrix if and only if $zw - uv$ and $vx - tz$ are both zero.*

Proof. The trace of $R2$ is equal to 0. It is therefore sufficient to consider the entries $R2[1,2], R2[2,1]$ and $R2[1,1]$. The entry $R2[2,1]$ is

$$\frac{1}{a^3b}(a^4p + a^4bv - 2a^3bpt - a^3u - a^3bst + a^2b^2p + 2a^2p + a^2s$$
$$+ a^2btu - abpt - au + b^2p + z - bv).$$

Using the identity for w and simplifying makes this equal to

$$vx - w - tz + \frac{tu}{a} - \frac{w}{a^2}.$$

We have from the first matrix that $xw = tu$. Dividing both sides by a gives

$$\left(1+\frac{1}{a^2}\right)w = \frac{tu}{a}, \text{ or } -w - \frac{w}{a^2} + \frac{tu}{a} = 0.$$

Therefore this entry is equal to $vx - tz$. The entry $R2[1,2]$ is

$$-a^2uv + (a^2+1)apv + azv - pvx + aptu + \frac{a^2su}{b} - a^2p^2t - \frac{as^2}{b} - \frac{aps}{b}(a^2+1)$$
$$- abps - p^2t + \frac{pz}{ab} - \frac{pu}{b} + p^2\left(\frac{a}{b}+\frac{b}{a}\right).$$

Using the identity for w and using $tu = xw$ to rewrite $aptu$ gives

$$-a^2uv + a^2pva + pva + azv + \frac{a^2uz}{b} - \frac{az^2}{b} - \frac{a^3ps}{b} - abpz - \frac{a^3p^2}{b}.$$

Using $vx = tz$ to rewrite $a^2pva + pva$ and the identity for w gives $a^2(zw - uv)$, which is 0 if and only if $zw = uv$.

The entry $R2[1,1]$ is

$$(-au + a^2p + z)v + ptu + \frac{azu}{b} - pxw - bzp - \frac{z^2}{b} - \frac{a^2pz}{b} + pv.$$

At this point we divide by a and use the identities $uv = zw$ and $vx = tz$ to see that this entry must be 0.

□

We have shown the following. Note that since the special case where $c = 0$ is a subset of this, we can unite the 2 subcases by considering just these polynomials.

Theorem 4.2.6. *A natural model for the character variety of the complement of the Borromean rings in S^3 is the vanishing set of*

$$tu = xw, tv = yw, uv = zw, vx = uy = tz$$

and

$$\mu(w,x,y,z,t,u,v) = w^2 - (tz + vx + uy - xyz)w + tuv + (t^2+u^2+v^2)$$
$$- (xyt + yzv + xzu) + (x^2+y^2+z^2) - 4$$

in $\mathbb{C}^7$.

4.3 A Nicer Model

In this section, we use a birational transformation to produce a nicer model for the character variety.

The polynomial that determines the character variety of the Borromean rings complement is given in Theorem 4.2.6 as

$$\begin{aligned}\mu(w,t,u,v,x,y,z) &= w^2 - (tz + vx + uy - xyz)w + tuv + (t^2 + u^2 + v^2) \\ &\quad - (xyt + yzv + xzu) + (x^2 + y^2 + z^2) - 4\end{aligned}$$

Making the substitutions $x = \frac{tu}{w}, y = \frac{tv}{w}, z = \frac{uv}{w}$ and multiplying by w^2 gives

$$\begin{aligned}\xi(w,t,u,v) &= w^4 + (t^2 + u^2 + v^2 - 2tuv - 4)w^2 + t^2u^2 + t^2v^2 + u^2v^2 \\ &\quad + t^2u^2v^2 - t^3uv - tu^3v - tuv^3\end{aligned}$$

which is biquadratic in w. If we set

$$\begin{aligned}b &= t^2 + u^2 + v^2 - 2tuv - 4 \\ c &= t^2u^2 + t^2v^2 + u^2v^2 + t^2u^2v^2 - t^3uv - tu^3v - tuv^3\end{aligned}$$

then $b^2 - 4c$ factors as

$$(t - u + v - 2)(t + u + v + 2)(t + u - v - 2)(t - u - v + 2)$$

which is not a square in $\mathbb{C}[t,u,v]$. Therefore $\xi(w,t,u,v)$ is irreducible.

To show that this substitution determines a birational mapping, we must show that the vanishing set of $w = 0$ has codimension at least 1. Suppose that $w = 0$. Then:

$$\mu(0,t,u,v,x,y,z) = x^2 + y^2 + z^2 + t^2 + u^2 + v^2 + tuv - xyt - yzv - xzu - 4$$

Therefore, if $w = 0$, the polynomial relations $wz = uv, wy = tv, wx = tu$ imply that either $v = 0$ or $t = u = 0$.

We first consider the case where $v \neq 0$. Then $t = u = 0$. Since t and u are zero and $vx = uy = tz$ we must have $vx = 0$. Since $v \neq 0$, we have $x = 0$, and the polynomial becomes:

$$\mu(0,0,0,v,0,y,z) = y^2 + z^2 + v^2 - yzv - 4$$

which has a vanishing set of dimension 2.

Now we consider the case where $v = 0$. Since $tu = 0$ at least one of t, u must be 0. If $t = u = v = 0$ we have

$$\mu(0,0,0,0,x,y,z) = x^2 + y^2 + z^2 - 4$$

which has a vanishing set of dimension 2. Suppose that $t \neq 0$. Then $u = 0$ and since $tz = 0$, we must have $z = 0$. In this case the polynomial reduces to:

$$\mu(0,t,0,0,x,y,0) = x^2 + y^2 + t^2 - xyt - 4$$

which has a vanishing set of dimension 2. If $u \neq 0$, then $t = 0$ and since $uy = 0$ we must have $y = 0$.The polynomial reduces to:

$$\mu(0,0,u,0,x,0,z) = x^2 + z^2 + u^2 - xzu - 4$$

which has vanishing set of dimension 2.

Therefore in all cases the vanishing set is of codimension 1, so this is a valid birational transformation. The fact that $\xi(w,t,u,v)$ is irreducible tells us that the character variety consists of exactly one component, which is of dimension 3.

We have shown the following.

Theorem 4.3.1. *The character variety of the Borromean rings complement is birational to the irreducible variety given by the vanishing set of*

$$\xi(w,t,u,v) = w^4 + (t^2 + u^2 + v^2 - 2tuv - 4)w^2 + t^2u^2 + t^2v^2 + u^2v^2 + t^2u^2v^2 - t^3uv - tu^3v - tuv^3$$

in $\mathbb{C}^4$.

4.4 Analyzing Symmetries of the Borromean Ring Complement

Let M be the Borromean ring complement and S the full symmetry group of M. Let σ be the $\frac{2\pi}{3}$ rotation sending C_1 to C_2, and let τ be the flip symmetry with $\tau(C_2) = C_3$ and $\tau(C_3) = C_2$. Then using Theorem 4.3.1, $V_\sigma = \langle t-u, u-v\rangle$ and $V_\tau = \langle t-u\rangle$. Note that since $V_\sigma \subset V_\tau$, we have $V_S = V_\sigma$.

Z_σ is the Zariski closure of the set of points $\{X_0(M(p/q,p/q,p/q))\}$ lying on $X_0(M)$. V_σ is the Zariski closure of the set of points on $X_0(M)$ with $t = u = v$. Using Theorem 4.3.1 again, we see that V_σ is birational to the vanishing set of

$$\psi(t,w) = w^4 + (-2t^3 + 3t^2 - 4)w^2 + t^6 - 3t^5 + 3t^4.$$

Note that $\psi(t,w)$ is biquadratic in w. Setting $b = -2t^3 + 3t^2 - 4$ and $c = t^6 - 3t^5 + 3t^4$ we calculate the determinant

$$b^2 - 4c = -3t^4 + 16t^3 - 24t^2 + 16 = -(3t+2)(t-2)^3$$

which is not a square in $\mathbb{C}[t]$. Therefore $\psi(t,w)$ is irreducible and V_σ consists of a single irreducible component of dimension 1. Since $Z_\sigma \subseteq V_\sigma$ and Z_σ has dimension 1, it follows that $V_\sigma = Z_\sigma$.

If $\psi(t,w) = 0$ we can write

$$\left(w^2 + \frac{1}{2}\Big(-2t^3 + 3t^2 - 4\Big)\right)^2 = -\frac{1}{4}(3t+2)(t-2)^3$$

so the geometric genus of $\psi(t,w)$ is 0.

Z_τ is the Zariski closure of the set of points $\{X_0(M(p/q,p'/q',p'/q'))\}$ lying on $X_0(M)$. We know from Theorem 2.3.2 that this set has dimension 2. V_τ is the Zariski closure of the set of points on $X_0(M)$ with $t = u$. Using Theorem 4.3.1 again, we see that V_τ is birational to the vanishing set of

$$\psi(u,v,w) = w^4 + (2u^2 + v^2 - 2u^2v - 4)w^2 + u^4 + 2u^2v^2 + u^4v^2 - 2u^4v - u^2v^3.$$

Note that $\psi(u,v,w)$ is biquadratic in w. Setting $b = 2u^2 + v^2 - 2u^2v - 4$ and $c = u^4 + 2u^2v^2 + u^4v^2 - 2u^4v - u^2v^3$ we calculate the determinant

$$b^2 - 4c = -(v-2)^2(2u+v+2)(2u-v-2)$$

which is not a square in $\mathbb{C}[u,v]$. Therefore $\psi(u,v,w)$ is irreducible and V_τ consists of a single irreducible component of dimension 2. Since $Z_\tau \subseteq V_\tau$ and Z_τ has dimension 2, it follows that $V_\tau = Z_\tau$.

BIBLIOGRAPHY

[1] Kenneth L. Baker and Kathleen L. Petersen, *Character varieties of once-punctured torus bundles with tunnel number one*, Internat. J. Math. **24** (2013), no. 6, 1350048, 57. MR 3078072

[2] S. Boyer, E. Luft, and X. Zhang, *On the algebraic components of the* $\mathrm{SL}(2,\mathbb{C})$ *character varieties of knot exteriors*, Topology **41** (2002), no. 4, 667–694. MR 1905834

[3] Steven Boyer, Cameron McA. Gordon, and Xingru Zhang, *Characteristic submanifold theory and toroidal Dehn filling*, Adv. Math. **230** (2012), no. 4-6, 1673–1737. MR 2927352

[4] Steven Boyer and Xingru Zhang, *A proof of the finite filling conjecture*, J. Differential Geom. **59** (2001), no. 1, 87–176. MR 1909249

[5] D. Cooper, M. Culler, H. Gillet, D. D. Long, and P. B. Shalen, *Plane curves associated to character varieties of* 3*-manifolds*, Invent. Math. **118** (1994), no. 1, 47–84. MR 1288467

[6] D. Cooper and D. D. Long, *Remarks on the A-polynomial of a knot*, J. Knot Theory Ramifications **5** (1996), no. 5, 609–628. MR 1414090

[7] Marc Culler, *Lifting representations to covering groups*, Adv. in Math. **59** (1986), no. 1, 64–70. MR 825087

[8] Marc Culler, C. McA. Gordon, J. Luecke, and Peter B. Shalen, *Dehn surgery on knots*, Ann. of Math. (2) **125** (1987), no. 2, 237–300. MR 881270

[9] Marc Culler and Peter B. Shalen, *Varieties of group representations and splittings of* 3*-manifolds*, Ann. of Math. (2) **117** (1983), no. 1, 109–146. MR 683804

[10] William D. Dunbar and G. Robert Meyerhoff, *Volumes of hyperbolic* 3*-orbifolds*, Indiana Univ. Math. J. **43** (1994), no. 2, 611–637. MR 1291531

[11] Stavros Garoufalidis, *On the characteristic and deformation varieties of a knot*, Proceedings of the Casson Fest, Geom. Topol. Monogr., vol. 7, Geom. Topol. Publ., Coventry, 2004, pp. 291–309 (electronic). MR 2172488

[12] Ben Klaff and Stephan Tillmann, *A birationality result for character varieties*, Math. Res. Lett. **23** (2016), no. 4, 1099–1110. MR 3554502

[13] Kouzi Kodama and Makoto Sakuma, *Symmetry groups of prime knots up to* 10 *crossings*, Knots 90 (Osaka, 1990), de Gruyter, Berlin, 1992, pp. 323–340. MR 1177431

[14] Emily Landes, *Identifying the canonical component for the Whitehead link*, Math. Res. Lett. **18** (2011), no. 4, 715–731. MR 2831837

[15] Melissa L. Macasieb, Kathleen L. Petersen, and Ronald M. van Luijk, *On character varieties of two-bridge knot groups*, Proc. Lond. Math. Soc. (3) **103** (2011), no. 3, 473–507. MR 2827003

[16] José Marí a Montesinos-Amilibia, *Around the Borromean link*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **102** (2008), no. 1, 75–87. MR 2416239

[17] Tomotada Ohtsuki, Robert Riley, and Makoto Sakuma, *Epimorphisms between 2-bridge link groups*, The Zieschang Gedenkschrift, Geom. Topol. Monogr., vol. 14, Geom. Topol. Publ., Coventry, 2008, pp. 417–450. MR 2484712

[18] Luisa Paoluzzi and Joan Porti, *Hyperbolic isometries versus symmetries of links*, Topology Appl. **156** (2009), no. 6, 1140–1147. MR 2493374

[19] Kathleen L. Petersen and Alan Reid, *Gonality and genus of canonical components of twist links.*

[20] Kathleen L. Petersen and Anh T. Tran, *Character varieties of double twist links*, Algebr. Geom. Topol. **15** (2015), no. 6, 3569–3598. MR 3450771

[21] Robert Riley, *An elliptical path from parabolic representations to hyperbolic structures*, Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977), Lecture Notes in Math., vol. 722, Springer, Berlin, 1979, pp. 99–133. MR 547459

[22] Peter B. Shalen, *Representations of 3-manifold groups*, Handbook of geometric topology, North-Holland, Amsterdam, 2002, pp. 955–1044. MR 1886685

[23] William P. Thurston, *The geometry and topology of three-manifolds*, 1979.

[24] Anh T. Tran, *The universal character ring of the* $(-2, 2m+1, 2n)$*-pretzel link*, Internat. J. Math. **24** (2013), no. 8, 1350063, 13. MR 3103879

[25] ———, *Character varieties of* $(-2, 2m+1, 2n)$*-pretzel links and twisted Whitehead links*, J. Knot Theory Ramifications **25** (2016), no. 2, 1650007, 16. MR 3463852

BIOGRAPHICAL SKETCH

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