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# Analysis of Two Partial Differential Equation Models in Fluid Mechanics: Nonlinear Spectral Eddy-Viscosity Model of Turbulence and Infinite-Prandtl-Number Model of Mantle Convection

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# ANALYSIS OF TWO PARTIAL DIFFERENTIAL EQUATION MODELS IN FLUID MECHANICS: NONLINEAR SPECTRAL EDDY-VISCOSITY MODEL OF TURBULENCE AND INFINITE-PRANDTL-NUMBER MODEL OF MANTLE CONVECTION

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This thesis is dedicated to my mother and father.

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# ABSTRACT

This thesis presents two problems in the mathematical and numerical analysis of partial differential equations modeling fluids. The first is related to modeling of turbulence phenomena. One of the objectives in simulating turbulence is to capture the large scale structures in the flow without explicitly resolving the small scales numerically. This is generally accomplished by adding regularization terms to the Navier-Stokes equations. In this thesis, we examine the spectral viscosity models in which only the high-frequency spectral modes are regularized. The objective is to retain the large-scale dynamics while modeling the turbulent fluctuations accurately. The spectral regularization introduces a host of parameters to the model. In this thesis, we rigorously justify effective choices of parameters.

The other problem is related to modeling of the mantle flow in the Earth's interior. We study a model equation derived from the Boussinesq equation where the Prandtl number is taken to infinity. This essentially models the flow under the assumption of a large viscosity limit. The novelty in our problem formulation is that the viscosity depends on the temperature field, which makes the mathematical analysis non-trivial. Compared to the constant viscosity case, variable viscosity introduces a second-order nonlinearity which makes the mathematical question of well-posedness more challenging. Here, we prove this using tools from the regularity theory of parabolic partial differential equations.

# CHAPTER 1

# INTRODUCTION

In this thesis, we will examine two different problems in the analysis of partial differential equations that arise from fluid dynamics. In the first part, we will examine some PDE's related to turbulence modeling. Fluid turbulence in three dimensions is usually modelled by the Navier-Stokes equations(NSE) with a large *Reynolds number*. At the present time, the simulation of this equation in this regime is a formidable task due to the need to resolve the small scale fluctuations or *eddies* that have subtle effects on the large-scale dynamics of the fluid. In order to make this problem computationally tractable, this effect must be modeled while the large-scale motion is simulated faithfully. In one idea, the velocity field is averaged over a small radius to derive equations in terms of the averaged velocity. In this process, the problem of *closure* arises in that the average of the nonlinear term in NSE, which is called the *Reynolds stress*, must be approximated and expressed solely in terms of the averaged quantities. The way in which this is done gives rise to a variety of models. The approach we will look at, called the *eddy-viscosity method*, treats the Reynolds stress as a viscous effect caused by the transport and dissipation of energy due to the small-scale eddies. For this reason, this additional viscosity is called the *eddy-viscosity* or *turbulent viscosity*. The turbulence model of Smagorinsky belongs to this type [32, Smagorinsky]. For an overall survey on issues related to these models, see a discussion by [22, Layton]. Unfortunately, a simple application of this idea leads to the over-smearing of the large-scale structures in the fluid. To remedy this unwanted effect, it has been proposed that the eddy-viscosity be added only to the *subgrid scales*. This means that we define appropriate notion of subgrid scales, and add an eddy-viscosity only to those subgrid scales. In this way, we hope to prevent the large-scale structure from being smeared away. In this thesis, we examine a particular class of models called the spectral eddy-viscosity models in which the scales are defined in terms of Fourier modes. The subgrid viscosity is simply realized as an addition of the artificial viscosity only to the high-frequency modes. A simple implementation of this is to insert a high-pass spectral filter into the normal artificial viscosity.

We consider two types of eddy-viscosities: the hyperviscosity and the nonlinear viscosity. The hyperviscosity models are considered by various researchers because of the simplicity of the idea [25, Lions], [14, Guermond]. An example of the nonlinear viscosity is the Smagorinsky model mentioned before. The difficulty of turbulence manifests itself mathematically in the unsolved conjecture of the well-posedness of 3D Navier-Stokes equations, which is one of the millennium prize problems set by the Clay mathematics institute. The hyperviscosity and typical nonlinear viscosity models are well-posed and overcome this difficulty. The catch is that we have now introduced several parameters to the model: eddy-viscosity coefficient, strength of the viscosity operator, and the cut-off frequency that distinguishes the small scales from the large scales. Hence, we would like to find some guiding principles for choosing these parameters.

A rigorous justification of a turbulence model is difficult partly because we do not have a good physical understanding of turbulence phenomena. However, given that the weak solution to the Navier-Stokes equations models the turbulence accurately in an ideal world where we have infinite computational power, we can split the problem of turbulence modelling into two parts. One is that the turbulence model should model the fluid. This can be undertaken by showing that the model is consistent with the Navier-Stokes equations in some way. For instance, this was undertaken by [14, Guermond] who analyzed and proved that the solution to the hyperviscosity model, under certain constraints on the parameters, converges to a suitable solution of the Navier-Stokes equations that satisfies the strongest partial regularity proved to date by [2, Caffarelli, Kohn and Nirenberg]. Another type of consistency result is to assume that the solution to NSE is smooth and show that in this regime, the turbulence model is "close" to the solution to NSE. Intuitively, the cut-off filter plays an important role here, as it gives us a spectral convergence rate to the solution of NSE assuming that the solution is smooth. The other part is that the turbulence model should be tractable for numerical simulations. This means that it should be well-posed, and if so the numerical method used to simulate it should be stable. The by-product of these investigation is the insight gained into how various parameters introduced into the equation such as the cut-off frequency, the strength of nonlinearity or hyperviscosity, affect the consistency and stability.

In the second chapter we will introduce various notations and formal definitions of our turbulence models. In the third chapter, we tackle the question of well-posedness. In the fourth chapter, we consider a semi-implicit discretization of the nonlinear viscosity model. We will derive a uniform-in-time stability estimate for a bounded power input. This gives us some insight as to the reason why we should not take a strongly nonlinear viscosity. The fifth chapter investigates the consistency question. We estimate the degree by which the nonlinear viscosity model is a perturbation of the NSE by estimating the error rate as the perturbation goes to zero. From this analysis, we realize that certain parameters depend on others and hence can be eliminated. In the sixth chapter, we prove that for some specific values of parameters, the hyperviscosity model and the nonlinear viscosity model have effectively finite dimensional dynamics in that any two solutions under the same forcing that agrees in the low-frequency part are exponentially contracted to a single path in the phase space; in other words, the high-frequency modes becomes irrelevant to the dynamics of the solutions. The hyperviscosity model case is shown to possess such an exponential contraction property already [37, Temam]. It is interesting that we can also estimate the dimension of such a finite dimensional *attractor*, in terms of the parameters.

In the second part of our thesis, we consider a model related to the mantle convection in the Earth's interior. A basic assumption is that in the geological time scales, the rocks will behave as a very viscous fluid and hence we can use the equation of fluid motion to model this phenomena. An equation that is often used to model fluid convection is the Boussinesq approximation, where we neglect the density variations in the Navier-Stokes equations while we preserve its gravitational effect by modeling the buoyancy forces that are proportional to the spatial variations in the temperature field. The temperature field itself is convected by the fluid and such a coupling provides the setting for fluid convection. In order to use Boussinesq approximation to model the mantle flow, we must take the rheology of the rocks into account. In one direction, this simplifies our model in that the high viscosity of the flow allows us to ignore the inertial term in the NSE: a large simplification from a mathematical point of view. Such a viscous limit of Boussinesq approximation is called the *infinite Prandtl number model* because of the way in which the nondimensional number in the Boussinesq approximation called the *Prandtl number* is taken to infinity in this process. This is a substantial simplification, and when the viscosity is uniform over the whole domain, an extensive analysis of this model can be undertaken [3, Doering and Constantin]. Unfortunately, in our model, the viscosity variation cannot be neglected. This is because the part of the earth closer to the core is hotter and hence the rocks deform more readily than the part closer to the surface. Therefore, we model such an effect by varying the viscosity of the flow according to the the temperature field. The infinite Prandtl number model with temperature dependent viscosity is among the most popular models in use by geophysical community to simulate tectonic dynamics and mantle flows [29, Moresi].

For the constant viscosity case, the well-posedness questions are relatively easy, and large efforts are spent on the estimation of a quantity called the *Nusselt number*, which quantifies the ratio of the heat transport due to the convection to that due to the conduction. For the temperature dependent viscosity case, the second-order nonlinearity gives an additional challenge even to the well-posedness question. In this thesis, we will show how the well-posedness can be proved for this equation. An important question that is left open is the gap that exists between the estimate of the Nusselt number between the constant viscosity case and our case.

# CHAPTER 2

# SPECTRAL EDDY-VISCOSITY MODELS

In the subsequent chapters, we will introduce two spectral eddy-viscosity models and discuss various mathematical problems they inspire. In this work we will consider a three-dimensional domain that is periodic in all directions, and thus limit ourselves to the investigation of isotropic turbulence.

## 2.1 The Navier-Stokes equations

We will start with the famous Navier-Stokes equations.

$$\begin{aligned} &\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla \pi = f, \\ &\nabla \cdot u = 0, \end{aligned}$$

where $u$ is the velocity field, $\pi$ is the pressure and $\nu = Re^{-1}$ is the nondimensional inverse Reynolds number. The modern theory of partial differential equations concerns itself extensively with different notions of what it means for a function to "solve" a PDE. The reason is that when we have stated the NSE as above, we a priori assume that the solution possess at least two spatial derivatives and one time derivative and that they are continuous. Such solutions are called *classical solutions*. It turns out that showing that classical solution exists for general data is a mathematically challenging task, or as in the case of NSE: unknown. If we cannot hope for finding a smooth, classical solution, can we find a different notion of a solution that relaxes this criteria? The answer lies in the introduction of the *weak solution* that uses integration by parts so that less regularity is required of a solution. In the NSE case, a weak solution is a vector field $u$ such that $\int u = 0$ and satisfies the following:

$$\int \langle \partial_t u, \phi \rangle \, dt + \iint (\nu \nabla u : \nabla \phi + u \cdot \nabla u \cdot \phi + \pi \nabla \cdot \phi) \, dx \, dt = \iint f \, dxdt$$
$$\iint \nabla \cdot u \psi = 0,$$

for all *test vector field* $\phi$ and a test function $\psi$ that is smooth and compactly supported in space and in time. $\langle , \rangle$ denotes the duality coupling.

As formulated, $u$ is only required to have just one derivative in space. For a technical reason, the time derivative is allowed to be even more irregular in that it can be a measure. More specifically, using the terminology of Sobolev spaces, $u$ is sought in the space $L^\infty((0,T);L^2) \cap L^2((0,T):H^1)$ and $\partial_t u, p \in L^{\frac{4}{3}}((0,T):H^{-1})$. Such spaces basically fall out naturally from the a priori analysis of the PDE.

## 2.2 Eddy-viscosity models

We will now discuss how a typical eddy-viscosity model is derived. As noted in the previous chapter, the large-scale structure of the velocity field can be extracted by filtering out the small fluctuations.

$$u_l(x) = g_\delta * u = \int g_\delta(x-y)u(y) \, dy$$

where $g_\delta$ is a smooth function of compact support of radius $\delta > 0$. If we average the NSE using this function, we get

$$\begin{aligned} &\partial_t u_l - \nu \Delta u_l + \nabla \cdot (u_l \otimes u_l) \\ &+\nabla \cdot (g_\delta * (u \otimes u) - u_l \otimes u_l) + \nabla \pi_l = f, \\ &\nabla \cdot u_l = 0. \end{aligned}$$

Notice how this averaging process introduces an additional stress term:

$$R_\delta(u,u) = g_\delta * (u \otimes u) - u_l \otimes u_l.$$

This is called the *Reynolds stress*. Clearly, we cannot yet solve for the average field, because $g_\delta * (u \otimes u)$ contains interactions between the large-scale eddies and the small-scale eddies where the latter is something we would like to avoid computing. Thus, the problem of turbulence modeling is that of a *closure* of the above model: we would like to find an appropriate approximation $S_\delta(u_l, u_l) \sim R_\delta(u, u)$.

$$\begin{aligned} &\partial_t w - \nu \Delta w + \nabla \cdot (w \otimes w) + \nabla \cdot S_\delta(w, w) + \nabla q = f, \\ &\nabla \cdot w = 0. \end{aligned}$$

The idea of the eddy-viscosity model is that the small eddies dissipate energy; therefore, their effect on the average velocity field can be modeled by a viscous dissipation. Such an eddy-induced viscous effect is called the *eddy-viscosity*:

$$\nabla \cdot S_\delta(w, w) = -\nabla \cdot (\nu_T \nabla w).$$

where $\nu_T$ is the eddy-viscosity coefficient.

Smagorinsky proposed $\nu_T = |\nabla u|^{p-2}$ and derived the following:

$$\begin{aligned} &\partial_t u - \nu \Delta u + u \cdot \nabla u - \nabla \cdot (\epsilon_\delta (|\nabla u|^{p-2} \nabla u)) + \nabla \pi = f, \\ &\nabla \cdot u = 0. \end{aligned}$$

The dependence of $\epsilon_\delta$ on $\delta$ should be determined appropriately by dimensional analysis by comparing with the dimensions of the Reynolds stress. We will circumvent this issue for now and simply denote this coefficient as $\epsilon$. Notice that the eddy-viscosity is modeled as a diffusion effect that is proportional in strength to the velocity gradient. The degree of such a proportionality is quantified by the parameter $p$. The nonlinear diffusion operator introduced here is called the *p-Laplacian*, and it is used extensively in the field of non-Newtonian fluid dynamics as well as in many other areas. The mathematical property of weak solutions to this equation is investigated by [20, Ladyzhenskaya]. It is known that the strong solution to the above equation exists globally and is unique on a periodic domain for $p \geq \frac{11}{5}$. [28, Malek et al]. Even though this equation is well-posed in this sense, the $p$-Laplacian adds some difficulty. In particular, we are unable to talk about the solution in a classical sense unless we know that the gradient of the velocity is continuous. The author does not know if such a result is proved. However, a local Holder regularity for an equation of the form $\partial_t u - \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$ is known [8, DiBenedetto]. The proof of the local regularity for this equation depends on maximum principle type arguments. The extension of this result to the Smagorinsky case is obstructed by the fact that the maximum principle techniques which are often the method of choice in proving regularity results do not apply, because of of the inherently global effect introduced by the incompressibility condition.

Another turbulence model that attains well-posedness is the *hyperviscosity model* which was analyzed by [25, Lions]:

$$\begin{aligned}
&\partial_t u - \nu\Delta u + u \cdot \nabla u - \epsilon(-\Delta)^\alpha u + \nabla \pi = f,\\
&\nabla \cdot u = 0.
\end{aligned}$$

The fractional differentiation operator is defined in the frequency space as follows:

$$(-\hat{\Delta})^\alpha u(k) = |k|^{2\alpha}\hat{u}(k).$$

where $\hat{u}$ denotes the $k$th Fourier coefficient of $u$. One can consider this as a certain kind of an eddy-viscosity model in which

$$S(u, u) = |\nabla|^{2\alpha-2}\nabla u.$$

The unique strong solution is known to exist for $\alpha > 5/4$ [25, Lions]. Unlike the nonlinear viscosity case, this equation has an advantage in that the strong solution is a classical solution. It is also illuminating to express this equation as an evolution equation in the frequency space.

These models are derived as attempts to obtain well-posedness, a property that is lacking for the 3D NSE. Unfortunately, it is numerically observed that they tend to smear large-scale structures too much.

## 2.3 Spectral eddy-viscosity models

In order to preserve large-scale structures, we would like to limit the regularization effect to the small-scales. In fact, this is essentially the idea in the spectral viscosity method due to [5, Tadmor] in the context of hyperbolic conservation laws, where we add a viscosity to only the high-frequency part. Inspired by this, we may propose the following spectral viscosity model for NSE:

$$\begin{aligned}
&\partial_t u - \nu\Delta u + u \cdot \nabla u - \epsilon Q(\nabla \cdot (Q(\nabla u)) + \nabla \pi = f,\\
&\nabla \cdot u = 0.
\end{aligned}$$

where $Q$ is a *high-pass filter*; that is, it erases all the low-frequency modes of the input. Therefore, the subgrid scale viscosity is effective only for the high frequency part of the solution. An important point is that for a smooth solution that can be expressed in terms of low modes, the spectral-viscosity operator becomes zero. Thus, low-mode solutions in fact satisfy NSE. This indicates how the filtered viscosity tries to model the large-scale structure in the fluid consistently. In the above, the turbulent viscosity was constant, but this will not make the model well-posed. For this reason, we will combine this spectral idea with eddy-viscosity.

Basically, we modify the Smagorinsky model and the hyperviscosity model so that the regularization only affects the high-frequency modes:

$$\partial_t u - \nu\Delta u + u\cdot\nabla u - \epsilon Q\nabla\cdot(|\nabla Qu|^{p-2}\nabla Qu) + \nabla\pi = f, \tag{2.1}$$

$$\partial_t u - \nu\Delta u + u\cdot\nabla u - \epsilon(-\Delta)^{\alpha}Qu + \nabla\pi = f. \tag{2.2}$$

As it turns out, it is beneficial to stabilize 2.1 by a linear filtered viscosity as well:

$$\partial_t u - \nu\Delta u + u\cdot\nabla u - \epsilon Q(\nabla\cdot((1+|Q(\nabla u)|^{p-2})Q(\nabla u)) + \nabla\pi = f. \tag{2.3}$$

Such an idea is used for the numerical simulation of turbulence in [16, Jansen] in the setting of two-grid finite element method and where the cut-off operator is defined by an appropriate orthogonal projection onto the fine scales. Notice that the spectral viscosity model inevitably introduces several free parameters. These parameters are important in quantifying the trade-offs in modeling turbulence. The large nonlinearity or hyperviscosity and low cut-off gives the model more stability at the expense of increasing the modeling error, while small nonlinearity or hyperviscosity and high cut-off increases the consistency for less stability. The purpose of our work is to analyze 2.3 and 2.2 and mathematically investigate such a trade-off. It is also worth noting that the model 2.2 is natural in our periodic setting because the equation can be formulated in the frequency space. This makes the analysis of 2.2 somewhat simpler than 2.3, where it is difficult to interpret the spectral nonlinear viscosity either in the frequency space or the physical space.

## 2.4 Formal definitions

We now formalize the models 2.3 and 2.2. Let $I = [0, T]$ and $Q_T = I \times \mathbb{T}^3$ denote the time-space cylinder in a periodic domain. $\mathbb{T}^3$ is the unit box $[0,1]^3$ with identification of the planes $x_i = 0$ with $x_i = 1$. To formalize the various aspects of this, we define the projection operator: $P_N$ by

$$P_N(f) = \sum_{|k|_\infty \leq N} \hat{f}(k) e^{ik\cdot x}.$$

Define $P$ to be the Leray projector, an orthogonal projector onto the space of divergence free vector fields. Define $X_N = P_N(L^2(\mathbb{T}^3))$, $V_N = PP_N((L^2(\mathbb{T}^3))^3)$. We define the filter $Q_M$ as $I - P_M$.

Thus, the nonlinear viscosity model is the following:

$$\begin{aligned} &\partial_t u - \nu\Delta u + u\cdot\nabla u - \epsilon Q_M(\nabla\cdot(|Q_M(\nabla u)|^{p-2}Q_M(\nabla u)) + \nabla\pi = f, \\ &\nabla\cdot u = 0. \end{aligned} \tag{2.4}$$

Clearly, this model is determined by three parameters $M$, $p$ and $\epsilon$. Thus we call the above model $NV(\epsilon, p, M)$.

The hyperviscosity model is the following:

$$\begin{aligned} &\partial_t u - \nu\Delta u + u\cdot\nabla u - \epsilon(-\Delta)^\alpha Q_M u + \nabla\pi = f, \\ &\nabla\cdot u = 0. \end{aligned} \tag{2.5}$$

Clearly, this model is determined by three parameters $M$, $\alpha$ and $\epsilon$. The operator $(-\Delta)^\alpha$ is defined as a multiplier in the Fourier space: $(-\hat{\Delta})^\alpha = |k|^{2\alpha}$, where $k$ is the wave-number. Note that this is basically a generalization of differential operators to the setting with fractional index. One caveat is that such operators, unlike the Laplacian, are global (in the physical space) for fractional index, hence presenting some additional subtlety in the analysis. We call the above model $HV(\epsilon, \alpha, M)$.

For a notational convenience, we set $\tilde{u} = Q_M u$, $\bar{u} = P_M u$ and refer to the former as the *high-frequency part* of $u$ and the latter as the *low-frequency part* of $u$.

Now, even though we can show that a classical solution exists for 2.5, this is not necessarily so for 2.4. Therefore, 2.4 does not make sense as stated. Hence, we must introduce the notion of *weak solutions* to $NV$ and $HV$.

We call $u$ a weak solution to $NV(\epsilon,p,M)$ if $\nabla\cdot u=0$ almost everywhere and for all $\phi\in(C^\infty(Q_T))^n$, $\int\phi=0$ and $\nabla\cdot\phi=0$, and for almost all time,

$$\int \partial_t u\cdot\phi+\nu\nabla u:\nabla\phi+u\cdot\nabla u\cdot\phi+\epsilon|\nabla\tilde{u}|^{p-2}\nabla\tilde{u}:\nabla\tilde{\phi}\,dx=\int f\cdot\phi\,dx,$$

and $HV(\epsilon,\alpha,M)$ if $\nabla\cdot u=0$ almost everywhere and for all $\phi\in(C^\infty(Q_T))^n$, $\int\phi=0$ and $\nabla\cdot\phi=0$, and for almost all time,

$$\int \partial_t u\cdot\phi+\nu\nabla u:\nabla\phi+u\cdot\nabla u\cdot\phi+\epsilon|\nabla|^\alpha\tilde{u}\cdot|\nabla|^\alpha\tilde{\phi}\,dx=\int f\cdot\phi,\,dx,$$

where $|\nabla|^\alpha$ is defined as:

$$\widehat{|\nabla|}^\alpha u(k)=|k|^\alpha\hat{u}(k).$$

Our objective is to solve these equations by approximating them by a finite-dimensional system that solves a similar problem. In other words, call $u_N$ a *Galerkin $N$-dimensional projection* of $u$ and call it the solution to the system $NV_N(\epsilon,p,M)$, if $u\in L^2((0,T):V_N)$, $\partial_t u\in L^s((0,T):V_N)$ for some $s>1$ and for all $\phi\in L^2((0,T):V_N)$ and almost all time,

$$\int \partial_t u_N\cdot\phi+\nu\nabla u_N:\nabla\phi+u_N\cdot\nabla u_N\cdot\phi+\epsilon|\nabla\tilde{u_N}|^{p-2}\nabla\tilde{u_N}:\nabla\tilde{\phi}\,dx=\int f\cdot\phi\,dx,$$

and $HV_N(\epsilon,\alpha,M)$ if $u\in V_N$ and for all $\phi\in L^2((0,T):V_N)$ and almost all time,

$$\int \partial_t u_N\cdot\phi+\nu\nabla u_N:\nabla\phi+u_N\cdot\nabla u_N\cdot\phi+\epsilon|\nabla|^\alpha\tilde{u_N}\cdot|\nabla|^\alpha\tilde{\phi}\,dx=\int f\cdot\phi\,dx.$$

Now that all the formal definitions are in place, we are in good shape to tackle, in the next chapter, the important mathematical question of well-posedness of our model.

# CHAPTER 3

# WELL-POSEDNESS

In this chapter we will discuss well-posedness issues for the spectral viscosity models. In general, the *well-posedness* of a given partial differential equation means that it possesses the following properties:

1. A solution to the equation exists in an appropriate (weak) sense.
2. This solution is unique.
3. The solution is regular in an appropriate sense.

The point of introducing a weak solution is that the weaker the class in which we seek a solution, the easier it is to find it by the compactness method. In certain cases such as hyperbolic conservation laws, the introduction of a weak solution is inevitable, because we know that there exist non-classical solutions. For the Navier-Stokes equations, we know that a weak solution exists, but we are neither sure of its uniqueness nor its regularity. For a detailed discussion of the well-posedness for the Navier-Stokes equations, we refer the reader to [36, Temam] and [4, Constantin and Foias].

The well-posedness is also important from a physical standpoint. The Navier-Stokes equations are derived from a microscopic model by statistical averaging; therefore, we assume that such a procedure is valid. However, if the macroscopic model that results possesses a weak solution with a singularity, then it may undermine the very process of "averaging" that took us from the microscopic to the macroscopic. This is one of the reasons why the regularity results for the equations of mathematical physics hold particular interest for mathematicians.

Proving the well-posedness of our model is also an important undertaking from a numerical simulation viewpoint. This is because of the various stability estimates that the well-posedness proof generates as a by-product. These stability estimates offer a skeleton for proving that the numerical scheme that is generated from the turbulence model is well-conditioned.

Now the classic result in the direction of showing well-posedness for NSE is the result of J. Leray on the existence of a weak solution to the Navier-Stokes problem.

**Theorem 3.0.1** *Let $f \in L^2(Q_T)$, $u_0 \in H^1$ and $\nabla \cdot u_0 = 0$, then there exists a weak solution $u \in L^2(I; H^1) \cap L^\infty(I; L^2)$ and $\partial_t u \in L^{\frac{4}{3}}(I; H^{-1})$, such that $\nabla \cdot u = 0$ almost everywhere and for all $\phi \in C^\infty(Q_T)$ with $\nabla \cdot \phi = 0$ and $\int_{\mathbb{T}^n} \phi \, dx = 0$,*

$$\int_{Q_T} \partial_t u \phi + \nu \nabla u : \nabla \phi + u \cdot \nabla u \cdot \phi \, dxdt = \int_{Q_T} f \cdot \phi \, dxdt.$$

We currently do not know whether a weak solution is unique. The question of uniqueness is intimately tied to the regularity question as there are a host of results that show that if a solution possesses a certain amount of regularity then it is unique. Examples of results in this direction are that of [30, Prodi] and [31, Serrin].

## 3.1 Energy dissipation estimate

To discuss the well-posedness issue in more detail, we note that one of the difficulties associated with the Navier-Stokes equations is its lack of many globally controlled quantities. What the turbulence model does is that it basically adds another globally controlled quantity to the Navier-Stokes equations so that the global well-posedness is restored.

Suppose $NV$ and $HV$ possess a smooth solution, then if we test $NV$ and $HV$ by $u$, we get the most important global differential inequality called the *energy dissipation estimate*:

$$\frac{1}{2}\partial_t \|u\|^2 + \nu \|\nabla u\|^2 + \epsilon(\|\nabla \tilde{u}\|^2 + \|\nabla \tilde{u}\|_p^p) = (f, u),$$

and

$$\frac{1}{2}\partial_t \|u\|^2 + \nu \|\nabla u\|^2 + \epsilon \||\nabla|^\alpha \tilde{u}\|^2 = (f, u).$$

Let

$$C_{0,f} = \|u(0)\| + \int_0^t \|f\| \, ds.$$

This implies the following energy balance for $NV(\epsilon, p, M)$

$$\|u(t)\|^2 + \nu \int_0^t \|\nabla u\|^2 + \epsilon \int_0^t (\|\nabla \tilde{u}\|^2 + \|\nabla \tilde{u}\|_p^p) \leq (3/2) C_{0,f}^2, \tag{3.1}$$

and for $HV(\epsilon, \alpha, M)$

$$\|u(t)\|^2 + \nu \int_0^t \|\nabla u\|^2 + \epsilon \int_0^t \||\nabla|^\alpha \tilde{u}\|^2 \leq (3/2) C_{0,f}^2. \tag{3.2}$$

Notice how the additional control on norms of $\tilde{u}$ is added in addition to the usual energy balance equation for the NSE.

Another important bound for both models as well as for NSE is that for uniformly bounded forcing, the $L^2$ norm of the solution remains bounded. This can be seen from the inequality:

$$\partial_t \|u\| + \nu \|u\| \leq \|f\|,$$

which follows from the energy dissipation estimate and application of the Poincaré inequality to the viscosity part. Consequently,

$$\|u(t)\| \leq e^{-\nu t}\|u(0)\| + \int_0^t e^{-\nu(t-s)}\|f(s)\|\, ds \leq e^{-\nu t}\|u(0)\| + \frac{\|f\|_{L^\infty((0,t):L^2}}{\nu}.$$

### 3.1.1 Why is it difficult to show well-posedness for NSE?

Terence Tao [34] indicates why the well-posedness problem for NSE is difficult by giving useful dimensional heuristics for intuitively grasping these types of global energy estimates. Suppose we take the forcing $f = 0$ and that $u$ is supported in the frequency support of order $N$ and hence has a physical support of order $N^{-1}$ by the uncertainty principle. Thus, the energy dissipation estimate tells us that the following global quantities are controlled: $\int_{\mathbb{T}^d} |u|^2 \sim U^2 N^{-d}$ and $\int_0^T \int_{\mathbb{T}^d} |\nabla u|^2 \sim T N^2 U^2 N^{-d}$ where we have designated their dimension. Due to the skew-symmetry of the nonlinear term, its effect on the globally conserved quantities is nil; however, it can have a local effect of order $\int_0^T \int_\Omega u \cdot \nabla u \cdot u \sim U^3 N N^{-d}$. Now, let us assume that $\|u(0)\| \sim O(1)$, then $\|u(t)\|^2 \sim U^2 N^{-d} \sim O(1)$ so $U \sim N^{d/2}$. It follows that the dissipative effect is of order $\int_0^T \int_{\mathbb{T}^d} \|\nabla u\|^2 \sim T N^2$, while the nonlinear effect is of order $\int_0^T \int_\Omega u \cdot \nabla u \cdot u \sim T N^{1+d/2}$.

When $d = 2$, the linear term and the nonlinear term have the same dimension. Hence, we say that our energy dissipation bound is *critical* for the 2D NSE, while for $d = 3$, the nonlinearity dominates and hence we call our energy bound *supercritical*. The problem with 3-dimensional turbulence is that the global energy bound is *supercritical* and does not provide enough control on the size of $u$ to prevent the local instability due to the nonlinearity from happening.

Notice that energy dissipation also says that $TN^2 \sim O(1)$ or $T \sim N^{-2}$, which is the well-known parabolic scaling $T \sim L^2$. This roughly means that the solution can have the $N$th mode staying large only for time interval of length $N^{-2}$. This implies that the nonlinearity is of the order $TN^{1+d/2} \sim N^{d/2-1}$. This is a positive power of $N$ for $d \geq 3$. Hence, it leaves the possibility for energy to be cascaded to higher and higher frequencies only staying within any one frequencies in time of interval length $N^{-2}$, and meanwhile increasing the nonlinearity. But sum of $N^{-2}$ as $N \to \infty$ is finite; therefore, the solution can blow up in finite time due to ever increasing nonlinear destabilizing effect. However, no such blowup is known to happen and hence this problem remains open.

On the other hand, our turbulence models add more dissipative terms. For $NV$ model we have $\int_0^T \int_{\mathbb{T}^d} |\nabla \tilde{u}|^p \sim TU^p N^p N^{-d} \sim TN^{d(\frac{p}{2}-1)+p}$ which for 3D becomes critical at $p = \frac{11}{5}$. Upon actual analysis, this equation is shown to possess a strong solution at this value of $p$. For the $HV$ model, we have $\int_0^T \int_{\mathbb{T}^d} ||\nabla|^\alpha \tilde{u}|^2 \sim TU^2 N^{2\alpha} N^{-d} \sim TN^{2\alpha}$ which becomes critical at $\alpha = \frac{5}{4}$, an index beyond which the well-posedness is established.

Notice how the simple dimensional analysis is a powerful tool in predicting the criticality of the equation, and therefore gives us forsight into the well-posedness question. In any case, from the dimensional analysis, our turbulence model can be thought of as adding a control on the critical quantities for the high-frequency modes of the equation so that the possibility of a blow-up due to cascading is prohibited. However, the nonlinearity is free to act on the low-frequency part due to filtering, and hence we expect that the low-frequency behavior models the fluid well.

### 3.1.2 Simple interpolation result

Note that the energy balance equation roughly says that $\|u\|_{L^\infty((0,T):L^2)}$ and $\nu^{1/2}\|\nabla u\|_{L^2((0,T):L^2)}$ are of the same order. This implies that we should have a bound on the spaces that are *in between* these spaces. This is the content of the following interpolation lemma.

**Lemma 3.1.1** *If $u$ is a smooth solution to $NV$, or $HV$, and $2 \leq r \leq 6$ then*

$$\|u\|_{L^{\frac{4r}{3(r-2)}}((0,T):L^r)} \leq \|u\|_{L^\infty((0,T):L^2)}^{\frac{6-r}{2r}} \|\nabla u\|_{L^2((0,T):L^2)}^{\frac{3(r-2)}{2r}}$$
$$\lesssim \nu^{\frac{3(r-2)}{4r}} C_{0,f}$$

**Proof** Let $s = \frac{4r}{3r-6}$, then

$$\int_0^t \|u\|_r^s \, dt \leq \int \|u\|_2^{s\frac{6-r}{2r}} \|u\|_6^{s\frac{3(r-2)}{2r}} \, dt$$
$$\leq \|u\|_{L^\infty((0,T):L^2)}^{\frac{2(6-r)}{3(r-2)}} \|\nabla u\|_{L^2((0,T):L^2)}^2,$$

where we have used the Hölder's inequality and the Sobolev inequality. ∎

## 3.2 Existence of a strong solution for $NV$

Having heuristically discussed the well-posedness issues, we will see in this section that precisely beyond the index $p \geq \frac{11}{5}$, as we saw from the dimensional analysis, $NV$ possesses a strong solution. We will also show that the weak solution to $NV$ converges to a weak solution of NSE as $\epsilon \to 0$. For weak solutions, regularity and convergence to NSE of the $HV$ model, we refer the reader to [14, Guermond]. To this end, we will proceed in a standard fashion by deriving a series of a-priori estimates. Besides the energy dissipation estimate 3.1, we must show regularity in time and in space. First, we will show regularity in time:

**Lemma 3.2.1** *Let $\frac{1}{q} + \frac{1}{p} = 1$. If $u_N$ is a solution to $NV_N$ then $\partial_t u_N \in L^{\min\{4/3,q\}}(I; W^{-1,q})$.*

**Proof** If $\phi \in V_N$, then,

$$\begin{aligned}
(\partial_t u_N, \phi) &= (f + \nu \Delta u_N - u_N \cdot \nabla u_N + \epsilon Q_M \nabla \cdot (|\nabla Q_M u_N|^{p-2} \nabla Q_M u_N), \phi) \\
&= (f + \nu \nabla \cdot \nabla u_N - \nabla \cdot (u_N \otimes u_N) + \epsilon Q_M \nabla \cdot (|\nabla Q_M u_N|^{p-2} \nabla Q_M u_N), \phi) \\
&= (f, \phi) + (\nu \nabla u_N + u_N \otimes u_N, \nabla \phi) - (\epsilon |\nabla Q_M u_N|^{p-2} \nabla Q_M u_N, \nabla Q_M \phi) \\
&\lesssim (\|f\|_{-1} + \nu \|\nabla u_N\| + \|u_N\|_4^2) \|\nabla \phi\|_2 + \epsilon \|\nabla Q_M u_N\|_p^{p-1} \|\nabla Q_M \phi\|_p \\
&\lesssim (\|f\|_{-1} + \nu \|\nabla u_N\| + \|u_N\|_4^2) \|\nabla \phi\|_2 + \epsilon \|\nabla Q_M u_N\|_p^{p-1} \|\nabla Q_M \phi\|_p,
\end{aligned}$$

where we have used the $(p,p)$-estimate A.1.6 for the operator $Q_M = I - P_M$.

After time integration,

$$\begin{aligned}
\int (\partial_t u_N, \phi) \, dt \;&\leq\; (\|f\|_{-1} + \nu \|\nabla u_N\|) \|\nabla \phi\|_{L^\infty(I;L^2)} \\
&+\; \|u_N\|_{L^{8/3}(I;L^4)}^2 \|\nabla \phi\|_{L^4(I;L^2)} + \epsilon \|\nabla Q_M u_N\|_{L^p(I;L^p)}^{p-1} \|\nabla \phi\|_{L^p(I;L^p)}.
\end{aligned}$$

Therefore, the above inequality tells us that $\partial_t u_N \in L^{\min\{4/3,q\}}(I; W^{-1,q})$. ∎

Now, in order to prove the existence, we will show that the Galerkin projection satisfies the following spatial regularity result uniformly in $N$. The proof of this theorem is almost identical to the special case of that given in [28, Malek et al] except that in our case we have a cut-off filter that needs to be handled in a special way.

**Theorem 3.2.2** *Let $u_N$ be the solution to $NV_N(\epsilon, p, M)$ where*

$$p \geq \frac{11}{5}.$$

*Let $\lambda = \frac{2(3-p)}{3p-5}$. Then if $p < 3$,*

$$\|\nabla u_N(t)\|^2 \lesssim \|\nabla u_N(0)\|^2 + \int_0^t (\frac{\|f\|^2}{\nu} + \epsilon M^2 \|\nabla u_N\|_p^p) + (\int_0^t \epsilon^{-\frac{2\lambda}{3}} \|\nabla u_N\|_p^p)^{1/(1-\lambda)},$$

*while if $p \geq 3$,*

$$\|\nabla u_N(t)\|^2 \lesssim \|\nabla u_N(0)\|^2 + \int_0^t \frac{\|f\|^2}{\nu} + \|\nabla u_N\|_3^3 \, dt.$$

*We also have that $u_N \in L^p(I; W^{1,3p}) \cap L^2(I; H^2) \cap L^\infty(I; H^1)$.*

**Proof** We multiply the equation by $-\Delta u_N$:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla u_N\|^2 &+ \epsilon(\nabla \cdot (1 + |\nabla \tilde{u_N}|^{p-2})\nabla \tilde{u_N}, \Delta \tilde{u_N}) \\ &+ \nu\|\Delta u_N\|^2 + b(u_N, u_N, -\Delta u_N) \\ &\leq \frac{\nu}{4}\|\Delta u_N\|^2 + \frac{1}{\nu}\|f\|^2. \end{aligned}$$

Note that

$$\int u^k \partial_k u^j (-\partial_{ll} u^j) = \int \partial_l u^k \partial_k u^j \partial_l u^j + \frac{1}{2}\int u^k \partial_k (\partial_l u^j)^2 = \int \partial_l u^k \partial_k u^j \partial_l u^j.$$

Consequently,

$$b(u_N, u_N, -\Delta u_N) \leq \|\nabla u_N\|_3^3.$$

Now, let $I_p(\tilde{u_N}) = \sum_{i,j,k} \int |\nabla \tilde{u_N}|^{p-2} (\partial_{kj} \tilde{u_N}_i)^2 \, dx$. Then due to the monotonicity formula A.2.3.

$$(\nabla \cdot |\nabla \tilde{u_N}|^{p-2} \nabla \tilde{u_N}, \Delta \tilde{u_N}) \geq I_p(\tilde{u_N})$$

thus, we have our inequality of the form

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_N\|_2^2 + \epsilon(I_p(\tilde{u_N}) + \|\Delta \tilde{u_N}\|^2) + \nu\|\Delta u_N\|^2 \lesssim \|\nabla u_N\|_3^3$$

The $p \geq 3$ case basically follows from the above equation. We must work a little bit harder for $p < 3$.

Note that by interpolation,

$$\|\nabla u_N\|_3 \leq \|\nabla u_N\|_2^{\frac{2(p-1)}{3p-2}} \|\nabla u_N\|_{3p}^{\frac{p}{3p-2}},$$

and

$$\|\nabla u_N\|_3 \leq \|\nabla u_N\|_p^{\frac{p-1}{2}} \|\nabla u_N\|_{3p}^{\frac{3-p}{2}}.$$

Thus given $0 < \alpha < 1$,

$$\|\nabla u_N\|_3^3 \leq \|\nabla u_N\|_2^{Q_1} \|\nabla u_N\|_p^{Q_2} \|\nabla u_N\|_{3p}^{Q_3},$$

where $Q_1 = 3\alpha\frac{(p-1)2}{3p-2}$, $Q_2 = 3(1-\alpha)\frac{p-1}{2}$, $Q_3 = 3(1-\alpha)\frac{3-p}{2} + 3\alpha\frac{p}{3p-2}$. Recall A.2.4:

$$\|\nabla u_N\|_{3p} \lesssim I(u_N)^{1/p}.$$

Using these inequalities, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u_N\|^2 \quad &+ \quad \epsilon(I_p(\tilde{u_N}) + \|\Delta\tilde{u_N}\|^2) + \nu\|\Delta u_N\|^2 \\
&\leq \quad \frac{1}{\nu}\|f\|^2 + \|\nabla u_N\|_2^{Q_1}\|\nabla u_N\|_p^{Q_2}\|\nabla u_N\|_{3p}^{Q_3} \\
&\lesssim \quad \frac{1}{\nu}\|f\|^2 + \|\nabla u_N\|_2^{Q_1}\|\nabla u_N\|_p^{Q_2}(\|\nabla\tilde{u_N}\|_{3p} + M^{\frac{2}{p}}\|\nabla u_N\|_p)^{Q_3} \\
&\lesssim \quad \frac{1}{\nu}\|f\|^2 + \|\nabla u_N\|_2^{Q_1}\|\nabla u_N\|_p^{Q_2}(I_p(\tilde{u_N})^{\frac{1}{p}} + M^{\frac{2}{p}}\|\nabla u_N\|_p)^{Q_3} \\
&\lesssim \quad \frac{1}{\nu}\|f\|^2 + (\epsilon^{-\frac{Q_3}{p}}\|\nabla u_N\|_2^{Q_1}\|\nabla u_N\|_p^{Q_2})^{\frac{p}{p-Q_3}} + \frac{\epsilon}{2}(I_p(\tilde{u_N})^{\frac{1}{p}} + M^{\frac{2}{p}}\|\nabla u_N\|_p)^p,
\end{aligned}$$

where we have used the Bernstein inequality A.1.7 on the third line and the previous lemma on the fourth line.

It is now clear which $\alpha$ must be chosen. We would like to set

$$Q_2\frac{p}{p-Q_3} = p.$$

We get that

$$1-\alpha = \frac{p(3p-5)}{6(p-1)},$$

and

$$\alpha = \frac{(3-p)(3p-2)}{6(p-1)}.$$

Consequently, $Q_1 = 3-p$, $Q_2 = \frac{p(3p-5)}{4}$, $Q_3 = \frac{(3-p)3p}{4}$.

We have

$$\frac{p}{p-Q_3} = \frac{p}{Q_2} = \frac{4}{3p-5}.$$

Now set

$$\lambda = \frac{Q_1}{2}\frac{p}{p-Q_3} = \frac{2(3-p)}{3p-5}.$$

Then note that

$$\frac{Q_3}{p}\frac{p}{p-Q_3} = \frac{3(3-p)}{3p-5} = \frac{2\lambda}{3}.$$

Thus, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u_N\|^2 \quad &+ \quad \epsilon I_p(Q_M u_N) + \epsilon\|\Delta Q_M u_N\|^2 + \nu\|\Delta u_N\|^2 \\ &\lesssim \quad \frac{1}{\nu}\|f\|^2 + C\epsilon^{-\frac{2\lambda}{3}}(\|\nabla u_N\|_2^2)^\lambda \|\nabla u_N\|_p^p + \frac{\epsilon}{2}(I_p(\tilde{u_N}) + M^2\|\nabla u_N\|_p^p)\end{aligned}$$

Therefore integrating in time,

$$\begin{aligned}&\|\nabla u_N(t)\|^2 \le \|\nabla u_N(0)\|^2 \\ + \quad & C\int_0^t \left(\frac{1}{\nu}\|f\|^2 + \epsilon^{-\frac{2\lambda}{3}}(\|\nabla u_N\|_2^2)^\lambda\|\nabla u_N\|_p^p + \frac{1}{2}\epsilon M^2\|\nabla u_N\|_p^p\right)\, dt.\end{aligned}$$

Let $A = \int_0^t \frac{1}{\nu}\|f\|^2 + \frac{1}{2}\epsilon M^2\|\nabla u_N\|_p^p\, dt$, and $B = \epsilon^{-\frac{2\lambda}{3}}(\|\nabla u_N\|_2^2)^\lambda\|\nabla u_N\|_p^p$. If $g = \|\nabla u_N\|^2$, the above inequality is of the form

$$g(t) - g(0) \le A + \int_0^t Bg^\lambda\, dt.$$

This inequality is solved to give the following:

$$f(t) \le ((f(0)+A)^{1-\lambda} + (1-\lambda)\int_0^t B\, dt)^{1/(1-\lambda)} \lesssim f(0) + A + (\int_0^t B\, dt)^{1/(1-\lambda)}.$$

Applying this to above we get,

$$\|\nabla u_N(t)\|^2 \lesssim \|\nabla u_N(0)\|^2 + \int_0^t (\frac{\|f\|^2}{\nu} + \epsilon M^2\|\nabla u_N\|_p^p) + (\int_0^t \epsilon^{-\frac{2\lambda}{3}}\|\nabla u_N\|_p^p)^{1/(1-\lambda)}.$$

This implies that $\nabla u_N \in L^\infty(I; L^2)$ given that $\lambda \le 1$ which translates to $p \ge \frac{11}{5}$.

This would then imply the further regularity stated in the theorem. ∎

From the energy dissipation estimate and the regularity of the time-derivative, we have that $u_N \in L^\infty(I;L^2)\cap L^2(I;H^1)$, $\partial_t u_N \in L^{\min\{4/3,q\}}(I;W^{-1,q})$ . We now use Aubin-Lions compactness theorem to obtain an appropriate subsequence which we still call $u_N$ such that $u_N \rightharpoonup u$ weakly in $L^2(I;H^1)$, $\nabla u_N \to \nabla u$ strongly in $L^2(I;L^2)$. The regularity result shows that in fact $u_N \in L^p(I;W^{1,3p})\cap L^\infty(I;H^1)\cap L^2(I;H^2)$ uniformly in $N$, and therefore the same holds for $u$. In particular, Aubin-Lions theorem implies the strong convergence in $L^2(I;W^{1,q})$ for $q<\min\{6,3p\}$.

Now let $\psi_j \in X_j$. Then, clearly by the weak convergence and incompressibility condition,

$$(\partial_t u_N,\psi_j)\to\langle\partial_t u,\psi_j\rangle,$$

and

$$\nu(\nabla u_N,\nabla\psi_j)\to\nu\langle\nabla u,\nabla\psi_j\rangle.$$

Finally, since $\nabla u_N \rightharpoonup \nabla u$ in $L^2(I;L^2)$, and $u_N\to u$ strongly in $L^2(I;L^2)$ we have,

$$(u_N\cdot\nabla u_N,\psi_j)\to\langle u\cdot\nabla u,\psi_j\rangle.$$

The convergence of the p-Laplacian part is slightly more challenging. If $p<6$ then, note that due to A.2.2,

$$\int\langle|\nabla\tilde{u_N}|^{p-2}\nabla\tilde{u_N}-|\nabla\tilde{u}|^{p-2}\nabla\tilde{u},\nabla\tilde{\psi_j}\rangle\le(p-2)\|\nabla\tilde{\psi_j}\|_p\|\nabla(\tilde{u_N}-\tilde{u})\|_p(\|\nabla\tilde{u}\|_p+\|\nabla\tilde{u_N}\|_p)^{p-2}.$$

Due to the fact that $p<6$, $u_N\to u$ in $L^2(I;W^{1,p})$ strongly. The Jackson inequality A.1.9 tells us that $Q_M$ is an $(p,p)$ type operator. Therefore, $\tilde{u_N}\to\tilde{u}$ in $L^2(I;W^{1,p})$. Thus, we have the required convergence.

For $p\ge 6$ we must consider a different approach. First, since $\nabla\tilde{u_N}\to\nabla\tilde{u}$ in $L^2(Q_T)$, due to A.3.1 there exists a subsequence $N_i$ such that $\nabla\tilde{u_{N_i}}\to\nabla\tilde{u}$ almost everywhere. Consequently, due to A.2.2, $|\nabla\tilde{u_{N_i}}|^{p-2}\nabla\tilde{u_{N_i}}\to|\nabla\tilde{u}|^{p-2}\nabla\tilde{u}$ almost everywhere.

Note then that for any set $M\subset Q_T$,

$$\int_M|\nabla\tilde{u_{N_i}}|^{p-2}\nabla\tilde{u_{N_i}}:\nabla\psi_j\le|M|^{1/p}\|\nabla\tilde{u_{N_i}}\|_p^{p-1}\|\nabla\psi_j\|_\infty.$$

Thus, the p-Laplacian term is uniformly integrable. Thus, the required convergence follows from A.3.2.

**Theorem 3.2.3** *If*

$$p \geq \frac{11}{5}.$$

*Then there exists a weak solution $u$ to $NV(\epsilon, p, M)$ given that $u(0) \in H^1$ and $f \in L^2$. In fact $u$ possesses a further regularity: $u \in L^p(I; W^{1,3p}) \cap L^2(I; H^2) \cap L^\infty(I; H^1)$.*

## 3.3 Further regularity

We will show in this section that $\partial_t u \in L^2(I; L^2)$ and $u \in L^\infty(I; L^p)$, given that $u(0) \in W^{1,p}$. Thus with this additional result, $u$ satisfies a host of regularity results and we may call such a solution the *strong solution* to $NV$.

**Theorem 3.3.1** *With the same hypothesis as in the previous section and suppose in addition that $u(0) \in W^{1,p}$. Then, the solution possesses a further regularity that $\partial_t u \in L^2(I; L^2)$ and $u \in L^\infty(I; L^p)$.*

**Proof** We multiply the equation by $\partial_t u$ to obtain,

$$\begin{aligned} \int \|\partial_t u\|^2 \quad &+ \quad \epsilon((1 + |\nabla \tilde{u}|^{p-2}) \nabla \tilde{u}, \partial_t \nabla \tilde{u}) \\ &+ \quad \frac{\nu}{2} \partial_t \|\nabla u\|^2 + b(u, u, \partial_t u)\, dt \\ &\leq \quad \int \frac{1}{4} \|\partial_t u\|^2 + 4\|f\|^2\, dt. \end{aligned}$$

Using A.2.2 we have

$$\begin{aligned} \frac{1}{2} \int \|\partial_t u\|^2 \quad &+ \quad \epsilon(\|\nabla \tilde{u}(t)\|^2 + \|\nabla \tilde{u}(t)\|_p^p) + \nu \|\nabla u(t)\|^2 \\ &\lesssim \quad \epsilon(\|\nabla \tilde{u}(0)\|^2 + \|\nabla \tilde{u}(0)\|_p^p) + \nu \|\nabla u(0)\|^2 \\ &\quad + \int \left( \int |u \cdot \nabla u|^2\, dx + 4\|f\|^2 \right) dt. \end{aligned}$$

We have that

$$\int |u \cdot \nabla u|^2\, dx \leq \|u\|^2_{\frac{6p}{3p-2}} \|\nabla u\|^2_{3p} \lesssim \|\nabla u\|_2^2 \|\nabla u\|_{3p}^2.$$

The last inequality is due to the Sobolev inequality and the fact that $\frac{6p}{3p-2} \leq 6$.

Thus, we have

$$\begin{aligned} \frac{1}{2} \int \|\partial_t u\|^2 \quad &+ \quad \epsilon(\|\nabla \tilde{u}(t)\|^2 + \|\nabla \tilde{u}(t)\|_p^p) + \nu \|\nabla u(t)\|^2 \\ &\leq \quad \epsilon(\|\nabla \tilde{u}(0)\|^2 + \|\nabla \tilde{u}(0)\|_p^p) + \nu \|\nabla u(0)\|^2 \\ &\quad + \|\nabla u\|^2_{L^\infty(I;L^2)} \|\nabla u\|^2_{L^2(I;L^{3p})} + 4 \int \|f\|^2\, dt. \end{aligned}$$

The right hand side is bounded due to the regularity result of the previous section. Therefore, the theorem follows. ∎

## 3.4 Uniqueness and stability

In this section we show that the solution to $NV(\epsilon, p, M)$ is unique. We will do this by deriving a stability estimate. Such an estimate also holds for the 3D NSE given that a solution is sufficiently regular. The key point is that due to the regularity result proved in the previous section, the uniqueness for the $NV$ follows.

**Theorem 3.4.1** *Let $u_1$ and $u_2$ be two distinct solutions to $NV(\epsilon, p, M)$, such that*

$$p \geq \frac{11}{5}.$$

*Then,*

$$\|(u_1 - u_2)(t)\|^2 \leq \|u_1(0) - u_2(0)\|^2 e^{\int_0^t \min\{\nu^{-3}\|\nabla u_1\|^4, \epsilon M^2 + \epsilon^{-3}\|\nabla u_1\|^4\}\, ds}$$

*Since $u_1 \in L^\infty(I; H^1)$ for the range of index satisfied by $p$, the solution to $NV(\epsilon, p, M)$ is unique.*

**Proof** We would like to derive an estimate for $w = u_1 - u_2$.

Since $u_1$ and $u_2$ satisfy

$$\partial_t u_i - \nu \Delta u_i - \epsilon P Q_M \nabla \cdot ((1 + |\nabla \tilde{u_i}|^{p-2}) \nabla \tilde{u_i}) + P(u_i \cdot \nabla u_i) = f_i,$$

we can see that $w$ satisfies

$$\partial_t w - \nu \Delta w - \epsilon P Q_M \nabla \cdot ((1 + |\nabla \tilde{u_1}|^{p-2}) \nabla \tilde{u_1} - ((1 + |\nabla \tilde{u_2}|^{p-2}) \nabla \tilde{u_2})$$
$$+ P(u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2) = 0.$$

We test this equation against $w$ and use the monotonicity A.2.2 to get,

$$\frac{1}{2}\partial_t \|w\|^2 + \nu \|\nabla w\|^2 + \epsilon(\|\nabla \tilde{w}\|^2 + \gamma \|\nabla \tilde{w}\|_p^p)$$
$$\leq -(w \cdot \nabla u_1, w).$$

First note that due to Gagliardo-Nirenberg, and $\int w = 0$ we have

$$\|w\|_4 \leq \|w\|^{1/4} \|\nabla w\|^{3/4}.$$

Thus,

$$
\begin{aligned}
(w\cdot\nabla u_1, w) &\le \|\nabla u_1\|\|\nabla\tilde{w}\|^{3/2}\|\tilde{w}\|^{1/2} + \|\nabla u_1\|\|\bar{w}\|_4^2\\
&\le \frac{\epsilon}{12}\|\nabla\tilde{w}\|^2 + C\epsilon^{-3}\|\nabla u_1\|^4\|\tilde{w}\|^2 + M^{(1/2-1/4)3*2}\|\nabla u_1\|\|\bar{w}\|^2\\
&\le \frac{\epsilon}{12}\|\nabla\tilde{w}\|^2 + (M^{3/2}\|\nabla u_1\| + \epsilon^{-3}\|\nabla u_1\|^4)\|w\|^2,
\end{aligned}
$$

where we have used the Bernstein inequalityA.1.7. We also have that

$$
\begin{aligned}
&(w\cdot\nabla u_1, w) \le \|\nabla u_1\|\|\nabla w\|^{3/2}\|w\|^{1/2} \le \frac{\nu}{12}\|\nabla w\|^2 + C\nu^{-3}\|\nabla u_1\|^4\|w\|^2\\
&\le \frac{\nu}{12}\|\nabla w\|^2 + \nu^{-3}\|\nabla u_1\|^4\|w\|^2.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\partial_t\|w\|^2 + \nu\|\nabla w\|^2 + \epsilon(\|\nabla\tilde{w}\|^2 + \gamma\|\nabla\tilde{w}\|_p^p)\\
&\le C\|\nabla\tilde{u}_1\|_p^p + C_2(\min\{\nu^{-3}\|\nabla u_1\|^4, \epsilon M^2 + \epsilon^{-3}\|\nabla u_1\|^4\})\|w\|^2.
\end{aligned}
$$

Therefore, the Gronwall inequality implies

$$
\partial_t(e^{-\int_0^t(\min\{\nu^{-3}\|\nabla u_1\|^4,\epsilon M^2+\epsilon^{-3}\|\nabla u_1\|^4\})\,ds}\|w\|^2) + \nu\|\nabla w\|^2 + \epsilon(\|\nabla\tilde{w}\|^2 + \gamma\|\nabla\tilde{w}\|_p^p) \le 0.
$$

Using forward inequality, this implies the following estimate:

$$
\begin{aligned}
&\|(u_1-u_2)(t)\|^2\\
&\le \|u_1(0)-u_2(0)\|^2 e^{\int_0^t \min\{\nu^{-3}\|\nabla u_1\|^4,\epsilon M^2+\epsilon^{-3}\|\nabla u_1\|^4\}\,ds}.
\end{aligned}
$$

∎

## 3.5 Convergence to a weak NSE solution

We have shown that the $NV$ model is well-posed. This indicates at least in part that solving the $NV$ equation numerically is tractable. We must however also show that somehow $NV$ models the turbulence well. For this, we show that the solution to $NV$ converges to the solution of NSE. The first issue we must clarify, however, is that there are two senses in which the sequence of solutions to $NV$ can be possibly taken to converge to a NSE solution. One is to take $M\to\infty$ and the other $\epsilon\to 0$. In fact, we are also interested in cases in which both of these occur at the same time. In light of the fact that we do not know much about the regularity of a weak solution to NSE, it is unlikely that as $M \to \infty$ independent of $\epsilon$, the $p$-Laplacian term goes to zero. Therefore, the most sensible thing to do is to take the sequence as $\epsilon \to 0$, and perhaps let $M \to \infty$ as a function of $\epsilon$. Thus, we will show that the sequence $u_\epsilon$ parametrized by $\epsilon$ contains a subsequence that converges to a weak solution of NSE, and let $M$ depend on $\epsilon$ by expressing it as $M(\epsilon)$.

Note that $u_\epsilon \in L^2(I; H^1) \cap L^\infty(I; L^2)$ and $\partial_t u_\epsilon \in L^{\min\{4/3,q\}}(I; W^{-1,q})$ uniformly in $\epsilon$ and therefore we can use the Aubin-Lions to obtain the weak limit $u$. We want to show that such a $u$ is a weak solution of NSE.

Note that,

$$(\epsilon|\nabla Q_{M(\epsilon)}u|^{p-2}\nabla Q_{M(\epsilon)}u, \nabla\psi) \leq \epsilon^{1/p}(\epsilon^{1/p}\|\nabla Q_{M(\epsilon)}u\|_p)^{p-1}\|\nabla\psi\|_p.$$

We know that $\int \epsilon^{1/p}\|\nabla Q_{M(\epsilon)}u\|_p$ is uniformly bounded in $\epsilon$ due to 3.1. Thus, we may take $\epsilon \to 0$ and see that the nonlinear term goes to zero. Therefore the limit of the solution to $NV(\epsilon, p, M(\epsilon))$ as $\epsilon \to 0$ satisfies the weak formulation for the NSE.

**Lemma 3.5.1** *Let $u_\epsilon$ be the solution to $NV(\epsilon, p, M(\epsilon))$. Then there exists $u$ such that a subsequence of $u_\epsilon$ converges weakly $u_{\epsilon_j} \rightharpoonup u$ in $L^2(I; H^1)$ and $u_{\epsilon_j} \to u$ strongly in $L^2(L^2)$ as $\epsilon_j \to 0$, and $u$ is a weak solution of the NSE.*

In this chapter we proved some preliminary results concerning the well-posedness. In the next chapter, we will investigate the $NV$ model further by discretizing the equation in time. We will derive a certain stability result that allows us to address the question of choosing the appropriate parameter for the model.

# CHAPTER 4

# STABILITY AND CONVERGENCE FOR SEMI-IMPLICIT SCHEME : P=3 CASE

In this section, we consider a semi-implicit scheme to solve the nonlinear spectral viscosity method for $p = 3$. We call this method semi-implicit in the sense that we only lag the nonlinear inertial term in time, but keep the p-Laplacian implicit.

First, assume that $f \in L^2((0,T) : L^2) \cap L^1((0,T) : L^2)$. Suppose $n$ is the number of mesh points. Let $\delta t > 0$ be given, and $t_i = \delta t i$. Given $u_0 \in W^{1,3}$, seek $u_{\delta t}(i) : \{i = 1, \ldots n\} \times \mathbb{T}^3 \to \mathbb{R}$, $i > 0$ and $\nabla \cdot u_{\delta t}(i) = 0$, $\hat{u}(0) = 0$ such that,

$$\begin{aligned}
&\int u_{\delta t}(i) \cdot \phi \, dx - \int u_{\delta t}(i-1) \cdot \phi \, dx \\
&+\delta t \left( \int \epsilon((1 + |\nabla \tilde{u_{\delta t}}(i)|)\nabla \tilde{u_{\delta t}}(i) : \nabla \tilde{\phi}) + \nu \nabla u_{\delta t}(i) : \nabla \phi + u_{\delta t}(i-1) \cdot \nabla u_{\delta t}(i) \cdot \phi \, dx \right) \\
&-\delta t \int f(i\delta t) \cdot \phi \, dx = 0.
\end{aligned} \tag{4.1}$$

We call the system of such equations for $i = 1, \ldots n$, $NV_{\delta t}(\epsilon, 3, M)$. Notice that we are lagging the nonlinear convection term, so that we are essentially solving a nonlinear Oseen type equation.

The goal of this section is to prove the stability and convergence of the semi-implicit scheme in this setting. $p = 3$ turns out to be especially nice since for the equation that is satisfied by $\nabla u_{\delta t}(i)$, the $L^3$ norm of the gradient has exactly the same dimension as the nonlinear form. We will use this property to derive estimates that allows us to show convergence as $\delta t \to 0$ to the solution of $NV$ and show the rate at which the convergence takes place.

We will note that the semi-implicit Euler scheme for the case $M = 0$ has been analyzed by [7, Diening]. However, their emphasis was on a more physically significant(and challenging) case of smaller $p$'s.

We first proceed to discretize the system and show existence to the finite dimensional approximation.

## 4.1 Existence for the finite-dimensional system

We will first show that the semi-implicit equation is solvable. To this end we will use the Galerkin projection of $u_{\delta t}(i)$ much as was done in the previous chapters. Let $f_{i,N} = P_N f(\delta i)$. In fact we want to solve for $u_{\delta t,N}(i) \in V_N$ such that for each $\phi \in V_N$.

$$\begin{aligned}
&\int u_{\delta t,N}(i) \cdot \phi \, dx - \int u_{\delta t,N}(i-1) \cdot \phi \, dx + \\
&\delta t \left( \int \epsilon((1 + |\nabla \tilde{u_{\delta t,N}}(i)|) \nabla \tilde{u_{\delta t,N}}(i) : \nabla \tilde{\phi}) + \nu \nabla u_{\delta t,N}(i) : \nabla \phi + u_{\delta t,N}(i-1) \cdot \nabla u_{\delta t,N}(i) \cdot \phi \, dx \right) \\
&-\delta t \int f_{i,N} \cdot \phi \, dx = 0. \qquad (4.2)
\end{aligned}$$

Let us call the above equation $NV_{\delta t,N}(\epsilon, p, M)$.

Alternatively, we can set $\mathbf{f}_i(\hat{u_{\delta t,N}}(i))_k$ with $\mathbf{f}_i : \mathbb{R}^N \to \mathbb{R}^N$ to be the left-hand-side of the above equation for each of the basis $\phi_k \in V_N$. So that the above equation can be stated as an algebraic equation:

Find $u_{\delta t,N}(i)$ so that,

$$\mathbf{f}_i(\hat{u_{\delta t,N}}(i)) = 0.$$

Now, if we test this equation against $\hat{u_{\delta t,N}}(i)$, we get

$$\begin{aligned}
&\mathbf{f}_i(\hat{u_{\delta t,N}}(i)) \cdot \hat{u_{\delta t,N}}(i) = \|u_{\delta t,N}(i)\|^2 \\
&+\delta t(\epsilon(\|\nabla \tilde{u_{\delta t,N}}(i)\|^2 + \|\nabla \tilde{u_{\delta t,N}}(i)\|_3^3) + \nu \|\nabla \tilde{u_{\delta t,N}}(i)\|^2) - (u_{\delta t,N}(i-1), u_{\delta t,N}(i)) \\
&\geq \frac{1}{2}\|u_{\delta t,N}(i)\|^2 + \delta t(\epsilon(\|\nabla \tilde{u_{\delta t,N}}(i)\|^2 + \|\nabla \tilde{u_{\delta t,N}}(i)\|_3^3) + \nu \|\nabla \tilde{u_{\delta t,N}}(i)\|^2) \\
&-\frac{1}{2}\|u_{\delta t,N}(i-1)\|^2 + \delta t \|f_N\| \|u_{\delta t,N}(i)\|.
\end{aligned}$$

Thus,

$$\mathbf{f}_i(\hat{u_{\delta t,N}}(i)) \cdot \hat{u_{\delta t,N}}(i) \geq \frac{1}{2}(\|u_{\delta t,N}(i)\|^2 - \|u_{\delta t,N}(i-1)\|^2) - \delta t \|f_N\| \|u_{\delta t,N}(i)\|.$$

We note then the following topological fixed-point theorem taken from [9, Evans] which is a consequence of the Brouwer's fixed point theorem. It says roughly that any continuous vector field that points outward on each point of a sufficiently large sphere must equal zero somewhere inside that sphere. Another name for this type of theorem is a *Hairy-ball* theorem.

**Lemma 4.1.1** *Assume the continuous function* $\boldsymbol{f} : \mathbb{R}^N \to \mathbb{R}^N$ *satisfies*

$$\boldsymbol{f}(v) \cdot v \geq 0,$$

*if* $|v| = r$ *for some* $r > 0$. *Then there exists a point* $v \in B(0,r)$ *such that*

$$\boldsymbol{f}(v) = 0.$$

Using such an assertion, together with an assumption that $\delta t$ is sufficiently small, we can prove the existence of $u_{\delta t,N}(i)$ for each $i = 1 \ldots n$ such that

$$\begin{aligned}&\frac{1}{2}\|u_{\delta t,N}(i)\|^2 + \delta t(\epsilon(\|\nabla \tilde{u_{\delta t,N}}(i)\|^2 + \|\nabla \tilde{u_{\delta t,N}}(i)\|_3^3 + \nu\|\nabla u_{\delta t,N}(i)\|^2) \\ &\leq \frac{1}{2}\|u_{\delta t,N}(i-1)\|^2 + \delta t\|f_N\|\|u_{\delta t,N}(i)\|. \end{aligned} \tag{4.3}$$

## 4.2 A-priori energy estimate

Now, having shown the existence for the finite-dimensional system we now want to see what happens as $N \to \infty$. For this, various a-priori estimates must be shown. The energy dissipation estimate in this setting appears as a discrete iteration formula; and therefore, we must introduce the following discrete Gronwall type inequality to solve it:

**Lemma 4.2.1** *Suppose we have a sequence* $c_i > 0, a_i > 0, b_i > 0$ *that satisfies*

$$c_i + a_i \leq \mu_i c_{i-1} + b_i,$$

*for each* $i = 1 \ldots n$. *Then for each* $j = 1 \ldots n$,

$$c_j + \sum_{i=1}^{j}(\prod_{l=i+1}^{j}\mu_l)a_i \leq (\prod_{l=1}^{j}\mu_l)c_0 + \sum_{i=1}^{j}(\prod_{l=i+1}^{j}\mu_l)b_i.$$

**Proof** We will proceed by induction. The base $i = 1$ case follows from the hypothesis. Suppose that the assertion is true for $j = 1 \ldots n-1$, we would like to prove it for $j = n$. We have

$$
\begin{aligned}
&c_n + \sum_{i=1}^{n}(\prod_{l=i+1}^{n} \mu_l)a_i = c_n + a_n + \mu_n \sum_{i=1}^{n-1}(\prod_{l=i+1}^{n-1} \mu_l)a_i \\
&\leq \mu_n(c_{n-1} + \sum_{i=1}^{n-1}(\prod_{l=i+1}^{n-1} \mu_l)a_i) + b_n \leq \mu_n(\mu_{n-1}c_0 + \sum_{i=1}^{n-1}(\prod_{l=i+1}^{n-1} \mu_l)b_i) + b_n \\
&= (\prod_{l=1}^{n} \mu_l)c_0 + \sum_{i=1}^{n}(\prod_{l=i+1}^{n-1} \mu_l)b_i.
\end{aligned}
$$

∎

We get as a result the existence of a solution to 4.2 that satisfies the discrete energy dissipation estimate.

**Lemma 4.2.2** *Let*

$$C_{0,f} = \|u_{0,N}\| + (\sum_{i=1}^{j} \|f_N(i)\|\delta t).$$

*Then, there exists a solution $u_{\delta t,N}(i)$ to 4.2 which satisfies,*

$$
\begin{aligned}
&\max_{k\leq j} \|u_{\delta t,N}(k)\|^2 + \sum_{i=1}^{j} 2\delta t(\epsilon(\|\nabla \tilde{u}_{\delta t,N}(i)\|^2 + \|\nabla \tilde{u}_{\delta t,N}(i)\|_3^3 + \nu\|\nabla u_{\delta t,N}(i)\|^2) \\
&\lesssim C_{0,f}^2.
\end{aligned}
$$

**Proof** First sum up the 4.3 for all $i = 1 \dots j$ to obtain

$$
\begin{aligned}
&\|u_{\delta t,N}(j)\|^2 + \sum_{i=1}^{j} 2\delta t(\epsilon(\|\nabla \tilde{u}_{\delta t,N}(i)\|^2 + \|\nabla \tilde{u}_{\delta t,N}(i)\|_3^3 + \nu\|\nabla u_{\delta t,N}(i)\|^2) \\
&\leq \|u_N(0)\|^2 + 2\sum_{i=1}^{j} \delta t\|f_N(i)\|\|u_{\delta t,N}(i)\|.
\end{aligned}
$$

Taking the maximum over $j$ on the left hand side, and canceling, we get

$$
\begin{aligned}
&\max_{j} \frac{1}{2}\|u_{\delta t,N}(j)\|^2 + \sum_{i=1}^{j} 2\delta t(\epsilon(\|\nabla \tilde{u}_{\delta t,N}(i)\|^2 + \|\nabla \tilde{u}_{\delta t,N}(i)\|_3^3 + \nu\|\nabla u_{\delta t,N}(i)\|^2) \\
&\leq \|u_N(0)\|^2 + 4(\sum_{i=1}^{j} \|f_N(i)\|\delta t)^2.
\end{aligned}
$$

∎

## 4.3 Regularity

In order to show existence, we would like to use a compactness method. To do this we must show that the solution has an additional regularity.

### 4.3.1 Space-regularity

**Theorem 4.3.1** *Let*

$$C_{sp} = \epsilon^{-1} C_{0,f}^2 + C_{0,f}^3 M^{\frac{5}{2}} \nu^{-1} + 4C_{0,f}^2,$$

*then a solution to 4.2 satisfies,*

$$\max_{j \le n} \frac{1}{4} \|\nabla u_{\delta t,N}(i)\|^2 + \sum \delta t(\epsilon(\|\Delta u_{\delta t,\tilde{N}}(i)\|^2 + \|u_{\delta t,\tilde{N}}(i)\|_{3p}^p) + \nu \|\Delta u_{\delta t,N}(i)\|^2)$$
$$\lesssim \frac{1}{2}\|\nabla u_N(0)\|^2 + C_{sp} + O(\delta t).$$

**Proof** We multiply 4.2 by $-\Delta u_{\delta t,N}(i)$.

$$\begin{aligned}\frac{1}{2}\|\nabla u_{\delta t,N}(i)\|^2 \quad &+ \quad \delta t(\epsilon(\nabla \cdot (1 + |\nabla u_{\delta t,\tilde{N}}(i)|)\nabla u_{\delta t,\tilde{N}}(i), \Delta u_{\delta t,\tilde{N}}(i)) \\ &+ \quad \delta \nu \|\Delta u_{\delta t,N}(i)\|^2 + \delta t(u_{\delta t,N}(i-1) \cdot \nabla u_{\delta t,N}(i), -\Delta u_{\delta t,N}(i))) \\ &\le \quad \frac{1}{2}\|\nabla u_{\delta t,N}(i-1)\|^2 + \delta t \|f_{i,N}\| \|\nabla u_{\delta t,N}(i)\|.\end{aligned}$$

Note then

$$\int u_{\delta t,N}(i-1) \cdot \nabla u_{\delta t,N}(i) \cdot (-\Delta u_{\delta t,N}(i)) = \sum_{l,k,j} \int \partial_l u_{\delta t,N}(i-1)^k \partial_k u_{\delta t,N}(i)^j \partial_l u_{\delta t,N}(i)^j.$$
$$\lesssim \|\nabla u_{\delta t,N}(i)\|_3^3 + \|\nabla u_{\delta t,N}(i-1)\|_3^3$$

We obtain our inequality of the form

$$\frac{1}{2}\|\nabla u_{\delta t,N}(i)\|_2^2 + \delta t(\epsilon(\|\Delta u_{\delta t,\tilde{N}}(i)\|^2 + I_3(u_{\delta t,\tilde{N}}(i))) + \nu \|\Delta u_{\delta t,N}(i)\|^2)$$
$$\lesssim \frac{1}{2}\|\nabla u_{\delta t,N}(i-1)\|^2 + \delta t(\|\nabla u_{\delta t,N}(i)\|_3^3 + \|\nabla u_{\delta t,N}(i-1)\|_3^3) + \delta t \|f_{i,N}\| \|\nabla u_{\delta t,N}(i)\|.$$

Now we have

$$\|\nabla \bar{u}\|_3^3 \lesssim M^{\frac{3}{2}} \|\nabla u\|_2^3 \lesssim M^{\frac{5}{2}} \|\nabla u\|^2 \|u\|$$

Therefore,

$$\frac{1}{2}\|\nabla u_{\delta t,N}(i)\|_2^2 + \delta t(\epsilon(\|\Delta u_{\delta t,\tilde{N}}(i)\|^2 + I_3(u_{\delta t,\tilde{N}}(i))) + \nu \|\Delta u_{\delta t,N}(i)\|^2)$$
$$\le \frac{1}{2}\|\nabla u_{\delta t,N}(i-1)\|^2 + \delta t(\|\nabla u_{\delta \tilde{t},N}(i)\|_3^3 + \|\nabla u_{\delta \tilde{t},N}(i-1)\|_3^3)$$
$$+\delta t M^{\frac{5}{2}}(\|\nabla u_{\delta t,N}(i)\|^2 + \|\nabla u_{\delta t,N}\|^2(i-1)) \max_j \|u_{\delta t,N}(j)\| + \delta t \|f_{i,N}\| \|\nabla u_{\delta t,N}(i)\|.$$

We now sum over $i$ to obtain,

$$\frac{1}{2}\|\nabla u_{\delta t,N}(i)\|^2 + \sum \delta t(\epsilon(\|\Delta u_{\delta t,\tilde{N}}(i)\|^2 + I_3(u_{\delta t,\tilde{N}}(i))) + \nu\|\Delta u_{\delta t,N}(i)\|^2)$$
$$\lesssim \frac{1}{2}\|\nabla u_N(0)\|^2 + \sum \|\nabla u_{\tilde{\delta t},N}\|_3^3 \delta t$$
$$+ \max_j \|u_{\delta t,N}(j)\| \sum M^{\frac{5}{2}}\|\nabla u_{\delta t,N}\|^2 + (\sum \|f_{i,N}\|\delta t) \max_j \|\nabla u_{\delta t,N}(j)\| + O(\delta t)$$

Consequently, using 4.2.2,

$$\max_{j\leq n} \frac{1}{4}\|\nabla u_{\delta t,N}(i)\|^2 + \sum \delta t(\epsilon(\|\Delta u_{\delta t,\tilde{N}}(i)\|^2 + \|u_{\delta t,\tilde{N}}(i)\|_{3p}^p) + \nu\|\Delta u_{\delta t,N}(i)\|^2)$$
$$\lesssim \frac{1}{2}\|\nabla u_N(0)\|^2 + \epsilon^{-1}C_{0,f}^2 + C_{0,f}^3 M^{\frac{5}{2}}\nu^{-1} + 4C_{0,f}^2 + O(\delta t)$$

∎

### 4.3.2 Regularity in time

To use the compactness method, we also need that $u_{\delta t,N}$ has some regularity in time. The following lemma provides such a result.

**Lemma 4.3.2** *Let*

$$C_{tm} = \epsilon^{-1}C_{sp} + C_{0,f}^6\nu^{-1} + M^3 C_{0,f}^2\nu^{-1} + 2\sum \|f_{i,N}\|^2\delta t.$$

*then,*

$$\sum_i^n (\delta t)^{-1}\|u_{\delta t,N}(i) - u_{\delta t,N}(i-1)\|^2$$
$$+(\epsilon(\frac{1}{2}\|\nabla\tilde{u}_{\delta t,N}(j)\|^2 + \frac{1}{3}\|\nabla\tilde{u}_{\delta t,N}(j)\|_3^3) + \frac{\nu}{4}\|\nabla u_{\delta t,N}(j)\|^2)$$
$$\lesssim (\epsilon(\frac{1}{2}\|\nabla\tilde{u}_{\delta t,N}(0)\|^2 + \frac{1}{3}\|\nabla\tilde{u}_{\delta t,N}(0)\|_3^3) + \frac{\nu}{4}\|\nabla u_{\delta t,N}(0)\|^2) + C_{tm}. + O(\delta t)$$

**Proof** We multiply 4.2 by $\frac{u_{\delta t,N}(i)-u_{\delta t,N}(i-1)}{\delta t}$.

$$(\delta t)^{-1}\|u_{\delta t,N}(i) - u_{\delta t,N}(i-1)\|^2 + \epsilon((1 + |\nabla u_{\delta t,\tilde{N}}(i)|)\nabla u_{\delta t,\tilde{N}}(i), \nabla u_{\delta t,\tilde{N}}(i) - u_{\delta t,N}(\tilde{i} - 1))$$
$$+\nu(\nabla u_{\delta t}(i), \nabla(u_{\delta t,N}(i) - u_{\delta t,N}(i-1)))$$
$$+\delta t(u_{\delta t,N}(i-1)\cdot\nabla u_{\delta t,N}(i), (\delta t)^{-1}(u_{\delta t,N}(i) - u_{\delta t,N}(i-1)))$$
$$-\delta t(f, (\delta t)^{-1}(u_{\delta t,N}(i) - u_{\delta t,N}(i-1))) = 0.$$

Consequently,noting that

$$\int |\nabla u|\nabla u : \nabla v \leq \frac{2}{3}\|\nabla u\|_3^3 + \frac{1}{3}\|\nabla v\|_3^3,$$

$$\frac{1}{2}(\delta t)^{-1}\|u_{\delta t,N}(i) - u_{\delta t,N}(i-1)\|^2 + (\epsilon(\frac{1}{2}\|\nabla \tilde{u}_{\delta t,N}(i)\|^2 + \frac{1}{3}\|\nabla \tilde{u}_{\delta t,N}(i)\|_3^3) + \frac{\nu}{2}\|\nabla u_{\delta t,N}(i)\|^2)$$
$$\leq (\epsilon(\frac{1}{2}\|\nabla \tilde{u}_{\delta t,N}(i-1)\|^2 + \frac{1}{3}\|\nabla \tilde{u}_{\delta t,N}(i-1)\|_3^3) + \frac{\nu}{2}\|\nabla u_{\delta t,N}(i-1)\|^2)$$
$$+\delta t(\|\nabla \tilde{u}_{\delta t,N}(i) u_{\delta t,N}(i-1)\|^2 + \|\nabla \bar{u}_{\delta t,N}(i) u_{\delta t,N}(i-1)\|^2 + \|f_i\|^2)$$

Note that by Hölder inequality,

$$\int |\nabla u|^2 |u|^2 \, dx \leq \|u\|_{\frac{18}{7}}^2 \|\nabla u\|_9^2 \lesssim \|u\|_{\frac{18}{7}}^6 + \|\nabla u\|_9^3$$

but we have by Gagliardo-Nirenberg,

$$\|u\|_{\frac{18}{7}}^6 \leq \|u\|^4 \|\nabla u\|^2.$$

We also have,

$$\int |\nabla \bar{u}|^2 |u|^2 \, dx \leq \|u\|_2^2 \|\nabla \bar{u}\|_\infty^2 \leq \|u\|^2 \|\nabla u\|^2 M^3$$

where we have used the Sobolev inequality and the Bernstein's A.1.7 inequality.

Therefore,

$$\frac{1}{2}(\delta t)^{-1}\|u_{\delta t,N}(i) - u_{\delta t,N}(i-1)\|^2 + (\epsilon(\frac{1}{2}\|\nabla \tilde{u}_{\delta t,N}(i)\|^2 + \frac{1}{3}\|\nabla \tilde{u}_{\delta t,N}(i)\|_3^3) + \frac{\nu}{2}\|\nabla u_{\delta t,N}(i)\|^2)$$
$$\lesssim (\epsilon(\frac{1}{2}\|\nabla \tilde{u}_{\delta t,N}(i-1)\|^2 + \frac{1}{3}\|\nabla \tilde{u}_{\delta t,N}(i-1)\|_3^3) + \frac{\nu}{2}\|\nabla u_{\delta t,N}(i-1)\|^2)$$
$$+\delta t(\|\nabla \tilde{u}_{\delta t,N}(i)\|_9^3 + \|\nabla u_{\delta t,N}(i-1)\|^2 \|u_{\delta t,N}(i-1)\|^4$$
$$+M^3 \|u_{\delta t,N}(i-1)\|^2 \|\nabla u_{\delta t,N}(i)\|^2 + \|f_i\|^2)$$

If we sum this successively, we obtain

$$\sum_i^n (\delta t)^{-1}\|u_{\delta t,N}(i) - u_{\delta t,N}(i-1)\|^2 + (\epsilon(\frac{1}{2}\|\nabla \tilde{u}_{\delta t,N}(i)\|^2 + \frac{1}{3}\|\nabla \tilde{u}_{\delta t,N}(i)\|_3^3) + \frac{\nu}{2}\|\nabla u_{\delta t,N}(i)\|^2)$$
$$\lesssim (\epsilon(\frac{1}{2}\|\nabla \tilde{u}_{\delta t,N}(0)\|^2 + \frac{1}{3}\|\nabla \tilde{u}_{\delta t,N}(0)\|_3^3) + \frac{\nu}{2}\|\nabla u_{\delta t,N}(0)\|^2)$$
$$+ \sum \|\nabla \tilde{u}_{\delta t,N}\|_9^3 \delta t + \max_j \|u_{\delta t,N}(j)\|^4 \sum \|\nabla u_{\delta t,N}(i)\|^2$$
$$+M^3 \max_j \|u_{\delta t,N}(j)\| \sum \|\nabla u_{\delta t,N}(i)\|^2 \delta t + 2\sum \|f_{i,N}\|^2 \delta t + O(\delta t).$$

We are now left on the right hand side the terms which are bounded due to 4.2.2. Substituting the bounds, we get,

$$\sum_i^n (\delta t)^{-1} \|u_{\delta t,N}(i) - u_{\delta t,N}(i-1)\|^2$$
$$+(\epsilon(\frac{1}{2}\|\nabla \tilde{u}_{\delta t,N}(j)\|^2 + \frac{1}{3}\|\nabla \tilde{u}_{\delta t,N}(j)\|_3^3) + \frac{\nu}{4}\|\nabla u_{\delta t,N}(j)\|^2)$$
$$\leq (\epsilon(\frac{1}{2}\|\nabla \tilde{u}_{\delta t,N}(0)\|^2 + \frac{1}{3}\|\nabla \tilde{u}_{\delta t,N}(0)\|_3^3) + \frac{\nu}{4}\|\nabla u_{\delta t,N}(0)\|^2)$$
$$\epsilon^{-1}C_{sp} + C_{0,f}^6\nu^{-1} + M^3 C_{0,f}^2\nu^{-1} + 2\sum \|f_{i,N}\|^2\delta t + O(\delta t).$$

∎

## 4.4 Existence and convergence to NV

Having shown the existence for 4.2, we would like to show that some subsequence of $u_{\delta t,N}(i)$ converges to a solution to 4.1. We have shown that $u_{\delta t,N}(i) \in W^{1,9} \cap H^2$ for each $i$. We will construct such a subsequence by successively reducing the sequence to a convergent sequence that converges at each of the time steps up to the current iteration. We start this process at $i = 1$. By compactness there exists $u_{\delta t}(1) \in W^{1,9} \cap H^2$ and a subsequence such that $u_{\delta t,N_k}(1)$ converges to $u_{\delta t}(1)$ strongly in $W^{1,3}$ and weakly in $H^2$. We now again take the subsequence of $u_{\delta t,N_k}$ to obtain $u_{\delta t}(2)$ such that this sub subsequence converges to it. This process will continue for all $i = 1 \dots n$. In this way, we can obtain a subsequence $u_{\delta t,N_l}$ and a function $u_{\delta t}\{i = 1, \dots n\} \times \mathbb{T}^3 \to \mathbb{R}$ such that for each discrete time steps $i = 1 \dots n$, $u_{\delta t,N_l}(i)$ converges to $u_{\delta t}(i)$ strongly in $W^{1,3}$ and weakly in $H^2$. It remains to show that $u_{\delta t}(i)$ each satisfy 4.1. This is done in almost exactly the same manner as was done in chapter 3. Thus we have,

**Theorem 4.4.1** *There exists a unique solution $u_{\delta t}(i)$, $i = 1, \dots n$ to $NV_{\delta t}(\epsilon, 3, M)$.*

Can we take a sequence of problems $NV_{\delta t}$ as $\delta t \to 0$ and conclude that such a sequence converges to a function that satisfies $NV$? To do this, we need to clarify a few things. First, $u_{\delta t}(i)$ is defined on a discrete grid and therefore not suitable when we discuss about its convergence to a function that is defined continuously in time. We must interpolate this sequence between the discrete timesteps to derive a function that is defined on space-time. Secondly, we should decide on the most convenient way by which $\delta t$ converges to zero.

To this end, we consider successively refining the mesh. That is, at the $k$th step, we take $\delta t = 2^{-k}$, so that the mesh at the $k$th step is a refinement of the mesh at the $k-1$th step. Then, to obtain an appropriate function from the sequence $u_{\delta t}(i)$ so that we can talk about it as a function in time, we interpolate $u_{\delta t}(i)$'s to define:

$$u_{\delta t}(t) = \frac{t - t_{i-1}}{\delta t} u_{\delta t}(i) + \frac{t_i - t}{\delta t} u_{\delta t}(i-1).$$

Notice that for any norm,

$$\|u_{\delta t}(t)\| \leq \|u_{\delta t}(i)\| + \|u_{\delta t}(i-1)\|.$$

This implies for instance that

$$(\int \|\nabla u_{\delta t}\|^p)^{\frac{1}{p}} \lesssim (\sum_i \|\nabla u_{\delta t}(i)\|^p \delta t)^{\frac{1}{p}},$$

thus, $u_{\delta t}(t) \in L^2(I; H^2) \cap L^3(I; W^{1,9}) \cap L^\infty(I; H^1)$. and that

$$\int \|\partial_t u_{\delta t}(t)\|^2 = \sum (\delta t)^{-1} \|u_{\delta t}(i) - u_{\delta t}(i-1)\|^2.$$

thus due to 4.3.2, $\partial u_{\delta t} \in L^2(I; L^2)$ and the bound is uniform in $k$.

Thus, by Aubin-Lions compactness theorem, there exists a subsequence that converges to a function $u$ strongly in $L^2(I; W^{1,3})$. This convergence also takes place for almost all time in $I$ say $J \subset I$. Take a time $t \in J \cap \{j2^{-k}$ j relatively prime to k$\}$, in another words, $t$ is a dyadic time that belongs to $J$. Now, notice that

$$\partial_t u_{\delta t}(t) = \frac{(u_{\delta t}(i-1) + u_{\delta t}(i))}{\delta t},$$

whenever $t \in (t_{i-1}, t_i]$.

We see, therefore that $u_{\delta t}$ almost satisfies $NV$ at time $t$, except that the nonlinearity has the dependence on time $t_{i-1}$. However, due to the convergence of $L^2$ norm of $\|u_{\delta t}(i) - u_{\delta t}(i-1)\|$ to zero as $\delta t \to 0$, the residual term in the nonlinear term goes to zero. Since $u_{\delta t}(t)$ satisfies $NV$ with residual of order $O(\delta t)$ it suffices to show that at $t$ $u$ also satisfies $NV$. But we know that the strong convergence takes place at $t$ so it can be shown that $u$ satisfies $NV$ by a similar method as was shown in chapter 3.

**Theorem 4.4.2** *Let $u_{\delta t}(i)$ be sequence of solutions to $NV_{\delta t}$ where $\delta t = 2^{-k}$. We define for each $k$ the interpolant:*

$$u_{\delta t = 2^{-k}}(t) = \frac{t - t_{i-1}}{\delta t} u_{\delta t}(i) + \frac{t_i - t}{\delta t} u_{\delta t}(i-1).$$

*It can be shown that a subsequence of $u_{\delta t=2^{-k}}$ exists that converges to $u \in L^2(I;H^2) \cap L^3(I;W^{1,9}) \cap L^\infty(I;H^1)$ and that $u$ satisfies $NV$.*

## 4.5 Stability and uniqueness

We will now show the stability estimate to prove uniqueness. The following stability estimate is interesting in that if the power input is finite so that $C_{0,f} < \infty$ for all time, then the two solutions stay boundedly close to each other.

**Lemma 4.5.1** *Let $u_{\delta t}(i)^1, u_{\delta t}(i)^2$ be two solutions to $NV_{\delta t}$.*

*Let $\delta t$ satisfy:*

$$\delta t(M + \epsilon^{-1})\nu^{-2}C_{tm} \leq \frac{1}{4}$$

*Then,*

$$\|u^1_{\delta t}(n) - u^2_{\delta t}(n)\|^2 \lesssim 2^{(M+\epsilon^{-1})(\nu^{-1}C_{0,f})^2}\|u^1_{\delta t}(0) - u^2_{\delta t}(0)\|^2 + O(\delta t\nu)\|\nabla(u^1_{\delta t}(0) - u^2_{\delta t}(0))\|^2.$$

*In particular a solution $u_{\delta t}$ to $NV_{\delta t}$ is unique.*

**Proof** Let $w_i = u_{\delta t}(i)^1 - u_{\delta t}(i)^2$.

Then, subtracting the two equations satisfied by $u^j_{\delta t}(i)$ $j = 1, 2$ and testing against $w_i$ we get,

$$\frac{1}{2}\|w_i\|^2 + \delta t(\epsilon(\|\nabla\tilde{w}_i\|^2 + \gamma\|\nabla\tilde{w}_i\|^p_p + \nu\|\nabla w_i\|^2) \leq \delta t|(w_{i-1} \cdot \nabla u^1_{\delta t}(i), w_i)| + \frac{1}{2}\|w_{i-1,N}\|^2,$$

where we have used A.2.2. Due to the Gagliardo-Nirenberg inequality,

$$\|w\|_{\frac{18}{5}} \leq \|w\|^{\frac{1}{3}}\|\nabla w\|^{\frac{2}{3}}$$

therefore,

$$\begin{aligned}
|(w_{i-1} \cdot \nabla \tilde{u_{\delta t}}(i), w_i)| &\leq \|\nabla \tilde{u_{\delta t}}(i)\|_{\frac{9}{4}}\|w_{i-1}\|_{\frac{18}{5}}\|w_i\|_{\frac{18}{5}} \\
&\lesssim \|\nabla \tilde{u_{\delta t}}(i)\|_{\frac{9}{4}}(\|w_i\|^2_{\frac{18}{5}} + \|w_{i-1}\|^2_{\frac{18}{5}}) \\
&\lesssim \|\nabla \tilde{u_{\delta t}}(i)\|_{\frac{9}{4}}(\|w_i\|^{\frac{2}{3}}\|\nabla w_i\|^{\frac{4}{3}} + \|w_{i-1}\|^{\frac{2}{3}}\|\nabla w_{i-1}\|^{\frac{4}{3}}) \\
&\lesssim \|w_i\|^2\nu^{-2}\|\nabla \tilde{u_{\delta t}}(i)\|^3_{\frac{9}{4}} + \frac{\nu}{12}\|\nabla w_i\|^2 \\
&+\|w_{i-1}\|^2\nu^{-2}\|\nabla \tilde{u_{\delta t}}(i)\|^3_{\frac{9}{4}} + \frac{\nu}{12}\|\nabla w_{i-1}\|^2 \\
&\lesssim \|w_i\|^2\nu^{-2}\|\nabla \tilde{u_{\delta t}}(i)\|^3_3 + \frac{\nu}{12}\|\nabla w_i\|^2 \\
&+\|w_{i-1}\|^2\nu^{-2}\|\nabla \tilde{u_{\delta t}}(i)\|^3_3 + \frac{\nu}{12}\|\nabla w_{i-1}\|^2.
\end{aligned}$$

where we used the Young's inequality in the last line.

For the low-frequency part, noting that by Gagliardo-Nirenberg,

$$\|u\|_3 \leq \|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}},$$

therefore,

$$\begin{aligned}
&|(w_{i-1}\cdot\nabla \bar{u_{\delta t}}(i), w_i)| \leq \|\nabla \bar{u_{\delta t}}(i)\|_3\|w_{i-1}\|_3\|w_i\|_3 \\
&\lesssim \|\nabla \bar{u_{\delta t}}(i)\|_3(\|w_{i-1}\|\|\nabla w_{i-1}\| + \|w_i\|\|\nabla w_{i-1}\|) \\
&\lesssim \|w_i\|^2\nu^{-1}\|\nabla \bar{u_{\delta t}}(i)\|_3^2 + \frac{\nu}{2}\|\nabla w_i\|^2 \\
&+\|w_{i-1}\|^2\nu^{-1}\|\nabla \bar{u_{\delta t}}(i)\|_3^2 + \frac{\nu}{2}\|\nabla w_{i-1}\|^2 \\
&\lesssim \|w_i\|^2\nu^{-1}M\|\nabla \bar{u_{\delta t}}(i)\|_2^2 + \frac{\nu}{12}\|\nabla w_i\|^2 \\
&+\|w_{i-1}\|^2\nu^{-1}M\|\nabla \bar{u_{\delta t}}(i)\|_2^2 + \frac{\nu}{12}\|\nabla w_{i-1}\|^2.
\end{aligned}$$

Let

$$A_i = \nu^{-1}M\|\nabla u_{\delta t}(i)\|^2 + \nu^{-2}\|\nabla \tilde{u_{\delta t}}(i)\|_3^3$$

Thus, summarising our calculations, we have that

$$|(w_{i-1}\cdot\nabla u^1_{\delta t}(i), w_i)| \leq A_i\|w_i\|^2 + A_i\|w_{i-1}\|^2 + \frac{\nu}{6}\|\nabla w_i\|^2 + \frac{\nu}{6}\|\nabla w_{i-1}\|^2$$

Then, due to 4.3.2,

$$A_i \leq (\nu^{-2}M + \nu^{-2}\epsilon^{-1})C_{tm}$$

Thus we have,

$$\begin{aligned}
&\|w_i\|^2 + \delta t(\epsilon(\|\nabla \tilde{w}_i\|^2 + \gamma\|\nabla \tilde{w}_i\|_3^3 + \frac{1}{2}\nu\|\nabla w_i\|^2) \\
&\lesssim 2A_i\delta t\|w_i\| + (1+2A_i\delta t)\|w_{i-1}\|^2 + \delta t\frac{1}{2}\nu\|\nabla w_{i-1}\|^2.
\end{aligned}$$

or,

$$\begin{aligned}
&\|w_i\|^2 + (1-2A_i\delta t)^{-1}\delta t(\epsilon(\|\nabla \tilde{w}_i\|^2 + \gamma\|\nabla \tilde{w}_i\|_3^3 + \frac{1}{2}\nu\|\nabla w_i\|^2) \\
&\leq (1+2A_i\delta t)(1-2A_i\delta t)^{-1}\|w_{i-1}\|^2 + (1-2A_i\delta t)^{-1}\delta t\frac{1}{2}\nu\|\nabla w_{i-1}\|^2.
\end{aligned}$$

Since $1 - 2A_i\delta t \geq \frac{1}{2}$ by hypothesis, we have due to 4.2.1,

$$\begin{aligned}
&\|w_j\|^2 \\
&+ \sum_{i=1}^{j}\Big(\prod_{l=i+1}^{j}(1+2A_i\delta t)(1-2A_i\delta t)^{-1}\Big)(1-2A_i\delta t)2\delta t\big(\epsilon(\|\nabla\tilde{w}_i\|^2 + \gamma\|\nabla\tilde{w}_i\|_3^3 + \frac{1}{2}\nu\|\nabla w_i\|^2\big) \\
&\leq \Big(\prod_{l=1}^{j}(1+2A_i\delta t)(1-2A_i\delta t)^{-1}\Big)\|w_0\|^2 + (1-2A_1\delta t)^{-1}2\delta\frac{\nu}{2}\|\nabla w_0\|^2 \\
&+ \sum_{i=1}^{j}\Big(\prod_{l=i+1}^{j}(1+2A_i\delta t)(1-2A_i\delta t)^{-1}\Big)(1-2A_i\delta t)2\delta t\frac{1}{2}\nu\|\nabla w_{i-1}\|^2\big).
\end{aligned} \tag{4.4}$$

Note then that for $\delta t A_i \leq \frac{1}{4}$, $(1+2\delta t A_i)(1-2\delta t A_i)^{-1} \leq 2^{2\delta t A_i}$. The lemma follows from this, together with the 4.2.2. ∎

# CHAPTER 5

# ERROR RATE ESTIMATE

In this chapter, we consider a solution to the 3D NSE, and the solution to $NV(\epsilon, p, M)$, and bound the norm of their difference in terms of the difference of the initial datum and the regularity of the solution to NSE. As was discussed in chapter 3, we will discuss the error rate in terms of $\epsilon$. There, we have indicated that $M$ is allowed to depend on $\epsilon$ and go to infinity when $\epsilon \to 0$. One question is to find whether there exists an optimal choice of $M(\epsilon)$. The estimation of the convergence rate offers one possibility to choose $M(\epsilon)$, as we will see that the optimization of the error estimate naturally gives us how $M$ should depend on certain inverse polynomial of $\epsilon$. We also note the role played by the parameter $p$. On the one hand, raising $p$ stabilizes the system; therefore, in the presence of sufficient regularity, we obtain a better rate. On the other, higher $p$ implies that $NV$ is a significantly perturbed version of NSE, and therefore if NSE solution does not have the required regularity, the rate does not apply. We also prove as was done in the previous chapter that the uniform in time rate estimate is possible for $p = \frac{5}{2}$ although the rate now depends on the exponential power of $\nu^{-\frac{5}{2}}$. In this way, we have indicated how $M$ should depend on $\epsilon$ and how $p$ should be chosen, effectively restricting the parameter choices.

## 5.1 Nonlinear viscosity case

### 5.1.1 Small viscosity case

We first make an assumption that $p \geq \frac{5}{2}$ as we have made in the last section. In this section, unlike the stability estimate obtained for the implicit scheme, we would like to obtain an estimate that is completely independent of $\nu$. The price to pay is that we no longer have an uniform in time error bound. This estimate, however, gives us an insight into the nature of the trade-off between the cut-off frequency $M$ and the artificial viscosity coefficient $\epsilon$.

Let $u_1$ be a strong solution to NSE, and $u_2$ be a solution to $NV(\epsilon, p, M)$ such that their initial condition agree: $u_1(0,x) = u_2(0,x)$. We would like to derive an estimate on $w = u_1 - u_2$.

We know that $u_1$ satisfies

$$\partial_t u_1 - \nu \Delta u_1 + P(u_1 \cdot \nabla u_1) = f,$$

while $u_2$ satisfies

$$\partial_t u_2 - \nu \Delta u_2 - \epsilon P Q_M \nabla \cdot ((1 + |\nabla \tilde{u_2}|^{p-2}) \nabla \tilde{u_2}), + P(u_2 \cdot \nabla u_2) = f.$$

Then, we can see that $w$ satisfies

$$\begin{aligned}
&\partial_t w - \nu \Delta w - \epsilon P Q_M \nabla \cdot ((1 + |\nabla \tilde{u_1}|^{p-2}) \nabla \tilde{u_1} - ((1 + |\nabla \tilde{u_2}|^{p-2}) \nabla \tilde{u_2}) \\
&+ P(u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2) = -\epsilon P Q_M \nabla \cdot ((1 + |\nabla \tilde{u_1}|^{p-2}) \nabla \tilde{u_1}.
\end{aligned}$$

We test this equation against $w$ to get,

$$\begin{aligned}
&\frac{1}{2} \partial_t \|w\|^2 + \nu \|\nabla w\|^2 + \epsilon (\|\nabla \tilde{w}\|^2 + \gamma \|\nabla \tilde{w}\|_p^p) \\
&\leq \epsilon (\|\nabla \tilde{u_2}\| |\nabla \tilde{w}\| + \|\nabla \tilde{u_2}\|_p^{p-1} \|\nabla \tilde{w}\|_p) - (w \cdot \nabla u_2, w) \\
&\leq C\epsilon (\|\nabla \tilde{u_1}\|^2 + \|\nabla \tilde{u_1}\|_p^p) + \frac{\epsilon}{12} (\|\nabla \tilde{w}\|^2 + \gamma \|\nabla \tilde{w}\|_p^p) + |(w \cdot \nabla u_2, w)|.
\end{aligned}$$

We estimate the convective part due to the high-frequency as

$$\begin{aligned}
&(w \cdot \nabla \tilde{u_2}, w) \lesssim \|\nabla \tilde{u_2}\|_p (\|\bar{w}\|^2_{\frac{2p}{p-1}} + \|\tilde{w}\|^2_{\frac{2p}{p-1}}) \\
&\leq \frac{\epsilon}{12} \|\nabla \tilde{w}\|^2 + C\epsilon^{-\frac{3}{2p-3}} \|\nabla \tilde{u_2}\|_p^{\frac{2p}{2p-3}} \|\tilde{w}\|^2 + M^{\frac{3}{2p}} \|\nabla \tilde{u_2}\|_p \|\bar{w}\|^2,
\end{aligned}$$

where we have used A.1.7. For the low-frequency part,

$$(w \cdot \nabla \tilde{u_2}, w) \leq M^{\frac{5}{2}} \|u_2\| \|w\|^2.$$

Consequently,

$$\begin{aligned}
&\partial_t \|w\|^2 + \nu \|\nabla w\|^2 + \epsilon (\|\nabla \tilde{w}\|^2 + \gamma \|\nabla \tilde{w}\|_p^p) \\
&\leq C_1 \epsilon (\|\nabla \tilde{u_1}\|^2 + \|\nabla \tilde{u_1}\|_p^p) + C(\epsilon^{-\frac{3}{2p-3}} \|\nabla \tilde{u_2}\|_p^{\frac{2p}{2p-3}} + M^{\frac{5}{2}} \|u_2\| + M^{\frac{3}{2p}} \|\nabla \tilde{u_2}\|_p) \|w\|^2.
\end{aligned}$$

Therefore, letting

$$f(t) = C(\epsilon^{-\frac{3}{2p-3}} \|\nabla \tilde{u_2}\|_p^{\frac{2p}{2p-3}} + M^{\frac{5}{2}} \|u_2\| + M^{\frac{3}{2p}} \|\nabla \tilde{u_2}\|_p)$$

we have

$$\begin{aligned}&\partial_t(e^{-f(t)}\|w\|^2)+e^{-f(t)}\nu\|\nabla w\|^2\\&+e^{-f(t)}\epsilon(\|\nabla\tilde{w}\|^2+\gamma\|\nabla\tilde{w}\|_p^p)\\&\lesssim e^{-f(t)}\epsilon(\|\nabla\tilde{u_1}\|^2+\|\nabla\tilde{u_1}\|_p^p).\end{aligned}$$

We have that

$$\begin{aligned}&\int_0^t\left(\epsilon^{-\frac{3}{2p-3}}\|\nabla\tilde{u_2}\|_p^{\frac{2p}{2p-3}}+M^{\frac{5}{2}}\|u_2\|+M^{\frac{3}{2p}}\|\nabla\tilde{u_2}\|_p\right)\\&\leq\epsilon^{-\frac{5}{2p-3}}C_{0,f}^{\frac{4}{2p-3}}T^{\frac{2p-5}{2p-3}}+M^{\frac{3}{2p}}T\epsilon^{-\frac{1}{p}}C_{0,f}^{\frac{1}{p}}T\\&+M^{\frac{5}{2}}TC_{0,f}.\end{aligned}$$

It can be seen that by choosing $M(\epsilon)=\epsilon^{-\frac{2}{2p-3}}$ the terms can be balanced to have the same order dependence on $\epsilon^{-1}$. Therefore, the above expression is bounded by $c(f,u_0)\max\{t,t^{\frac{2p-5}{2p-3}}\}$ where $c(f,u_0)$ is a constant that depends on the initial condition and the forcing. Thus,

$$\begin{aligned}&\|(u_1-u_2)(t)\|^2\\&\leq e^{\epsilon^{-\frac{5}{2p-3}}c(f,u_0)\max\{t,t^{\frac{2p-5}{2p-3}}\}}\int_0^t\epsilon(\|\nabla\tilde{u_1}\|^2+\|\nabla\tilde{u_1}\|_p^p)dt.\end{aligned}$$

Therefore, we can summarize our result in the following.

**Theorem 5.1.1** *Let $u_1$ be a strong solution to the 3D NSE such that $u_1\in L^p((0,T);W^{1,p})$, and $u_2$ be a solution to $NV(\epsilon,p,\epsilon^{-\frac{2}{2p-3}})$, such that their initial conditions agree and the forcing $f\in L^1([0,T]:L^2)$. Then the following estimate holds for $t\leq T$,*

$$\begin{aligned}&\|(u_1-u_2)(t)\|^2\\&\leq e^{\epsilon^{-\frac{5}{2p-3}}c(f,u_0)\max\{t,t^{\frac{2p-5}{2p-3}}\}}\int_0^t\epsilon(\|\nabla\tilde{u_1}\|^2+\|\nabla\tilde{u_1}\|_p^p)\,dt,\end{aligned}$$

Note that the error estimate depends on the $L^p$ in time of the solution. Therefore, the error rate for the nonlinear viscosity is sensitive to the singularity that may develop in time. This is intuitive since the nonlinearity should in principle regularize such a singularity and hence keep the turbulence model away from the singular NSE solution. We also note that a larger $p$ also implies smaller power dependence on $\epsilon^{-1}$. Thus, for smooth NSE solution, the stability of a larger $p$ implies that the $NV$ solution stays closer to the NSE solution.

Another notable byproduct of the estimate is that the optimization of the estimate gave us the best value for $M$. The result is that larger $p$ means we can choose a smaller cut-off frequency. This is consistent with the fact that when the solution is smooth, we can use a more stable scheme.

### 5.1.2 Uniform in time estimate

We note in passing that as in the last section, we can obtain a uniform in time error estimate for $p = \frac{5}{2}$. The cost is that the estimate now depends on $\nu$.

Now, if $u_1$ is a solution to NSE and $u_2$ solution to $NV$, we have for $w = u_1 - u_2$,

$$\begin{aligned}&\partial_t \|w(t)\|^2 + \gamma\epsilon(\|\nabla w\|_p^p + \nu\|\nabla w\|^2)\\&\leq \epsilon(\|\nabla\tilde{w}\|^2 + \gamma\|\nabla\tilde{w}\|_p^p) + |(w, \nabla u_2, w)|\end{aligned}$$

Notice that letting $p = \frac{5}{2}$, we can bound the nonlinearity as

$$\begin{aligned}&|(w, \nabla\tilde{u}_2, w)| \leq \|\nabla\tilde{u}_2\|_{\frac{5}{2}}\|w\|_{\frac{10}{3}}^2\\&\leq \|\nabla\bar{u}_2\|_3\|w\|^{\frac{4}{5}}\|\nabla w\|^{\frac{6}{5}} \leq \nu^{\frac{-3}{2}}\|\nabla\tilde{u}_2\|_{\frac{5}{2}}^{\frac{5}{2}}\|w\|^2 + \frac{\nu}{12}\|\nabla w\|^2\end{aligned}$$

Due to A.1.7,

$$\begin{aligned}&|(w, \nabla\bar{u}_2, w)| \leq \|\nabla\bar{u}_2\|_3\|w\|_3^2\\&\leq \|\nabla\bar{u}_2\|_3\|w\|\|\nabla w\| \leq \nu^{-1}M\|\nabla u_2\|_2^2\|w\|^2 + \frac{\nu}{12}\|\nabla w\|^2\end{aligned}$$

We see that the energy estimate gives

$$\int_0^t \left(\nu^{-1}M\|\nabla u_2\|_2^2 + \nu^{\frac{-3}{2}}\|\nabla\tilde{u}_2\|_{\frac{5}{2}}^{\frac{5}{2}}\right) \leq \nu^{-2}MC_{0,f}^2 + \nu^{-\frac{3}{2}}\epsilon^{-1}C_{0,f}^2$$

$$\|w(t)\|^2 \leq e^{(\nu^{-2}MC_{0,f}^2 + \nu^{-\frac{3}{2}}\epsilon^{-1}C_{0,f}^2)}\|w(0)\|^2.$$

**Theorem 5.1.2** *Let $p = \frac{5}{2}$, if $u_1$ is a solution to NSE, $u_1 \in L^p((0,T); W^{1,p})$ and $u_2$ solution to $NV$ such that their initial condition agree and that the forcing $f \in L^1([0,\infty), L^2)$. Then the following uniform estimate holds:*

$$\|w(t)\|^2 \leq e^{(\nu^{-2}MC_{0,f}^2 + \nu^{-\frac{3}{2}}\epsilon^{-1}C_{0,f}^2)}\int_0^t \epsilon(\|\nabla\tilde{u_1}\|^2 + \|\nabla\tilde{u_1}\|_p^p)\,dt,$$

The above theorem states that as long as the solution to NSE is smooth and its total fluctuation is bounded in a certain appropriate time-space norm, then the error between the NSE solution and $NV$ solution is bounded in time. Thus, they will stay within a “tube” of constant radius about the origin.

## 5.2 Hyperviscosity case

In this section we will estimate the error for the hyperviscosity case. Let $u_1$ be a strong solution to NSE, and $u_2$ be a solution to $HV(\epsilon, \alpha, M)$ such that their initial condition agree: $u_1(0,x) = u_2(0,x)$. We would like to derive an estimate on $w = u_1 - u_2$.

We have that $u_1$ satisfies

$$\partial_t u_1 - \nu \Delta u_1 + P(u_1 \cdot \nabla u_1) = f,$$

while $u_2$ satisfies,

$$\partial_t u_2 - \nu \Delta u_2 - \epsilon(-\Delta)^\alpha \tilde{u_2}. + P(u_2 \cdot \nabla u_2) = f$$

Then, we can see that $w$ satisfies

$$\begin{aligned} &\partial_t w - \nu \Delta w - \epsilon(-\Delta)^\alpha \tilde{w} \\ &+P(u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2) = -\epsilon(-\Delta)^\alpha \tilde{u_1}. \end{aligned}$$

We test this equation against $w$ to get

$$\begin{aligned} &\frac{1}{2}\partial_t \|w\|^2 + \nu \|\nabla w\|^2 + \epsilon \||\nabla|^\alpha \tilde{w}\|^2 \\ &\leq \epsilon \||\nabla|^\alpha \tilde{u_1}\| \||\nabla|^\alpha \tilde{w}\| - (w \cdot \nabla u_1, w) \\ &\leq C\epsilon \||\nabla|^\alpha \tilde{u_1}\|^2 + \frac{\epsilon}{12} \||\nabla|^\alpha \tilde{w}\|^2 + |(w \cdot \nabla u_1, w)|. \end{aligned}$$

First note that due to Gagliardo-Nirenberg, and $\int w = 0$ we have

$$\|w\|_4 \leq \|w\|^\theta \||\nabla|^\alpha w\|^{1-\theta},$$

for $\theta = \frac{4\alpha - 3}{4\alpha}$. Thus,

$$\begin{aligned} &(w \cdot \nabla u_1, w) \leq \|\nabla u_1\| \||\nabla|^\alpha w\|^{2(1-\theta)} \|w\|^{2\theta} \leq \frac{\epsilon}{12} \||\nabla|^\alpha w\|^2 + C\epsilon^{-(1-\theta)/\theta} \|\nabla u_1\|^{1/\theta} \|w\|^2 \\ &\leq \frac{\epsilon}{12} \||\nabla|^\alpha \tilde{w}\|^2 + (\epsilon M^{2\alpha} + \epsilon^{-\frac{3}{4\alpha-3}} \|\nabla u_1\|^{\frac{4\alpha}{4\alpha-3}}) \|w\|^2, \end{aligned}$$

where we have used A.1.7 and A.1.8.

We can also bound the nonlinearity using the physical viscosity term as the regularizing effect as done in the previous section. Consequently,

$$\begin{aligned}&\partial_t\|w\|^2+\nu\|\nabla w\|^2+\epsilon\||\nabla|^\alpha\tilde{w}\|^2\\&\leq C_1\epsilon\||\nabla|^\alpha\tilde{u_1}\|^2+C_2(\epsilon M^{2\alpha}+\epsilon^{-\frac{3}{4\alpha-3}}\|\nabla u_1\|^{\frac{4\alpha}{4\alpha-3}})\|w\|^2.\end{aligned}$$

Alternatively,

$$\partial_t(e^{-\int_0^t(\|\nabla u_1\|^4,\epsilon M^{2\alpha}+\epsilon^{-\frac{3}{4\alpha-3}}\|\nabla u_1\|^{\frac{4\alpha}{4\alpha-3}})\,ds}\|w\|^2)+\nu\|\nabla w\|^2+\epsilon\||\nabla|^\alpha\tilde{w}\|^2\leq C\epsilon\||\nabla|^\alpha\tilde{u_1}\|^2.$$

We can again balance the terms so that it can be bounded by a common power of $\epsilon^{-1}$. For this, we choose $M\sim\epsilon^{\frac{-2}{4\alpha-3}}$. Solving the above equation gives the following estimate:

$$\begin{aligned}&\|(u_1-u_2)(t)\|^2\\&\leq e^{\epsilon^{-\frac{3}{4\alpha-3}}\int_0^t(1+\|\nabla u_1\|^{\frac{4\alpha}{4\alpha-3}})\,ds}\int_0^t\epsilon\||\nabla|^\alpha\tilde{u_1}\|^2,dt.\end{aligned}$$

In summary, we have the following theorem.

**Theorem 5.2.1** *Let $u_1$ be a strong solution to the 3D NSE such that $u_1\in L^2((0,T);H^\alpha)\cap L^{\frac{4\alpha}{4\alpha-3}}((0,T);H^1)$, and $u_2$ be a solution to $HV(\epsilon,\alpha,\epsilon^{\frac{-2}{4\alpha-3}})$, such that their initial conditions agree. Then the following estimate holds:*

$$\begin{aligned}&\|(u_1-u_2)(t)\|^2\\&\leq e^{\epsilon^{-\frac{3}{4\alpha-3}}\int_0^t(1+\|\nabla u_1\|^{\frac{4\alpha}{4\alpha-3}})\,ds}\int_0^t\epsilon\||\nabla|^\alpha\tilde{u_1}\|^2.\end{aligned}$$

Note the trade-off at work here. We have reduced the exponential dependence on $t\epsilon^{-1}$, which is significant for small $\epsilon$. Also, the balancing of the term allowed us to choose the cut-off $M\sim\epsilon^{\frac{-2}{4\alpha-3}}$ which is smaller than the nonlinear viscosity method.

# CHAPTER 6

# CONTRACTION IN PHASE SPACE

It is known that for the 2-dimensional Navier-Stokes equations, there is a frequency scale beyond which the molecular dissipation becomes dominant and the exponential contraction of the phase space results. In this section, we show that the $NV$ model and $HV$ model in 3D also possess such a characteristic frequency scale when we take the cut-off frequencies to exceed such a scale. In this section, we will estimate the dimension of this finite-dimensional attractor for $NV$ and $HV$ models in 3D. We will assume that the forcing is bounded (i.e. $f \in L^\infty([0,\infty), L^2)$); however, we do not assume that the total energy input is bounded.

## 6.1 Contraction for the $NV$ model at $p = \frac{5}{2}$

We obtain the phase space contraction estimate for the $NV$ model. This is essentially done by considering two distinct solutions to the high-frequency part of $NV$ problem that agree in their low-frequency parts. Because dissipation acts much more strongly on high-frequency modes, we can show that such solutions converge to each other exponentially fast in time. This implies that the dynamics of the equation is mainly concentrated to low modes. Note that lowering the dimension of this *exponential attractor* without affecting the large-scale dynamics too much is exactly the goal of turbulence modeling.

We will see that for $NV$ model, the exponential attractor dimension can be obtained for $p = \frac{5}{2}$ because of the same reason that we were able to obtain uniform in time bounds in the previous chapters. We will see that the attractor dimension is bounded by $\epsilon^{-\frac{3}{2}}\nu^{-\frac{1}{2}}$.

We set the low frequency part: $P_M u = \bar{u}$ and the high-frequency part: $Q_M u = \tilde{u}$. Then,

$$\partial_t \bar{u} - \nu\Delta\bar{u} + P_M P(u \cdot \nabla u) = 0, \tag{6.1}$$

and

$$\partial_t \tilde{u} - (\nu+\epsilon)\Delta\tilde{u} - \epsilon Q_M P\nabla\cdot(|\nabla\tilde{u}|^{p-1}\nabla\tilde{u}) + Q_M P(u\cdot\nabla u) = 0. \tag{6.2}$$

We now focus on the high-frequency equation. Let us assume that there is a function $v$ that solves the following equation.

$$\partial_t v - (\nu+\epsilon)\Delta v - \epsilon Q_M P\nabla\cdot(|\nabla v|^{p-1}\nabla v) + Q_M P(v\cdot\nabla v + \bar{u}\cdot\nabla v + v\cdot\nabla\bar{u} + \bar{u}\cdot\nabla\bar{u}) = 0.$$

Subtracting, we have the following equation for $w = \tilde{u} - v$ ,

$$\partial_t w - (\nu+\epsilon)\Delta w - \epsilon Q_M P\nabla\cdot(|\nabla\tilde{u}|^{p-1}\nabla\tilde{u} - |\nabla v|^{p-1}\nabla v) + Q_M P(\tilde{u}\cdot\nabla\tilde{u} - v\cdot\nabla v + \bar{u}\cdot\nabla w + w\cdot\nabla\bar{u}) = 0.$$

Now, we test this equation by $w$ to obtain,

$$\frac{1}{2}\partial_t\|w\|^2 + (\nu+\epsilon)\|\nabla w\|^2 + \gamma\epsilon\|\nabla w\|_p^p + (w\cdot\nabla\tilde{u}, w) + (w\cdot\nabla\bar{u}, w) \le 0.$$

Now, due to the fact that $w$ lives in the high-frequency space, we have

$$\|\nabla w\| \ge M\|w\|.$$

Then, notice that

$$\begin{aligned}
&(w\cdot\nabla\tilde{u}, w) \le \|\nabla\tilde{u}\|_{\frac{5}{2}}\|w\|_{\frac{10}{3}} \\
&\lesssim \|\nabla\tilde{u}\|_{\frac{5}{2}}\|w\|^{\frac{4}{5}}\|\nabla w\|^{\frac{6}{5}} \lesssim \|w\|^2(\epsilon+\nu)^{-\frac{3}{2}}\|\nabla\tilde{u}\|_{\frac{5}{2}}^{\frac{5}{2}} + \frac{\nu+\epsilon}{12}\|\nabla w\|^2
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\frac{1}{2}\partial_t\|w\|^2 + (\nu+\epsilon)\|\nabla w\|^2 \\
&\le \frac{\epsilon+\nu}{2}\|\nabla w\|^2 + ((\epsilon+\nu)^{-\frac{3}{2}}\|\nabla\tilde{u}\|_{\frac{5}{2}}^{\frac{5}{2}} + (\epsilon+\nu)^{-1}\|\nabla\bar{u}\|_3^2)\|w\|^2.
\end{aligned}$$

Therefore,

$$\frac{1}{2}\partial_t\|w\|^2 \le -((\epsilon+\nu)M^2 - 2(\epsilon+\nu)^{-\frac{3}{2}}\|\nabla\tilde{u}\|_{\frac{5}{2}}^{\frac{5}{2}} - (\epsilon+\nu)^{-1}M\|\nabla u\|^2)\|w\|^2.$$

Now we let

$$f(t) = 2((\epsilon+\nu)M^2 - \frac{1}{t}\int_0^t ((\epsilon+\nu)^{-\frac{3}{2}}\|\nabla\tilde{u}\|_{\frac{5}{2}}^{\frac{5}{2}} + M(\epsilon+\nu)^{-1}\|\nabla u\|^2)\,ds.$$

We know that

$$t^{-1}\int_0^t \|\nabla\tilde{u}\|_{\frac{5}{2}}^{\frac{5}{2}} \le \epsilon^{-1}C_{f,\nu,0}^2,$$

where

$$C_{f,\nu,0}^2 = \|u_0\|^2 + \frac{1}{\nu}\|f\|^2_{L^\infty([0,\infty),L^2)}.$$

Therefore,

$$f(t) \geq 2(\nu+\epsilon)M^2 - ((\epsilon+\nu)^{-\frac{3}{2}}\epsilon^{-1}C_{f,\nu,0}^2 + M(\epsilon+\nu)^{-1}\nu^{-1}C_{f,\nu,0}^2).$$

Using the fact that $C_{f,\nu,0} \sim \nu^{-\frac{1}{2}}$, we see

$$2(\nu+\epsilon)M^2 - (\nu+\epsilon)^{-\frac{3}{2}}(\epsilon\nu)^{-1} - M(\epsilon+\nu)^{-1}\nu^{-2} \lesssim f(t).$$

Thus we may take

$$(\epsilon+\nu)^{-2}\nu^{-2} \lesssim M,$$

so that there exists a $\delta > 0$ such that

$$f(t) \geq \delta.$$

Then,

$$\partial_t(e^{f(t)t}\|w\|^2) \leq 0.$$

Therefore,we can summarize what we found as follows:

**Theorem 6.1.1** *Let $u$ solve $NV(\epsilon, \frac{5}{2}, M)$ and $v$ solve the high-frequency equation for $NV(\epsilon, p, M)$ with low-frequency forcing by $\bar{u}$, then we have the following contraction estimate:*

$$\|\tilde{u}(t) - v(t)\|^2(t) \leq e^{-t\delta}\|\tilde{u}_0 - v_0\|^2, \tag{6.3}$$

*when*

$$(\epsilon+\nu)^{-2}\nu^{-2}C < M,$$

*for sufficiently large constant $C$.*

## 6.2 Attractor for $HV$ with $\alpha = \frac{3}{2}$

We now consider the hyperviscous model for $\alpha = \frac{3}{2}$ and show that in this case, we can obtain the contraction estimate even in 3D, and therefore can estimate the dimension of the attractor. The high-frequency part has the following form:

$$\partial_t\tilde{u} - \nu\Delta\tilde{u} - \epsilon(-\Delta)^\alpha\tilde{u} + Q_M P(u\cdot\nabla u) = 0.$$

Let us assume that there is a function $v$ that solves the above where $v(0) = \tilde{u}$ and with the low-frequency part $\bar{u}$ used as the forcing. In other words,

$$\partial_t v - \nu\Delta v - \epsilon(-\Delta)^\alpha v + Q_M P(v \cdot \nabla v + \bar{u} \cdot \nabla v + v \cdot \nabla \bar{u} + \bar{u} \cdot \nabla \bar{u}) = 0.$$

Notice that if we took $P_M$ of the above equation, $\bar{v}$ satisfies

$$\partial_t v - \nu\Delta v - \epsilon(-\Delta)^\alpha v = 0$$

. Since $\bar{v}(0) = 0$, $\bar{v}(t) = 0$ for all time. Thus, $v$ consists only of the high-frequency part.

Subtracting, we have the following equation for $w = \tilde{u} - v$ ,

$$\partial_t w - \nu\Delta w - \epsilon(-\Delta)^\alpha w + Q_M P(\tilde{u} \cdot \nabla \tilde{u} - v \cdot \nabla v + \bar{u} \cdot \nabla w + w \cdot \nabla \bar{u}) = 0.$$

We test the equation by $w$ to obtain,

$$\frac{1}{2}\partial_t \|\tilde{w}(t)\|^2 + \epsilon \||\nabla|^\alpha \tilde{w}\|^2 + (w \cdot \nabla u, w) = 0.$$

Note that

$$\|\nabla w\|_{\frac{6}{5-2\alpha}} \leq \||\nabla|^\alpha w\|_2.$$

Since $\frac{\alpha}{3} + \frac{5-2\alpha}{6} = \frac{5}{6}$,

$$\|w \cdot \nabla w\|_{\frac{6}{5}} \leq \|w\|_{\frac{3}{\alpha}} \|\nabla w\|_{\frac{6}{5-2\alpha}}$$

we get

$$(w \cdot \nabla u, w) = -(w \cdot \nabla w, u) \lesssim \|u\|_6 \|w \cdot \nabla w\|_{\frac{6}{5}} \lesssim \|u\|_6 \|w\|_{\frac{3}{\alpha}} \||\nabla|^\alpha w\|_2,$$

but we have,

$$\|w\|_{\frac{3}{\alpha}} \lesssim \|w\|_2^\theta \||\nabla|^\alpha w\|^{1-\theta}.$$

where

$$\theta = \frac{4\alpha - 3}{2\alpha}.$$

and hence $\alpha \leq \frac{3}{2}$. Thus,

$$\begin{aligned}\|u\|_6 \|w\|_{\frac{3}{\alpha}} \||\nabla|^\alpha w\|_2 &\lesssim \|u\|_6 \|w\|^\theta \||\nabla|^\alpha w\|^{2-\theta} \\ &\leq \epsilon^{-\frac{2-\theta}{\theta}} \|u\|_6^{\frac{2}{\theta}} \|w\|^2 + \epsilon \||\nabla|^\alpha w\|^2.\end{aligned}$$

Now, note that due to the Poincare(forward) inequality and that $w \in Ran(Q_M)$ we have

$$M^\alpha \|w\| \lesssim \||\nabla|^\alpha w\|.$$

Consequently,

$$\partial_t \|w(t)\|^2 \lesssim -(\epsilon M^{2\alpha} - 2\|u\|_6^{\frac{4\alpha}{4\alpha-3}} \epsilon^{-\frac{3}{4\alpha-3}})\|w\|^2.$$

Then, to respect the energy dissipation we must set $\alpha = \frac{3}{2}$ so that

$$(t^{-1}\int_0^t \|u\|_6^{\frac{4\alpha}{4\alpha-3}})^{\frac{4\alpha-3}{4\alpha}} \lesssim (t^{-1}\int_0^t \|\nabla u\|_2^2)^{1/2}.$$

Summarizing,

**Theorem 6.2.1** *Let $u$ solve $HV(\epsilon, \frac{3}{2}, M)$ in $n = 3$, and $v$ solve the high-frequency equation for $HV(\epsilon, \alpha, M)$ with low-frequency forcing by $\bar{u}$, then we have the following contraction estimate:*

$$\|\tilde{u}(t) - v(t)\|^2(t) \le e^{-t\delta}\|\tilde{u}_0 - v_0\|^2, \tag{6.4}$$

*where*

$$M > C(\nu\epsilon^2)^{-1/3}.$$

*for sufficiently large $C$.*

Note the small dependence of $M$ on power of $\nu$. We can see from this that the dimension of the exponential attractor is rather manageable. For more information about attractors and its finite dimensionality in the setting of hyperviscosity, see [35, Temam].

# CHAPTER 7

# INTRODUCTION TO INFINITE PRANDTL-NUMBER EQUATION

Consider the generalized non-dimensional Boussinesq model: (for instance, see [38, Turcotte])

$$\begin{aligned}
&\frac{1}{Pr}(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \nabla\cdot(\nu(\theta)\nabla\mathbf{u}) + \nabla p = Ra\theta\mathbf{k},\\
&\nabla\cdot\mathbf{u} = 0,\\
&\partial_t\theta - \nabla\cdot(k(\theta)\nabla\theta) + \mathbf{u}\cdot\nabla\theta = 0,
\end{aligned} \tag{7.1}$$

on the domain $\Omega = \mathbb{T}^2 \times [0,1]$. The temperature satisfies the Dirichlet boundary condition:

$$\theta(x,y,0) = 1, \theta(x,y,1) = 0,$$

while velocity satisfies the free-slip condition [38, Turcott]:

$$\begin{aligned}
&u_3(x,y,1) = u_3(x,y,0) = 0,\\
&\partial_3 u_1(x,y,1) = \partial_3 u_1(x,y,0) = 0,\\
&\partial_3 u_2(x,y,1) = \partial_3 u_2(x,y,0) = 0
\end{aligned}$$

We use the following convention for labeling the variables $x = (x_1, \ldots, x_n) = (x', x_n)$.

$Pr$ is the Prandtl number and $Ra$ is the Rayleigh number. Notice that the viscosity $\nu$ and the conductivity $k$ depend on the temperature. Such dependence can model various physical phenomena especially in geophysical applications.

In this work, we only consider a very viscous flow in which the inertial term can be neglected. The notable physical example of such a flow is the mantle flow in the interior of the Earth. For simplicity, we also neglect the temperature dependence of the conductivity:

$$
\begin{aligned}
&-\nabla \cdot (\nu(\theta)\nabla \mathbf{u}) + \nabla p = Ra\theta\mathbf{k}, \\
&\nabla \cdot \mathbf{u} = 0, \\
&\partial_t\theta - \Delta\theta + \mathbf{u}\cdot\nabla\theta = 0. \qquad (7.2)
\end{aligned}
$$

In the mantle flow, the viscosity depends strongly on the temperature. For example, a typical function that is used in the geophysics literature is $\nu(\theta) = \exp(-a|\theta|)$ [38, Turcotte pg. 319], where $a$ quantifies the viscosity contrast, e.g. $\eta = exp(-a) = exp(-5log(10)) = 10^{-5}$.

The temperature independent viscosity case has been analyzed extensively; for instance by [3, Constantin and Doering]. A rather detailed analysis is possible in that case since the Stokes equation is linear and hence techniques from potential theory can be used. In fact, in the constant viscosity setting, the real challenge is the estimation of a constant called the *Nusselt number* which quantifies the ratio of heat transported due to the convection and that due to the conduction and is defined as follows:

$$Nu = \limsup_{T\to\infty} \frac{1}{T}|\Omega|^{-1}\int_0^T\int_\Omega |\nabla\theta|^2.$$

Since convection is a nonlinear transport process in our equation, the *Nusselt number* also describes the degree of nonlinearity present in our equation and is typically of order some power of $Ra$. Estimating the exact power occupies a large part of the current interests of the researchers in this area.

We would like to aim for such an extensive analysis for our equation. However, there is a significant mathematical challenge in our case, since the viscosity depends on the temperature which introduces a second order nonlinearity. On top of this, unlike the well-studied second order nonlinearity such as the $p$-Laplacian, what we have here is not monotonic in the temperature variable. Therefore, a variation in the temperature can potentially cause a large variation in the velocity field. Nevertheless, the regularity of the solution to this equation, and therefore the well-posedness can be shown. We can not, however, be satisfied with just a regularity result. What matters here is the actual constants in the estimation of the regularity of the solutions as these constants can be used to bound the Nusselt number.

Before plunging into the regularity, we note that the defining feature of the flow we study is the boundary layer behavior. In the next section, we will examine a simplified one-dimensional model from which we obtain some insights into the behavior of the boundary layer. We learn that the velocity has a strong decay toward the surface, effectively modeling the transition from the upper mantle layer to the lithosphere. Various estimates are expressed in terms of the important parameters $\eta$ and $Ra$.

## 7.1 One dimensional model

In the mantle, the viscosity is known to depend strongly on the temperature. This has an important effect on the dynamics of the flow since it implies a formation of a stagnant layer at the surface that models the transition from the hot and fluid upper mantle to the cool and elastic lithosphere. The mathematical challenge is to predict such a behavior rigorously from the equation.

We will first analyze an artificially simplified model in 1D:

$$\begin{aligned} &-\partial_x(\nu(\theta)\partial_x u) + \partial_x p = Ra\theta, \\ &\partial_t\theta - \partial_{xx}\theta + u\partial_x\theta = 0, \end{aligned} \tag{7.3}$$

where $p$ is given, $\theta(0) = 1, \theta(1) = 0$, $u(0) = u(1) = 0$, and $\theta|_{t=0} = \theta_0$.

### 7.1.1 Boundary Decay Estimate by Comparison Principle

In what follows we assume $0 \leq \theta_0 \leq 1$, and hence by the maximum principle the temperature lies between 0 and 1. We start with the velocity equation:

$$-\partial_x(\nu(\theta)\partial_x u) + \partial_x p = Ra\theta, \tag{7.4}$$

with $u(0) = 0, u(1) = 0$.

The goal is to understand the effect of temperature dependent viscosity in the boundary layer. To do so, we dominate the solution from above and below by carefully chosen super(sub)solutions that clarify the role of $\nu(\theta)$ on the boundedness of $u$. The consequence is that we obtain an estimate on the Lipshitz norm of $u$, and the boundedness of $u$ in terms of $\nu(\theta)^{-1}$.

**Lemma 7.1.1** *We have for solution $u$ of 7.4*

$$|u(x) - u(x_0)| \leq C(Ra + \max|\partial_x p|)\left|\int_x^{x_0} \nu(\theta(s))^{-1} s\right|.$$

**Proof** Let $x_0 \in (0,1)$, $u_0 = u(x_0)$ and consider

$$u_{x_0,+(-)\infty}(x) = \begin{cases} +(-)A_l \frac{\int_x^{x_0} \nu(\theta(s))^{-1}s}{\int_0^{x_0} \nu(\theta(s))^{-1}s} + u_0, & x \le x_0 \\ -(+)A_r \frac{\int_{x_0}^{x} \nu(\theta(s))^{-1}s}{\int_{x_0}^{1} \nu(\theta(s))^{-1}s} + u_0 & x > x_0 \end{cases},$$

where

$$\begin{array}{l} A_l = \max\{(Ra + \max|\partial_x p|) \int_0^{x_0} \nu(\theta(s))^{-1}s, |u_0|\} \\ A_r = \max\{(Ra + \max|\partial_x p|) \int_{x_0}^{1} \nu(\theta(s))^{-1}s, |u_0|\} \end{array}.$$

We claim that $u_{x_0,+(-)\infty}$ is a super(sub)solution to 7.3 at $x_0$. For the supersolution, we plug $\theta_\infty$ into the equation for $x < x_0$ to get

$$-\partial_x(\nu(\theta)\frac{A_l}{\int_0^{x_0} \nu(\theta(s))^{-1}s}(-\nu(\theta)^{-1}x)) + \partial_x p = A_l(\int_0^{x_0} \nu(\theta(s))s)^{-1} + \partial_x p \ge Ra \ge Ra\theta,$$

by maximum principle. Also, $u_{x_0,+\infty}(x_0) = u_0$ and $u_{x_0,+\infty}(0) = A_l + u_0 \ge 0$. Other cases follow similarly. Thus, due to the comparison principle for linear elliptic equation [13, Gilbarg and Trudinger, pg.268-270],

$$u_{x_0,-\infty} \le u(x) \le u_{x_0,+\infty}.$$

Suppose $x_0 < 1/2$, we can then bound $u(x_0)$ by $u_{0,+(-)\infty}$, a barrier function at $x = 0$

$$|u(x_0)| \le A_r \frac{\int_0^{x_0} \nu(\theta(s))^{-1}s}{\int_0^1 \nu(\theta(s))^{-1}s} \le c(Ra + \max|\partial_x p|) \int_0^{x_0} \nu(\theta(s))^{-1}s.$$

Thus, we see that for $x < x_0$,

$$\begin{aligned} & u(x) - u(x_0) \le A_l(\int_x^{x_0} \nu(\theta)^{-1}s)(\int_0^{x_0} \nu(\theta)^{-1}s)^{-1} \\ & \le \max\{(Ra + \max|\partial_x p|) \int_0^{x_0} \nu(\theta(s))^{-1}s, |u_0|\}(\int_x^{x_0} \nu(\theta)^{-1}s)(\int_0^{x_0} \nu(\theta)^{-1}s)^{-1} \\ & \le \max\{(Ra + \max|\partial_x p|) \int_0^{x_0} \nu(\theta(s))^{-1}s, c(Ra + \max|\partial_x p|) \int_0^{x_0} \nu(\theta(s))^{-1}s\} \\ & (\int_x^{x_0} \nu(\theta)^{-1}s)(\int_0^{x_0} \nu(\theta)^{-1}s)^{-1} \\ & \le C(Ra + \max|\partial_x p|) \int_x^{x_0} \nu(\theta)^{-1}s. \end{aligned}$$

The case $x > x_0$ is easier since $\int_{x_0}^1 \nu(\theta)^{-1}s$ does not become singular as $x_0 \le 1/2$. The same reasoning works for $x_0 > 1/2$ by using the barrier function at $x = 1$ instead. ∎

Notice that since super(sub)solution depends on the temperature, we expect that by establishing a good bound on the temperature at the surface, we can expect a better than a linear decay there.

To do this, we focus on bounding the temperature:

$$\partial_t \theta - \partial_{xx}\theta + u\partial_x \theta = 0. \tag{7.5}$$

We establish a supersolution.

**Lemma 7.1.2** *We have for solution $\theta$ of 7.5,*

$$\theta(x) \leq \sqrt{C(Ra + \max|\partial_x p|)} \int_x^1 e^{-C(Ra+\max|\partial_x p|)(1-t)^2}\, dt.$$

**Proof** Note that

$$\nu(\theta)^{-1} \leq e^a$$

hence, due to 7.1.1,

$$|u| \leq C(Ra + \max|\partial_x p|)(1-x).$$

We use an error function, a useful tool in the theory of boundary layers.

$$f(x) = \sqrt{C(Ra + \max|\partial_x p|)} \int_x^1 e^{-C(Ra+\max|\partial_x p|)(1-t)^2}\, dt,$$

we claim that $f$ is a supersolution of 7.5. Note that $f$ is a supersolution if

$$\partial_t f - \partial_{xx} f \geq C(Ra + \max|\partial_x p|)(1-x)|\partial_x f| \geq |u||\partial_x f|,$$

which is equivalent to the condition, with the explicit ansatz, that

$$2(C(Ra + \max|\partial_x p|))^{3/2}(1-x) \geq C(Ra + \max|\partial_x p|)(1-x)(C(Ra + \max|\partial_x p|))^{1/2},$$

and if $f(0) > \theta(0) = 1$. Thus, the claim follows. ∎

Consequently, we can bootstrap the bound on the velocity,

**Lemma 7.1.3**

$$|u(x)| \leq C(Ra + \max|\partial_x p|) \int_x^1 e^{(a\sqrt{C(Ra+max|\partial_x p|)}(1-s))} s\, ds.$$

**Proof** Note that due to 7.1.2,

$$\nu(\theta)^{-1} = e^{a\theta} \leq e^{(a\sqrt{C(Ra+max|\partial_x p|)}(1-x))}$$

Notice that the function on the right hand side shows a strongly decaying profile at $x = 1$, illustrating the strong decay of heat transport in the vertical direction at the surface.

# CHAPTER 8

# WELL-POSEDNESS OF INFINITE PRANDTL MODEL

In this section, we show the well-posedness for the given equation. The well-posedness can be proved in many ways. We will first give a priori estimates that guarantee the existence of a weak solution. Then, in successive steps, we improve the regularity of the solution until the uniqueness can be shown.

Here we consider the following system of convective heat equation coupled with Stokes system:

$$\begin{aligned} &-\nabla \cdot (\nu(\theta)\nabla \mathbf{u}) + \nabla p = \mathbf{k}Ra\theta, \\ &\nabla \cdot \mathbf{u} = 0. \end{aligned} \tag{8.1}$$

$$\partial_t \theta - \Delta\theta + \mathbf{u} \cdot \nabla\theta = 0. \tag{8.2}$$

on the domain $\Omega = \mathbb{T}^2 \times [0, 1]$. The temperature satisfies the Dirichlet boundary condition:

$$\theta(x, y, 1) = 0, \theta(x, y, 0) = 1,$$

while velocity satisfies the free-slip condition:

$$\begin{aligned} &u_3(x, y, 1) = u_3(x, y, 0) = 0, \\ &\partial_3 u_1(x, y, 1) = \partial_3 u_1(x, y, 0) = 0, \\ &\partial_3 u_2(x, y, 1) = \partial_3 u_2(x, y, 0) = 0 \end{aligned}$$

We assume that $\eta \leq \nu \leq 1$ .

In particular we will be especially interested in expressing the various estimates in terms of two key quantities: $Ra$ and $\eta$. Let $w = \theta - (1 - x_3)$ be a perturbative variable. Then our equation can be re-expressed in terms of $w$:

$$
\begin{aligned}
&-\nabla \cdot (\nu(w + (1 - x_3))\nabla \mathbf{u}) + \nabla p = \mathbf{k} Ra(w + (1 - x_3)), \\
&\nabla \cdot \mathbf{u} = 0.
\end{aligned}
\tag{8.3}
$$

$$
\partial_t w - \Delta w + \mathbf{u} \cdot \nabla w - u_3 = 0. \tag{8.4}
$$

with $w = 0$ on $\partial\Omega$.

## 8.1 Extension to the perodic domain

In this section we will show that due to the free-slip boundary condition, a solution to 8.3 and 8.4 can be extended to the periodic domain and satisfy a slightly modified version of the original equation. On $\Omega$, this new version reduces to the original.

We extend $\mathbf{u}$ and $w = \theta - (1 - x_3)$ to $\mathbb{T}^2 \times [-1, 0]$ in the following way:

$$
\begin{aligned}
w(x_1, x_2, -x_3) &= -w(x_1, x_2, x_3), \\
u_3(x_1, x_2, -x_3) &= -u_3(x_1, x_2, x_3), \\
u_2(x_1, x_2, -x_3) &= u_2(x_1, x_2, x_3), \\
u_1(x_1, x_2, -x_3) &= u_1(x_1, x_2, x_3), \\
p(x_1, x_2, -x_3) &= p(x_1, x_2, x_3).
\end{aligned}
$$

We then identify the two ends of $[-1, 1]$ to complete the periodic extension. In other words, we make odd extensions for $w$ and $u_3$ and even extensions for $u_1$ , $u_2$ and $p$. This means that if $w$ and $u_3$ are smooth, then they have up to the first derivative that is continuous across the boundary points $x_3 \in \mathbb{Z}$ due to the Dirichlet boundary conditions. For $u_1$ and $u_2$, they are continuous on $\mathbb{Z}$ due to the even extension. Since their normal derivative is zero, we also have that they are continuous up to the first derivatives.

Note that the $k$th order normal derivative of an odd(even) function at a point $x_0$ has the same(different) sign as its value at the point $-x_0$ if $k$ is odd(even), and the sign flips if $k$ is even(odd). It can be seen that then, $\mathbf{u}$ and $w$ satisfy the following on $\mathbb{T}^2 \times [-1, 0]$. For ease of notation we only express the variable $x_3 \in [-1,0]$.

$$\begin{aligned}
&\partial_t w(x_3) - \Delta w(x_3) + \mathbf{u}(x_3)\cdot\nabla w(x_3) - u_3(x_3)\\
&= -\partial_t w(-x_3) + \Delta w(-x_3)\\
&-u_1(-x_3)\partial_1 w(-x_3) - u_2(-x_3)\partial_2 w(-x_3) - u_3(-x_3)\partial_3 w(-x_3)\\
&+u_3(-x_3) = 0.
\end{aligned}$$

Thus, the extended function satisfy the original heat equation for $x_3 \in [-1,0]$.

To extend $\nu$, we must extend $1 - x_3$ in the following way:

$$g(x_3) = 1 - x_3$$

$$g(x_3) = \begin{cases} 1 - x_3 & 0 \le x_3 \le 1 \\ -1 - x_3 & -1 \le x_3 \le 0, \end{cases}$$

An important remark is that extension of $w$ extends $\nu(w+g)$ in way that it preserves Hölder continuity and Lipshitz continuity. Now, let $x_3 \in [-1,0]$ then,

$$\begin{aligned}
&\nabla\nu(w+g)(x_3)\cdot\nabla u_1(x_3) = \partial_1\nu(w+g)(-x_3)\partial_1 u_1(-x_3)\\
&+\partial_2\nu(w+g)(-x_3)\partial_2 u_1(-x_3) + (-\partial_3\nu(w+g)(-x_3))(-\partial_3 u_1(-x_3)).
\end{aligned}$$

Also for $u_2$, while,

$$\begin{aligned}
&\nabla\nu(w+g)(x_3)\cdot\nabla u_3(x_3) = \partial_1\nu(w+g)(-x_3)(-\partial_1 u_3(-x_3))\\
&+\partial_2\nu(w+g))(-x_3)(-\partial_2 u_3(-x_3)) + (-\partial_3\nu(w+g)(-x_3))\partial_3 u_3(-x_3).
\end{aligned}$$

So, for instance for the first component,

$$\begin{aligned}
&-\nabla\nu(w+g)(x_3)\cdot\nabla u_1(x_3) - \nu((w+g)(x_3))\Delta u_1(x_3) + \partial_1 p(x_3)\\
&= -\nabla\nu(w+(1-x_3))(-x_3)\cdot\nabla u_1(-x_3) - \nu((w+g)(-x_3))\Delta u_1(-x_3)\\
&+\partial_1 p(-x_3) = 0.
\end{aligned}$$

while for the third component,

$$\begin{aligned}
&-\nabla\nu(w+g)(x_3)\cdot\nabla u_3(x_3) - \nu(w+g(x_3))\Delta u_3(x_3) + \partial_3 p(x_3)\\
&= \nabla\nu(w+g)(-x_3)\cdot\nabla u_1(-x_3) + \nu(w+g(-x_3))\Delta u_3(-x_3)\\
&-\partial_3 p(-x_3) - (-\mathbf{k}Ra(w(-x_3) + x_3(-x_3)) = 0.
\end{aligned}$$

Thus, the velocity equation is satisfied. For incompressibility, the signs of $\partial_1 u_3$ and $\partial_2 u_2$ do not change when the point is reflected due to their evenness, while $\partial_3 u_3$ also has the same sign since the normal derivative do not change sign by reflection.

## 8.2 Existence

In this section, we obtain various estimates for the equations 8.1 and 8.2, with a goal for showing existence.

Here's our first a priori estimate.

**Lemma 8.2.1** *If $\boldsymbol{u}$ solves 8.1, then*

$$\left(\int |\nabla \boldsymbol{u}|^2\right)^{1/2} \leq c\eta^{-1} Ra \|\theta\|_{L^2}.$$

**Proof** We test the Stokes system by $\mathbf{u}$ to get

$$\begin{aligned} \eta \int |\nabla \mathbf{u}|^2 \leq \int \nu(\theta) |\nabla \mathbf{u}|^2 \leq Ra \int \theta \mathbf{u}_3 \\ \leq Ra \|\theta\|_{L^2} \left(\int (\int_0^{x_3} \partial_3 \mathbf{u}_3)^2\right)^{1/2} \\ \leq cRa \|\theta\|_{L^2} \left(\int |\partial_3 \mathbf{u}_3|^2\right)^{1/2}. \end{aligned}$$

∎

We now compute the standard a priori estimates.

**Lemma 8.2.2**

$$\partial_t \int \theta^2 + \int |\nabla \theta|^2 \leq C\eta^{-1} Ra.$$

**Proof** Set $w = \theta - \tau(z)$ for $\tau$ to be chosen later. We get that $w$ satisfies

$$\partial_t w - \Delta w - \tau'' + u_3 \tau' + \mathbf{u} \cdot \nabla w = 0.$$

Testing against $w$, we have

$$\int \frac{1}{2} \partial_t w^2 + \int u_3 \tau' w + \int |\nabla w|^2 + \int \tau' \partial_3 w = 0.$$

Thus by Cauchy-Schwartz,

$$\int \partial_t w^2 + \int |\nabla w|^2 \leq 2 \int |u_3 \tau' w| + \int (\tau')^2,$$

but

$$\begin{aligned} \int |\nabla \theta|^2 = \int |\nabla w + \tau' \mathbf{k}|^2 \\ = \int |\nabla w|^2 + 2 \int \partial_3 w \tau' + \int (\tau')^2. \end{aligned}$$

By adding two times the first equation with the second, we get,

$$\int \partial_t(w^2) + 2\int \mathbf{u}_3 w\tau' + \int(|\nabla w|^2 + |\nabla\theta|^2)$$
$$= \int(\tau')^2.$$

We now choose a specific background profile $\tau \in C^\infty([0,1])$ where

$$\tau(x_3) = \begin{cases} 1 & 0 \le x_3 \le 1-\delta \\ 0 & x_3 = 1, \end{cases}$$

and

$$\tau'(x_3) \sim \frac{1}{\delta}.$$

when $1-\delta \le x_3 \le 1$. We then bound the following term:

$$\begin{aligned}
\int u_3\tau' w &\le \frac{1}{\delta}\int_{1-\delta\le x_3\le 1}\int_{x_3}^1 \partial_3 w \int_{x_3}^1 \partial_3 u_3 \\
&\le \frac{1}{\delta}\int_{1-\delta\le x_3\le 1}\sqrt{\int(\partial_3 w)^2\delta\int(\partial_3 u_3)^2\delta} \\
&\le \left(\int_{1-\delta\le x_3\le 1}\int_{x_3}^1 |\partial_3 u_3|^2\right)^{1/2}\left(\int_{1-\delta\le x_3\le 1}\int_{x_3}^1 (\partial_3 w)^2\right)^{1/2} \\
&\le c\eta^{-1}Ra\delta\|\theta\|_{L^2}\|\nabla w\|_{L^2} \\
&\le c\eta^{-1}Ra\delta(\|\nabla\theta\|_{L^2}^2 + \|\nabla w\|_{L^2}^2),
\end{aligned}$$

where Poincaré works for $\theta$ since it vanishes at $x_n = 0$. We then choose $c\eta^{-1}Ra\delta = 1/4$, so that

$$\int \partial_t(w^2) + \frac{1}{2}\int(|\nabla w|^2 + |\nabla\theta|^2) \le C\eta^{-1}Ra.$$

∎

**Lemma 8.2.3**

$$\|\partial_t\theta\|_{L^2(0,T;H^{-1})} \le \|\nabla\theta\|_{L^2(0,T;L^2)} + C\|D\boldsymbol{u}\|_{L^\infty(0,T;L^2)}\|\theta\|_{L^2(0,T;H^1)}.$$

**Proof** Next we obtain the time derivative estimate for $\theta$. Let $\phi \in H_0^1(\Omega)$.

$$\begin{aligned}
\int \partial_t \theta \phi &= -\int \nabla\theta \cdot \nabla\phi + \int \mathbf{u}\theta\nabla\phi \\
&\leq \left(\left(\int |\nabla\theta|^2\right)^{1/2} + \left(\int (\mathbf{u}\theta)^2\right)^{1/2}\right)\left(\int |\nabla\phi|^2\right)^{1/2} \\
&\leq \left(\left(\int |\nabla\theta|^2\right)^{1/2} + \left(\int |\mathbf{u}|^6\right)^{\frac{1}{3}}\left(\int |\theta|^3\right)^{1/3}\right)\left(\int |\nabla\phi|^2\right)^{1/2} \\
&\leq \left(\left(\int |\nabla\theta|^2\right)^{1/2} + C\left(\int |D\mathbf{u}|^2\right)^{1/2}\|\theta\|_{H^1}\right)\left(\int |\nabla\phi|^2\right)^{1/2}.
\end{aligned}$$

Integrating in time,

$$\iint \partial_t\theta\phi \leq \left(\|\nabla\theta\|_{L^2(0,T;L^2)} + C\|D\mathbf{u}\|_{L^\infty(0,T;L^2)}\|\theta\|_{L^2(0,T;H^1)}\right)\|\nabla\phi\|_{L^2(0,T;L^2)}.$$

∎

We now use the standard Galerkin method in which we solve the following discrete system. Let $\{\psi_k\}$ be dense in $H_0^1(\Omega)$ and $\{\mathbf{v}_k\}$ be dense in $(H^1(\Omega))^n \cap \{\mathbf{v} : \nabla\cdot\mathbf{v} = 0\}$. Let $H_N = \text{span}\{\psi_k\}_1^N$, $V_N = \text{span}\{\mathbf{v}_k\}_1^N$.

We also modify $\nu$ as follows:

$$\nu_c(\theta) = e^{-a\min\{|\theta|,1\}}$$

The approximate problem is as follows:

Find $(\mathbf{u}_N, \theta_N - \tau) \in V_N \times H_N$ such that

$$\begin{aligned}
\int \nu_c(\theta_N)\nabla\mathbf{u}_N : \nabla\mathbf{v}_j &= \int Ra\theta_N\mathbf{v}_{j,3} \\
\int \partial_t\theta_N\psi_j + \int \nabla\theta_N\cdot\nabla\psi_j &+ \mathbf{u}_N\cdot\nabla\theta_N\psi_j = 0,
\end{aligned} \tag{8.5}$$

for all $(\mathbf{v}_j, \psi_j) \in V_N \times H_N$.

The existence of a unique solution to this ODE follows from the following facts. The first equation for a fixed temperature admits a unique solution because Stokes equation on the divergence-free space is symmetric and positive definite. Also, if $\theta_1$ and $\theta_2$ are two data to the first equation, then the difference $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$ satisfies:

$$\eta\int \|\nabla\mathbf{w}\|^2 \leq \int \nu_c(\theta_1)|\nabla\mathbf{w}|^2 \lesssim \|\nu(\theta_1) - \nu(\theta_2)\|_\infty^2\|\nabla\mathbf{u}_2\|_2^2 + Ra^2\|\theta_1 - \theta_2\|_2^2.$$

But due to the equivalence of norms in finite-dimensional vector space, this implies that the solution map of the first equation is Lipshitz. Thus, we can express the above system as a single equation by representing $\mathbf{u}_N$ in the second equation in terms of the temperature using the solution map of the first equation. The local existence now follows from the Picard theorem using the fact that the nonlinearity is locally Lipshitz.

The a-priori estimates are valid for the approximate problem as well, and hence we have

1. $\theta_N \in L^\infty(0,T;L^2) \cap L^2(0,T,H^1)$,
2. $\partial_t \theta_N \in L^2(0,T;H^{-1})$,
3. $\mathbf{u}_N \in L^\infty(0,T;H^1)$.

This implies that the local solution is in fact global.

Now, we can use Aubin-Lion's compactness theorem to obtain a subsequence $\{\theta_{N_i}\}$ converging weakly in $L^2(H^1)$, and strongly in $L^p(L^2)$ to a function $\theta$ for some $p$. We also have that $\mathbf{u}_{N_i}$ converging to $\mathbf{u}$ weakly in $L^l(H^1)$ for any $l < \infty$.

Clearly, for the linear terms, the weak convergence suffices to gurantee the convergence of the inner product in the Galerkin formulation to the corresponding one in the weak formulation. For nonlinear terms, notice that for any sequence $a_i b_i$, if $a_i \to a$ strongly in $L^2$ and $b_i \to b$ weakly in $L^2$ then, $\int a_i b_i \to \int ab$. Thus, strong convergence of $\theta_N$ and weak convergence of $\nabla \mathbf{u}_N$ implies convergence of the form

$$\int \nu_c(\theta_N) \nabla \mathbf{u}_N : \nabla \mathbf{v}_j$$

while the strong convergence of $u_N$ and weak convergence of $\theta_N$ implies the convergence of the convective term. In this way the limit can be taken in each terms of the approximate problem and the limit satisfies the weak formulation. Thus we have proved the following theorem.

**Theorem 8.2.4** *There exists* $(\boldsymbol{u}, \theta - \tau)$ *such that,*

1. $\theta \in L^\infty(0,T;L^2) \cap L^2(0,T,H^1)$,
2. $\partial_t \theta \in L^2(0,T;H^{-1})$,
3. $\boldsymbol{u} \in L^\infty(0,T;H^1)$.

*and satisfies the weak formulation:*

$$\int \nu_c(\theta)\nabla \boldsymbol{u} : \nabla \boldsymbol{v} = \int Ra\theta \boldsymbol{v}_3$$
$$\int \partial_t \theta\psi + \int \nabla\theta \cdot \nabla\psi + \boldsymbol{u} \cdot \nabla\theta\psi = 0, \tag{8.6}$$

*for almost all* $t$ *and all* $(\boldsymbol{v}, \psi) \in H_0^1 \times L^2$ *such that* $\nabla \cdot \boldsymbol{v} = 0$ *and* $\int \psi = 0$.

## 8.3 Maximum Principle Estimate

The Galerkin method does not allow us to take advantage of the maximum principle. However, having constructed a solution, we can now use the maximum principle to improve the regularity. As a side product, we can improve the dependence of $L^\infty(L^2) \cap L^2(H^1)$ norms on $\eta^{-1}Ra$, which is needed for bounding the Nusselt number.

**Lemma 8.3.1** $\min_{\partial Q_T} \theta \leq \theta(x,t) \leq \max_{\partial Q_T} \theta$ *for all* $(x,t) \in Q_T$.

**Proof** Assume the contrary, and let $\max_{\partial Q_T} \theta \leq l < \max_{Q_T} \theta$. Let $v = (\theta - l)_+$, which by above hypothesis belongs to $L^2((0,T) : H_0^1)$. We test the equation by $v$, to get

$$\iint \partial_t(\theta - l)_+(\theta - l)_+ + \iint |\nabla(\theta - l)_+|^2 + \iint \mathbf{u}\nabla(\theta - l)_+(\theta - l)_+ = 0.$$

Thus, $v = 0$ and so the claim follows. ∎

Note that this implies that we can replace $\nu_c$ back by $\nu$ in the weak formulation.

**Lemma 8.3.2** *If* $(\boldsymbol{u}, p)$ *solves* 8.1*, then*

$$\left(\int |\nabla \boldsymbol{u}|^2\right)^{1/2} \leq c\eta^{-1}Ra,$$

**Proof** The velocity bound is similar to 8.2.1, except that we use the Ł$^\infty$ norm bound on the temperature. ∎

The maximum principle allows us to improve on the temperature estimate:

**Lemma 8.3.3**

$$\partial_t \int \theta^2 + \int |\nabla\theta|^2 \leq C(\eta^{-1}Ra)^{2/3}.$$

**Proof** Set $w$, $\tau$ as in the proof of 8.2.2. We then use the maximum principle to get $\|w\|_\infty \leq c$, so that

$$\int u_3 \tau' w \leq \frac{c}{\delta} \int_{1-\delta \leq x_3 \leq 1} \int_{x_3}^{1} \partial_3 u_3$$
$$\leq c \int |\partial_3 u_3|$$
$$\leq c \left( \int |\partial_3 u_3|^2 \right)^{1/2} \delta^{1/2}$$
$$\leq c_2 \eta^{-1} Ra \delta^{1/2}.$$

We then choose $c\eta^{-1} Ra \delta^{1/2} = \delta^{-1}$, so that

$$\int \partial_t (w^2) + \frac{1}{2} \int (|\nabla w|^2 + |\nabla \theta|^2) \leq C(\eta^{-1} Ra)^{2/3}.$$

∎

The important corollary of the above is that we have the following bound on the Nusselt number.

**Corollary 8.3.4**

$$Nu = \limsup_{T \to \infty} \frac{1}{T} |\Omega|^{-1} \int_0^T \int_\Omega |\nabla \theta|^2 \leq c(\eta^{-1} Ra)^{2/3}.$$

We note that in the case of the temperature independent viscosity, the best bound obtained is due to [3, Doering and Constantin], and is of logarithmic factor times $Ra^{1/3}$.

## 8.4 Stability and Uniqueness

In the case of a temperature independent viscosity, the well-posedness can be shown easily from what has so far been proved. Not necessarily so for the temperature dependent case, as the fluctuation of the velocity now depends strongly on the fluctuation of the temperature. To see this, we show the following stability result.

**Lemma 8.4.1** *We have that for any* $(\theta_i, \boldsymbol{u}_i, p_i)$, $i = 1, 2$ *solutions to 8.2 and 8.1, the following inequality holds:*

$$\|(\theta_1 - \theta_2)(t)\|^2 \leq \|(\theta_1 - \theta_2)(0)\|^2 \exp(t((\eta^{-1} Ra)^2 + (a\eta^{-1})^2 \|\nabla^2 \boldsymbol{u}_2\|^4)).$$

**Proof** Let $(\theta_i, \mathbf{u}_i, p_i)$, $i = 1, 2$ be solutions to 8.2 and 8.1. Let $\zeta = \theta_1 - \theta_2$ and $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$. Then,

$$\partial_t \zeta + \mathbf{u}_1 \cdot \nabla\zeta + \mathbf{w} \cdot \nabla\theta_2 - \Delta\zeta = 0.$$

Test against $\zeta$ to get,

$$\int \frac{1}{2}\partial_t|\zeta|^2 + \int |\nabla\zeta|^2 = -\int \mathbf{w}\theta_2 \cdot \nabla\zeta.$$

Thus,

$$\int \partial_t|\zeta|^2 + \int |\nabla\zeta|^2 \leq \int |\mathbf{w}\theta_2|^2.$$

From the velocity equation we have

$$-\nabla \cdot (\nu(\theta_1)\nabla\mathbf{w}) - \nabla \cdot ((\nu(\theta_2) - \nu(\theta_1))\nabla\mathbf{u}_2) + \nabla(p_1 - p_2) = Ra\mathbf{k}\zeta,$$

We see first that,

$$|\nu(\theta_1) - \nu(\theta_2)| = |e^{-a|\theta_1|} - e^{-a|\theta_2|}| = |\int_{|\theta_1(x)|}^{|\theta_2(x)|} e^{-as}\, ds|a \leq |\theta_1(x) - \theta_2(x)|a$$

So $|\nu|_{Lip} \leq a$. Note that due to Gagliardo-Nirenberg and Sobolev inequalities,

$$\int |\zeta|^2|D\mathbf{u}|^2 \leq \|\zeta\|_3^2\|\nabla\mathbf{u}\|_6^2 \leq \|\zeta\|\|\nabla\zeta\|\|\nabla^2\mathbf{u}\|^2$$

so, testing the difference of the velocity equations against $w$,

$$\begin{aligned}
&\int \nu(\theta_1)|\nabla\mathbf{w}|^2 \\
&\lesssim Ra\int \zeta w_3 + a^2\int \zeta^2|\nabla\mathbf{u}_2|^2 + \frac{1}{4}\int \nu(\theta_1)|\nabla\mathbf{w}|^2 \\
&\lesssim \eta^{-1}Ra^2\int \zeta^2 + \frac{\eta}{4}\int |\nabla w_3|^2 + \eta^{-1}a^2\|\zeta\|\|\nabla\zeta\|\|\nabla^2\mathbf{u}_2\|^2 + \frac{\eta}{4}\int \nu(\theta_1)|\nabla\mathbf{w}|^2.
\end{aligned}$$

Therefore,

$$\int \partial_t|\zeta|^2 + \int |\nabla\zeta|^2 \lesssim \left((\eta^{-1}Ra)^2 + (a\eta^{-1})^2\|\nabla^2\mathbf{u}_2\|^4\right)\int \zeta^2.$$

The result follows by Gronwall's inequality. ∎

Notice the terms in the exponential factor. The presence of the $\|\nabla^2\mathbf{u}_2\|$ is specific to the temperature dependent viscosity case; and therefore, our aim in the next section is to show that this term is bounded.

## 8.5 Regularity

In this section we show that a weak solution has a further regularity. We can approach this in many different ways. Our hope is that in each estimation process we strive for a bound that is as tight as possible, since such estimates might be used to improve the Nusselt number bound. This goal will not be met by this thesis. Therefore, this remains as one of the future goals of our research.

We will first consider the global regularity of the temperature field. We will use the Galerkin method.

**Lemma 8.5.1**

$$\|\nabla^2 w(t)\|^2 + \int_0^t e^{-Ra^4(s-t)}\|\nabla^3 w\|\, ds \leq e^{Ra^4 t}(\|\nabla^2 w(0)\|^2 + Ra^2)$$

**Proof** In the following, we assume that in fact $w$ is an $N$ dimensional Galerkin projection. We take the Laplacian of the Galerkin formulation to get

$$\begin{aligned}
&\int (\partial_t \Delta w \psi + \nabla \Delta w \cdot \nabla \psi \\
&+\mathbf{u} \cdot \nabla \Delta w \psi + 2\nabla \mathbf{u} : \nabla^2 w \psi \\
&\Delta \mathbf{u} \cdot \nabla w \psi - \Delta u_3\ \psi)\ dx = 0
\end{aligned}$$

We test against the finite difference $\Delta w$, we get,

$$\begin{aligned}
&\frac{1}{2}\partial_t \|\Delta w\|^2 + \|\nabla \Delta w\|^2 \\
&+ \int (2\nabla \mathbf{u} : \nabla^2 w \Delta w + \Delta \mathbf{u} \cdot \nabla w \Delta w - \Delta u_3 \Delta w) = 0
\end{aligned}$$

Integrating by parts on the finite-difference,

$$\begin{aligned}
&\frac{1}{2}\partial_t \|\Delta w\|^2 + \|\nabla \Delta w\|^2 \\
&+ \int (2\nabla \cdot (\nabla \mathbf{u} : \nabla w)\Delta w \\
&+\nabla \mathbf{u} : \nabla^2 w \Delta w + \nabla \mathbf{u} \nabla w \cdot \nabla \Delta w - \Delta u_3 \Delta w) = 0
\end{aligned}$$

Integrating by parts, and Cauchy-Schwartz we get

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\Delta w\|^2 + \|\nabla\Delta w\|^2 \\
&\lesssim \int \left(|\nabla\mathbf{u}\nabla w|^2 + |\nabla\mathbf{u}\nabla w|^2 + |\nabla\mathbf{u}||\nabla^2 w|^2 + |\nabla u_3|^2\right) dx \\
&+\frac{1}{2}\int |\nabla\Delta w| \\
&\lesssim \|\nabla w\|_\infty^2\|\nabla\mathbf{u}\|^2 + \|\nabla\mathbf{u}\|\|\nabla^2 w\|_4^2 + \|\nabla\mathbf{u}\|^2 \\
&+\frac{1}{2}\int |\nabla\Delta w|
\end{aligned}$$

Also, due to Gagliardo-Nirenberg,

$$\|\nabla w\|_\infty \le \|\nabla^2 w\|^{\frac{1}{2}}\|\nabla^3 w\|^{\frac{1}{2}},$$

$$\|\nabla^2 w\|_4^2 \le \|\nabla^2 w\|^{\frac{1}{4}}\|\nabla^3 w\|^{\frac{3}{4}}.$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\Delta w\|^2 + \|\nabla\Delta w\|^2 \\
&\lesssim \|\nabla^2 w\|\|\nabla^3 w\|\|\nabla\mathbf{u}\|^2 + \|\nabla\mathbf{u}\|\|\nabla^2 w\|^{\frac{1}{2}}\|\nabla^3 w\|^{\frac{3}{2}}. + \|\nabla\mathbf{u}\|^2 \\
&\lesssim Ra^4\|\nabla^2 w\|^2 + Ra^2 + \frac{1}{2}\|\nabla^3 w\|^2
\end{aligned}$$

Due to elliptic regularity, $\|\nabla^3 w\| \lesssim \|\nabla\Delta w\|$

Thus use Gronwall the temperature estimate and the previous lemma:

$$\|\nabla^2 w(t)\|^2 + \int_0^t e^{-Ra^4(s-t)}\|\nabla^3 w\|\, ds \le e^{Ra^4 t}(\|\nabla^2 w(0)\|^2 + Ra^2)$$

We note that $\|\nabla^2 w\|$ is remains bounded under the extension since the first derivative of $w$ matches at the boundary. Thus, the result follows. ∎

Now, we can understand the following regularity result in this setting.

**Lemma 8.5.2**

$$\eta\int |\nabla^2\boldsymbol{u}|^2 \lesssim ((aRa)^2\|\nabla^2 w\|^4 + (\eta^{-1}Ra)^2) \tag{8.7}$$

**Proof** Note that since $\nu$ has a matching derivative at the origin, we can take the weak derivative of $\nu(\theta)$. We will again use the Galerkin projection. We take the partial derivative to obtain:

$$\int \nu_c(w+g(x_3))\nabla\partial_k\mathbf{u} : \nabla\mathbf{v} + \partial_k\nu(w+g(x_3))\nabla\mathbf{u} : \nabla\mathbf{v} = \int Ra\partial_k w v_3$$

Testing against $\partial_k u$ we get,

$$\eta\int |\nabla\partial_k\mathbf{u}|^2 \lesssim \int |\partial_k\nu(w+g)|^2|\nabla\mathbf{u}|^2 + (\eta^{-1}Ra)^2\int |w|^2 + \frac{\eta}{4}\int |\partial_{kk}u_3|^2$$

Exactly as in the proof of stability 8.4.1, we bound

$$\int |\partial_k\nu(w+g)|^2|\nabla\mathbf{u}|^2 \leq a^2\|\nabla w\|_6^2\|\nabla\mathbf{u}\|_3^2 \leq a^2\|\nabla^2 w\|^2\|\nabla\mathbf{u}\|\|\nabla^2\mathbf{u}\|$$

Consequently,

$$\eta\int |\nabla\partial_k\mathbf{u}|^2 \lesssim ((a\eta^{-1}Ra)^2\|\nabla^2 w\|^4 + (\eta^{-1}Ra)^2) + \frac{\eta}{2}\|\nabla^2\mathbf{u}\|^2$$

Combining the estimates for all directions $k$ gives the result. ∎

Thus, we have bounded $\|\nabla^2\mathbf{u}\|$, and therefore the uniqueness follows:

**Theorem 8.5.3** *Suppose that $\|\nabla^2 w(0)\| < \infty$, then the weak solution to the initial value problem for the infinite Prandtl number equation is unique.*

# CHAPTER 9

# CONCLUSION AND FUTURE WORK

## 9.1 Nonlinear viscosity model

In this thesis, we have examined two different problems in fluid dynamics. The first was about a particular turbulence model in which an artificial spectral viscosity was used to make the simulation of turbulence tractable. The model introduced various parameters and we posed a question whether an effective choice of a parameter can be made using the mathematical analysis. First, to show its basic effectiveness we proved that the resulting partial differential equation is well-posed. Then, we considered a semi-implicit discretization of the equation, and investigated a particular model in which the forcing dissipates in time. In this case, we were able to prove the uniform-in-time stability of the resulting model for a very specific $p = \frac{5}{2}$. We have also proved its consistency by showing that it converges to $NV$ as the time-step goes to zero.

To address the question of consistency, we derived an error estimate between the solution to $NV$ and a smooth solution to NSE. The same task was undertaken also for the $HV$ model. We discussed why $\epsilon$ should be taken to zero instead of $M \to \infty$. In fact, we have shown that the convergence to a weak solution to NSE occurs if $M$ depends on $\epsilon$ and is taken to infinity as $\epsilon \to 0$. The error rate gave us an additional insight in that $M(\epsilon) = \epsilon^{-\frac{5}{2p-3}}$ gives us the optimal rate for $NV$, while we obtained $M(\epsilon) = \epsilon^{-\frac{2}{4\alpha-3}}$ for $HV$. For the $NV$ model, we have also derived a uniform-in-time estimate for $p = \frac{5}{2}$. Thus, out of the infinity many models we have considered parametrized by $\epsilon$, $p$ and $M$, $NV(\epsilon, \frac{5}{2}, \epsilon^{-\frac{5}{2}})$ seems to offer a good choice in terms of the various properties that it is shown to possess.

### 9.1.1 Future work

For the semi-implicit scheme, we left the question of convergence rate estimate as a problem for future work. It is interesting to see if we could obtain a rate that is uniform in time. It is also interesting from turbulence point of view, to try to obtain a local estimate rather than global one. For example, in some cases, we may want to stabilize the system only at those locations in space where the fluctuation occurs. One difficulty in this direction for our model is that because of the $p$-Laplacian, the pressure may not have a good local regularity and may cause problem for local estimates.

## 9.2 Infinite Prandtl number model

In the second problem, we considered a model of mantle convection derived from a viscous limit of the Boussinesq model. The novelty in our case was the temperature dependent viscosity that provides a strong coupling between the Stokes equation and the heat equation. In order to understand such a strong coupling, we considered a simple 1D model that exhibits a boundary layer behavior caused by such a coupling. Some intuitions were gained by such a process since maximum principle type arguments were applicable for 1D. For 3D, we approached the question of well-posedness by using the tools from parabolic regularity theory. The local estimates are obtained and well-posedness shown. Unfortunately, these local analysis did not provide understanding for the long-time transport behavior of this equation quantified by the Nusselt number. The investigation into this aspect of the problem will be left for a future research.

# APPENDIX A

# MATHEMATICAL BACKGROUND

In this section we will gather a set of mathematical results that are used frequently in our analysis.

## A.1 Multiplier theory, filter operator, and fractional differentiation and integration

The analysis of the spectral viscosity equation involves an interaction of a rather broad range of mathematical concepts. The filter operator is defined in the frequency space, while the nonlinear viscosity involves effects on the physical space. The hyperviscosity brings the issue of the fractional differentiation and how it interacts with the filter. In this section, we will present a set of results that can serve as a common framework in which these concepts can be manipulated. Due to the use of spectral filtering, it is no surprise that the tools from harmonic analysis will be used extensively.

We denote $n$ to be the number of dimensions. First, recall that we defined the operator $P_N$ as follows:

$$P_N f = \sum_{|k|_\infty \leq N} f(k) e^{ik\cdot x}.$$

Thus, we can consider $P_N$ as an operator that is multiplied by $\chi_{|k|_\infty \leq N}$ in the frequency space. An operator that is obtained in this manner is called a *multiplier* operator. For a notational convenience, we denote $T_m$ as an operator obtained by the multiplication by $m$ in the frequency space. Therefore,

$$P_N = T_{\chi_{|k|_\infty \leq N}}.$$

We will also be interested in how this operator interacts with the fractional differentiation operator:

$$|\nabla|^s P_N = T_{|k|^s \chi_{|k|_\infty \leq N}},$$

and also how $I - P_N$ interacts with the fractional integration operator:

$$|\nabla|^{-s}(I - P_N) = T_{|k|^{-s} \chi_{|k|_\infty > N}},$$

both under which $s \geq 0$.

These operators will become fundamental to the discussion that follows. We need to be able to manipulate these operators analytically; that is, we need estimates in various norms. We call a linear operator $T$ that maps a measure space into another is of type $(p, q)$ if

$$\|T\|_{L^p \to L^q} < \infty.$$

Thus, the goal is to obtain $(p, q)$ estimates for various values of $p$ and $q$.

As we will see later, choosing the cube as the frequency region to which we project: $|k|_\infty \leq N$, is important. We could not have chosen $|k|_2$ there for a rather deep reason in harmonic analysis. We will briefly mention this issue later. We can immediately get that $|\nabla|^s P_N$ is an operator of type $(2, 2)$. Let $d$ denote the dimension.

**Lemma A.1.1** *For $s \geq 0$,*

$$\||\nabla|^s P_N\|_{L^2 \to L^2} \lesssim N^s$$

**Proof**

$$\||\nabla|^s P_N f\|_2 = \sum_{|k|_\infty \leq N} |k|^s |\hat{f}|^2 \leq d^{s/2} N^s \|\hat{f}\|_2 = d^{s/2} N^s \|f\|_2$$

∎

The $(2, 2)$ estimate is not flexible enough for our analysis. We want to obtain a $(p, q)$ type estimates. One convenience for choosing $|k|_\infty$ is that the cut-off operator can be expressed in the physical space as a convolution with a Dirichlet kernel:

$$D_N(x) = \prod_l \left(\frac{\sin((N + 1/2)x_l)}{\sin(x_l/2)}\right),$$

so that,

$$P_N(f) = D_N * f.$$

Obtaining a $(p,q)$ estimates for each values of $p,q$ may be tedious. The well-known *Riesz-Thorin interpolation theorem*, allows us to obtain a $(p,q)$ type estimates when $(1/p, 1/q)$ is a convex combination of two end-point types. Thus, we can simplify the task of obtaining an infinite number of estimates to just two [1, Bergh and Lofstrom]:

**Theorem A.1.2 (Riesz-Thorin)** *Let $T$ be an operator satisfying*

$$\|T\|_{L^{p_1}\to L^{q_1}} < \infty,$$

*and*

$$\|T\|_{L^{p_2}\to L^{q_2}} < \infty.$$

*Let*

$$(\frac{1}{p}, \frac{1}{q}) = (1-\theta)(\frac{1}{p_1}, \frac{1}{q_1}) + \theta(\frac{1}{p_2}, \frac{1}{q_2}),$$

*then we have*

$$\|T\|_{L^p\to L^q} \leq \|T\|^{1-\theta}_{L^{p_1}\to L^{q_1}} \|T\|^{\theta}_{L^{p_2}\to L^{q_2}}.$$

We will now show that $\|P_N\|_{L^1\to L^\infty}$ and $\|P_N\|_{L^p\to L^p}$ can be estimated. The desired estimate follows by an interpolation. First, the former is easy to prove:

**Lemma A.1.3**

$$\|P_N g\|_\infty \leq cN^d \|g\|_1,$$

**Proof** We will simply estimate the maximum norm of the Dirichlet kernel. Note that if $0 \leq t \leq \pi$ then, $|t|/\pi \leq |\sin(t/2)|$ and $|\sin(t/2)| \leq \min\{|t|, 1\}$ for all $t$. Therefore,

$$|\frac{\sin((N+1/2)t)}{\sin(t/2)}| \leq \frac{\min\{(N+1/2)|t|, 1\}\pi}{t} \lesssim N$$

by symmetry this holds for all $0 \leq t \leq 2\pi$. Thus,

$$\|D_N\|_\infty \lesssim N^d.$$

From which it follows that

$$\|D_N * g\|_\infty \leq \|D_N\|_\infty \|g\|_1 \leq cN^d \|g\|_1.$$

∎

The $L^p \to L^p$ bound is more difficult. Actually, since $\|D_N\|_1 \sim log(N)$, we can show such a bound up to a logarithmic factor by simply using the Young's inequality. However, the logarithmic dependence can be eliminated. The trade-off is that we need to use the theory of *singular integral operators* and *multiplier theory* which implies the non-triviality of such a bound. In fact, it is a result of [10, Fefferman] that had we chosen to use a cut-off filter with a square norm: $\sum_{|k|_2 \leq N} \hat{u}(k)$, no such bound can exist. However, for the partial sum operator we use, such a bound is true. The relevant tool to show this has been of historical significance in the development of harmonic analysis.

The tool to use is to show that the question about the boundedness in $L^p$ $1 < p < \infty$ of operator on a torus can be reduced to the corresponding question on $\mathbb{R}^d$ due to the following transference principle which is proved in [19, Krantz]:

**Theorem A.1.4 (Transference Principle)** *Let $T_m$ be an operator associated with a multiplier $m : \mathbb{R}^d \to \mathbb{C}$ such that $m$ is continuous at each point of $\mathbb{Z}^d$, then the restriction $\bar{m} = m|_{\mathbb{Z}^d}$ defines a multiplier operator on $L^2(\mathbb{T}^d)$, and,*

$$\|T_{\bar{m}}\|_{L^p(\mathbb{T}^d) \to L^p(\mathbb{T}^d)} \leq \|T_m\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)}.$$

Thus, we can transfer the question about a multiplier operator on a torus to a corresponding question on $\mathbb{R}^n$. The boundedness of the multiplier operator on $\mathbb{R}^d$ is answered by another fundamental theorem in harmonic analysis (see [1, Bergh and Lofstrom]):

**Theorem A.1.5 (Hormander-Mikhlin)** *Let $m : \mathbb{R}^d \to \mathbb{C}$ satisfy the homogeneous symbol estimates of order 0:*

$$|\nabla^k m(\zeta)| \lesssim |\zeta|^{-k},$$

*for all $\zeta \neq 0$ and $0 \leq k \leq d+2$. Then, $\|T_m\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \lesssim c_p$ for all $1 < p < \infty$.*

We will not give the detail of the proof, but using this principle, the boundedness of $P_N$ in $L^p$ follows by noting that it can be expressed as a combination of modulation and a Hilbert transform. Then using Hormander-Mikhlin, the Hilbert transform can be shown to be bounded in $L^p$. Then the following theorem follows by the transference:

**Theorem A.1.6** *If $1 < p < \infty$, We have*

$$\|P_N\|_{L^p \to L^p} \lesssim c_p$$

The following inequality is called the *Bernstein type* inequality, or *reverse inequality*.

**Theorem A.1.7** *Let* $1 < p \leq q < \infty$ *or* $1 < p < q \leq \infty$*, then*

$$\|P_N f\|_q \leq c_p N^{d(\frac{q-p}{pq})} \|f\|_p$$

**Proof** We simply interpolate between A.1.3 and A.1.6. We select $p \leq r \leq q$, such that $1 < r < \infty$ then there exists $0 \leq \theta \leq 1$ such that

$$\frac{1}{q} = \frac{\theta}{r},$$

and

$$\frac{1}{p} = \frac{\theta}{r} + 1 - \theta.$$

or $1 - \theta = \frac{1}{p} - \frac{1}{q}$. Thus, noting that $\|P_N\|_{L^r \to L^r} \leq c$ and $\|P_N\|_{L^1 \to L^\infty} \leq N^n$ , the conclusion follows by A.1.2. ∎

The Bernstein inequality is just the equivalence of a pair of norms in the finite dimensional vector space with an explicit constant. However, it is also instructive to view this as one manifestation of *the uncertainty principle*, which states that if a function is localized about the origin in the frequency space, then it must have a large physical support. But note that larger the $p$, the less sensitive the $L^p$ norm becomes to the size of the physical support. Thus, the $L^p$ norm of the frequency localized functions becomes less sensitive to $p$ as $p$ becomes larger, which is indeed shown by the constant dependence in the Bernstein's inequality.

We now turn to a host of inequalities that involve fractional derivatives. We define the fractional differentiation operator using multiplication by $|k|_2^s$ in the frequency space:

$$|\nabla|^s f = \sum |k|_2^s \hat{f} e^{ik\cdot x}$$

for each $f$ for which the right-hand-side is in $L^2$. We note in particular that $-\Delta = |\nabla|^2$.

We note that the fractional differentiation behaves in the following manner when composed with $P_N$. This is also another type of Bernstein inequality.

**Theorem A.1.8** *Let* $s \geq 0$*, then*

$$\||\nabla|^s P_N f\|_p \leq N^s c_p \|f\|_p.$$

**Proof** Define

$$\phi_j(x_j) = \begin{cases} 1 & |x_j| \le 1 \\ 0 & |x_j| \ge 2 \end{cases},$$

and $\phi_j$ is in $C^\infty$.

Let $\phi = \Pi_j \phi_j$.

We can express $|k|^s \chi_{|k|_\infty \le N} = N^s (|k| N^{-1})^s \phi(N^{-1}k) \chi_{|k|_\infty \le N}$.

Note that due to the dilation symmetry of the Fourier transform,

$$((|k|N^{-1})^s \phi(N^{-1}k))^\vee (x) = (|k|^s \phi(k))^\vee (Nx) N^n.$$

Thus, it suffices to consider the multiplier $|k|^s \phi(k)$, which is a symbol of order 0. Hence, by the Hormander-Mikhlin theorem, it is bounded in $L^p$. Thus,

$$\begin{aligned} \||\nabla|^s P_N\|_{L^p \to L^p} &= \|T_{N^s(|k|N^{-1})^s \phi(N^{-1}k)} P_N\|_{L^p \to L^p} \\ &\le N^s \|T_{|k|^s \phi(k)}\|_{L^p \to L^p} \|P_N\|_{L^p \to L^p} \le N^s c_p. \end{aligned}$$

and the claim follows by the transference principle. ∎

On the side note, instead of using the Hormander-Mikhlin theorem, for example in the case $s > 1$ we can show the bound for which the constant depends on $s$ using a more elementary method. This is because for $s > 1$, $(|k|^s \phi(k))^\vee$ belongs to $L^1$, so Young's inequality suffices.

We thus see that both versions of Bernstein inequality is a way to trade-in the localization in frequency space, for either a gain in integrability or differentiability.

The result that is in a sense opposite to reverse inequalities are the *Jackson-type* inequalities from the approximation theory. The following theorem is one version of this.

**Theorem A.1.9** *With the same hypothesis as in A.1.8,*

$$\||\nabla|^{-s}(I - P_N) f\|_p \le c_p N^{-s} \|f\|_p.$$

**Proof** Let

$$\phi_j(x_j) = \begin{cases} e^{-x_j^{-2}} & |x_j| \to 0 \\ 1 & |x_j| \ge 1 \end{cases}$$

and $\phi_j$ is in $C^\infty$.

Let $\phi = \Pi_j \phi_j$.

The multiplier can be expressed as $|k|^{-s}\chi_{|k|_\infty > N} = N^{-s}(|k|N^{-1})^{-s}\phi(N^{-1}k)\chi_{|k|_\infty > N}$. Again, due to the dilation symmetry,

$$((|k|N^{-1})^{-s}\phi(N^{-1}k))^\vee(x) = (|k|^{-s}\phi(k))^\vee(Nx)N^d.$$

Note that due to the rapid decay of $\phi(k)$ at the origin, $|k|^{-s}\phi(k) \in C^\infty$. But we have

$$\int |k|^{-s}\phi(k)e^{ik\cdot x} = \int |k|^{-s}\phi(k)e^{ik\cdot x}(\frac{-ix}{|x|}\cdot\nabla_k)^{d+1}e^{ik\cdot x}$$
$$= (\frac{-ix}{|x|})^{d+1}\int \nabla_k^{d+1}(|k|^{-s}\phi(k))e^{ik\cdot x} \leq c(|x|^{-(d+1)}),$$

for $|x| > t$ and is bounded for $|x| < t$ since $|k|^{-s}\phi(k)$ is bounded. Therefore, $(|k|^{-s}\phi(k))^\vee \in L^1(\mathbb{R}^d)$. Consequently, $\|T_{|k|^{-s}\phi(k)}\|_{L^p\to L^p} \leq c$ by Young's inequality. Thus,

$$\||\nabla|^{-s}(I-P_N)\|_{L^p\to L^p} = \|T_{N^{-s}(|k|N^{-1})^{-s}\phi(N^{-1}k)}(I-P_N)\|_{L^p\to L^p}$$
$$\leq N^{-s}\|T_{|k|^{-s}\phi(k)}\|_{L^p\to L^p}\|I-P_N\|_{L^p\to L^p} \leq N^{-s}c_p.$$

and the claim follows by the transference principle. ∎

Significance of these two inequalities can be illustrated with one example. We will use these results to prove the Gagliardo-Nirenberg inequality. The more general, Besov version of the inequality is proved by [27, Machihara].

**Lemma A.1.10** *Let $\lambda$, $\mu$, $p$, $q$, $r$, $\theta$ satisfy, $1 < q, p \leq r < \infty$, $0 < \theta < 1$.*

*1.*
$$0 > \frac{r-q}{rq}n - \lambda,$$

*2.*
$$0 \leq \frac{r-p}{rp}n + \mu,$$

*3.*
$$\theta(\lambda - \frac{n}{p} + \frac{n}{r}) + (1-\theta)(\mu - \frac{n}{q} + \frac{n}{r}) = 0.$$

*Then,*

$$\|f\|_{L^r} \lesssim |\hat{f}(0)| + \||\nabla|^\lambda f\|_{L^q}^\theta \||\nabla|^\mu (f - P_0 f)\|_{L^p}^{1-\theta}.$$

**Proof** We introduce the operator $\tilde{P}_{2^k} = P_{2^{k+1}}(I - P_{2^k})$. Then,

$$\|f\|_r \leq \|P_0 f + \sum_{k=0}^{\infty} \tilde{P}_{2^k} f\| \leq |\hat{f}(0)| + \sum_{k=0}^{\infty} \|\tilde{P}_{2^k} f\|_r.$$

Let $t > 0$ to be chosen later. We split the sum into high-frequency and low-frequency parts and estimate them differently.

First, using the first condition in our hypothesis, Jackson and Bernstein inequality,

$$\begin{aligned}\sum_{k \geq \log t} \|\tilde{P}_{2^k} f\|_r &\lesssim \sum_{k \geq \log t} 2^{k(\frac{r-q}{rq} n - \lambda)} \||\nabla|^\lambda f\|_q \\ &\lesssim t^{\frac{r-q}{rq} n - \lambda} \||\nabla|^\lambda f\|_q.\end{aligned}$$

Similarly, for the low-frequency part we use the second condition in our hypothesis,

$$\begin{aligned}\sum_{0 \leq k < \log t} \|\tilde{P}_{2^k} f\|_r &\lesssim \sum_{0 \leq k < \log t} 2^{k(\frac{r-p}{rp} n + \mu)} \||\nabla|^\mu (f - P_0 f)\|_p \\ &\lesssim t^{\frac{r-p}{rp} n + \mu} \||\nabla|^\mu (f - P_0 f)\|_p.\end{aligned}$$

Let $a = -(\frac{r-q}{rq} n - \lambda)$, $b = (\frac{r-p}{rp} n + \mu)$, then we want to minimize

$$t^{-a} \||\nabla|^\lambda f\|_q + t^b \||\nabla|^\mu f\|_p,$$

with respect to $t$. Optimizing this, we choose $t = (\frac{a\||\nabla|^\lambda f)\|_q}{b\||\nabla|^\mu (f - P_0 f)\|_q})^{1/(a+b)}$. Plugging this in, we get

$$\|f\|_r \lesssim |\hat{f}(0)| + \||\nabla|^\lambda f\|_q^{\frac{b}{a+b}} \||\nabla|^\mu (f - P_0 f)\|_p^{\frac{a}{a+b}}.$$

Choosing $\theta = \frac{b}{a+b}$ we see that

$$\theta(-a) + (1 - \theta) b = 0,$$

which is the third condition in our hypothesis. ∎

## A.2 Nonlinear monotone operator

We set $p \geq 2$ to be the degree of nonlinearity of our viscosity operator. Also, for convenience, introduce the number $q$ where $p + q = pq$, i.e. $q$ is the Holder conjugate of $p$. The nonlinear viscosity operator is defined as follows:

$$-\nabla \cdot (|\nabla u|^{p-2} \nabla u) = -\nabla \cdot ((\nabla u : \nabla u)^{(p-2)/2} \nabla u)$$

By $(A : B)$ we mean a componentwise inner product for the matrix $A$ and $B$.

The nonlinear viscosity operator is a type of *monotone operator* and satisfies a host of important inequalities that become important in the proof of well-posedness.

First, we consider some algebraic inequalities for vectors. See, [8, DiBenedetto]

**Lemma A.2.1** *Let $p \geq 2$, then for all $a, b \in \mathbb{R}^d$, there exists $\gamma > 0$ independent of $a, b$ such that*

1.
$$(|a|^{p-2}a - |b|^{p-2}b, a - b) \geq \gamma |a - b|^p$$

2.
$$(|a|^{p-2}a - |b|^{p-2}b, c) \leq (p-1)|c||a - b||a| + |b||^{p-2}$$

Some of the consequences of such a vector inequality are the following:

**Lemma A.2.2** 1.
$$\gamma \|\nabla(u - v)\|_p^p \leq (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla(u - v))$$

2.
$$(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla w) \leq (p-1)\|\nabla w\|_p \|\nabla(u - v)\|_p (\|\nabla u\|_p^2 + \|\nabla v\|_p^2)^{p-2}$$

**Proof**

$$\begin{aligned}
\int \quad \langle \quad & |\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla w\rangle \, dx \\
\leq \quad & \int |\nabla w||\nabla(u - v)|(|\nabla u| + |\nabla v|)^{p-2} \, dx \\
\leq \quad & (\int |\nabla w|^p)^{1/p} (\int |\nabla(u - v)|^p)^{1/p} (\int ||\nabla u| + |\nabla v||^{(p-2)p/(p-2)})^{(p-2)/p} \\
= \quad & \|\nabla w\|_p \|\nabla(u - v)\|_p \||\nabla u| + |\nabla v|\|_p^{p-2}
\end{aligned}$$

∎

Another remarkable property of this monotone operator is that it remains a coercive operator when tested against $-\Delta u$ in the following sense:

**Lemma A.2.3** *Let $u \in (C^2)^n$ then*

$$(\nabla \cdot |\nabla u|^{p-2}\nabla u, \Delta u) \geq \sum_{i,j,k} \int |\nabla u|^{p-2}(\partial_{kj}u^i)^2$$

**Proof** Note that we have,

$$\begin{aligned}
(\nabla \cdot |\nabla u|^{p-2}\nabla u, \Delta u) &= (|\nabla u|^{p-2}\nabla u, -\nabla\Delta u) \\
&= -\sum_{i,j,k} \int |\nabla u|^{p-2}\partial_j u^i \partial_{kk}\partial_j u^i \, dx \\
&= \sum_{i,j,k} \int |\nabla u|^{p-2}(\partial_{kj}u^i)^2 \\
&+ \sum_k \int (p-2)|\nabla u|^{p-4} \sum_{i,j,k,l} \partial_{kl}u^m \partial_l u^m \partial_{kj}u^i \partial_j u^i \, dx \\
&\geq \sum_{i,j,k} \int |\nabla u|^{p-2}(\partial_{kj}u^i)^2 \, dx
\end{aligned}$$

The last line is due to the fact that the sum inside the integral is non-negative. ∎

Define

$$I_p(u) = \sum_{i,j,k} \int |\nabla u|^{p-2}(\partial_{kj}u^i)^2.$$

We then have the following embedding theorem:

**Lemma A.2.4**

$$\|\nabla u\|_{3p} \leq I_p(u)^{1/p}.$$

**Proof**

$$\begin{aligned}
&\int |\nabla u|^{p-2}(\partial_{kj}u^i)^2 \geq \int |\partial_k u^i|^{p-2}(\partial_{kj}u^i)^2 \\
&= \int (\frac{2}{p}\partial_j|\partial_k u^i|^{p/2})^2 \geq (\int (|\partial_k u^i|^{p/2})^6)^{\frac{1}{3}}, \qquad \text{(A.1)}
\end{aligned}$$

where we have used the Sobolev inequality in the last line. ∎

## A.3 Compactness and measure theory results

The following is a standard measure theory result [24, Lieb and Loss].

**Lemma A.3.1** *Let $u_i$ be a Cauchy sequence in $L^1$, then there exists $u \in L^1$ and a subsequence $u_{i_j}$ such that $u_{i_j} \to u$ almost everywhere.*

**Proof** The point is to choose a subsequence for which the successive difference becomes thin very fast. Since $u_i$ is Cauchy, we can choose a subsequence $u_{i_j}$ so that $\|u_{i_{j+1}} - u_{i_j}\|_{L^1} \leq 2^{-j}$. Clearly,

$$u_{i_N}(x) = u_{i_0}(x) + \sum_{i=1}^{N} u_{i_j}(x) - u_{i_{j-1}}(x).$$

Let $F(x) = |u_{i_0}(x)| + \sum |u_{i_j}(x) - u_{i_{j-1}}(x)|$, then

$$\int u_{i_N}(x)\,dx \leq \int F(x)\,dx = \int |u_{i_0}(x)| + \sum |u_{i_j}(x) - u_{i_{j-1}}(x)|\,dx \leq C.$$

Thus $F$ is integrable, and hence finite almost everywhere. Thus, for almost every $x$, the sum $u_{i_0}(x) + \sum_{i=1}^{N} u_{i_j} - u_{i_{j-1}}(x)$ is absolutely convergent. Therefore, for such $x$ there exists $u(x)$ such that $u_{i_N}(x) \to u(x)$. Since $u_{i_N} \leq F$, dominated convergence theorem implies that $u \in L^1$. ∎

Given a sequence of functions $u_i$ in $L^1(\Omega)$, we call this sequence *uniformly integrable* if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $M \subset \Omega$ such that $|M| < \delta$,

$$|\int_M u_i\,dxdt| < \epsilon,$$

for all $i$.

A nice property of the uniformly integrable sequence of functions is that if they converge pointwise, then the integral also converges.

**Lemma A.3.2** *Let $u_i$ be uniformly integrable. $|\Omega| < \infty$. Suppose there exists $u \in L^1(\Omega)$ such that $u_i \to u$ almost everywhere. Then,*

$$\int u_i \to \int u.$$

**Proof** Let $\delta > 0$ be as in the definition of uniform integrability. By Egoroff's theorem, there exists a set $M$ such that $|M| \leq \delta$ and $u_i$ converges uniformly to $u$ on $M^c$. Now, by Fatou's lemma,

$$\int_M |u| \leq \liminf \int_M |u_i| \leq \epsilon.$$

now, take $i$ large enough so that $|u_i - u| \leq \epsilon |M^c|^{-1}$ on $M^c$, then

$$\int_M (u_i - u) + \int_{M^c} (u_i - u) \leq 3\epsilon.$$

Since $\epsilon$ was arbitrary,the lemma follows. ∎

We also use the Aubin-Lions compactness theorem [28, Malek et al] extensively:

**Theorem A.3.3 (Aubin-Lions)** *Let $1 < \alpha, \beta < \infty$. Let $X$ be a Banach space, and let $X_0$, $X_1$ be separable and reflexive Banach spaces. If $X_0 \subset\subset X \subset X_1$, Then the set*

$$\{v \in L^\alpha(I; X_0); \frac{dv}{dt} \in L^\beta(I; X_1)\}$$

*is compactly embedded in $L^\alpha(I; X)$.*

## A.4 Local averages and Hölder continuity

We summarize some of the mathematical results that are used in the analysis of the infinite Prandtl-number equation. We first define the parabolic cylinder $Q_\rho = \{(x,t) : |x - x_0| < \rho, |t - t_0| < \rho^2\} \cap \Omega$.

First we note the characterization of the Holder norm by the growth of local integrals due to Campanato.

**Theorem A.4.1 (Campanato)** *We have the following characterization of the Hölder semi-norm:*

$$[f]_{C^\alpha(\mathbb{T}^n)} \sim (\sup_r r^{-n-\alpha p} \int |f - (f)_r|^p)^{1/p}$$

A corollary of this theorem is that we have the following characterization of the Hölder continuity that is more suitable in the parabolic setting. [23, p. 50]

**Lemma A.4.2** *Let $0 < \alpha < 1$. Suppose we have that for all $0 < r < R$.*

$$|Q_r|^{-1} \iint_{Q_r} |u - (u)_{Q_R}|^2 \leq C r^{2\alpha}$$

*Then, for all $0 < r < cR$.*

$$|u(x,t) - u(y,s)| \leq C_2(|x - y|^\alpha + |t - s|^{\alpha/2})$$

*Thus, $u$ is $\alpha$-Hölder continuous in space and $\alpha/2$-Hölder continuous in time. We let the smallest constant on the right hand side the $C^{\alpha,\alpha/2}$ semi-norm of $u$.*

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# BIOGRAPHICAL SKETCH

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