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Geometry Optimization of Elemental Flow Channels with Asymmetric Bifurcations

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GEOMETRY OPTIMIZATION OF ELEMENTAL FLOW CHANNELS
WITH ASYMMETRIC BIFURCATIONS

By

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I dedicate this dissertation to

*My parents Sarat Chandra Prasad and Rajeswari*
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# TABLE OF CONTENTS

Nomenclature | vii
---|---
List of Figures | ix
List of Tables | xiv
Abstract | xv

1. **INTRODUCTION**
   1.1 Introduction | 1
   1.2 Dissertation Objectives | 1
   1.3 Intellectual Merit | 1
   1.4 Broader Impact | 3
   1.5 Literature Review | 4

2. **CONSTRUCTAL DENDRITIC GEOMETRY AND THE EXISTANCE OF ASYMMETRIC BIFURCATION**
   2.1 Effect of resistance on mass flow rate in a pipe | 7
   2.2 Symmetric dendritic networks when the flow is divided in half at every bifurcation node | 9
   2.3 Assembling symmetrical dendritic trees with square cross sectional shape channels | 12
   2.4 Asymmetry of bifurcation (the elemental Y and V shapes) | 16
   2.5 Asymmetry of dendritic tree networks | 22

3. **ASYMMETRIC BIFURCATION OF T-SHAPED CHANNELS**
   3.1 Introduction | 25
   3.2 Mass induced asymmetry in T-shaped channels | 27
   3.3 Mass and geometry induced asymmetry in T-shaped channels | 31
   3.4 T-shaped Flow Channel Networks | 37
   3.4 Conclusions | 41

4. **ASYMMETRIC BIFURCATION OF Y-SHAPED CHANNELS**
   4.1 Introduction | 42
   4.2 Y-shaped channels with mass and geometry induced asymmetry | 42
   4.3 Pressure drop and non-dimensional flow resistance | 44
   4.4 Symmetric Y-shaped channels | 46
   4.5 Y-shaped channels with mass induced asymmetry | 50
   4.6 Y-shaped channels with geometry induced asymmetry | 58
   4.7 Y-shaped configurations with g > 0.5 | 63
   4.8 Y-shaped Flow Channel Networks | 65
   4.9 Conclusions | 68
NOMENCLATURE
(in the order of appearance)

\( n, n_p \) - level of branching

\( D, D_n, D_{n+1} \) - diameter of the channel, diameter at \( n \) and \( n+1 \) level, respectively

\( \Delta P \) - pressure drop

\( \dot{m} \) - inlet mass flow rate

\( \nu \) - kinematic viscosity of the fluid

\( L \) - length of the channel

\( \gamma \) - mass fraction

\( R_v \) - flow resistance

\( n_0 \) - number of channels from the source/inlet

\( b \) - radius of the disc

\( V_c \) - volume occupied by the channels

\( \mu \) - dynamic viscosity of the fluid

\( u \) - velocity in the \( x \)-direction

\( a \) - width of the square cross section of channels

\( \beta \) - angle of bifurcation

\( C \) - multiplication constant

\( S \) - area of the rectangle enclosing the 2D channels

\( x/y, F \) - aspect ratio

\( N \) - number of outlets

\( R_f \) - non-dimensional flow resistance
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{f,\text{min}}$</td>
<td>minimum non-dimensional flow resistance</td>
</tr>
<tr>
<td>$x$</td>
<td>ratio of channel lengths</td>
</tr>
<tr>
<td>$y$</td>
<td>ratio of channel diameters</td>
</tr>
<tr>
<td>$g$</td>
<td>geometric deviation from the center</td>
</tr>
<tr>
<td>$p$</td>
<td>geometric deviation from the edge</td>
</tr>
<tr>
<td>$\dot{W}$</td>
<td>pumping power</td>
</tr>
<tr>
<td>$R$</td>
<td>radius of the circle enclosing the outlets in 3D channels</td>
</tr>
<tr>
<td>$V$</td>
<td>volume of tetrahedral/pyramid enclosing the 3D channels</td>
</tr>
</tbody>
</table>
## LIST OF FIGURES

| Figure 1.1 | Flow systems in nature | 2 |
| Figure 1.2 | Flow systems in engineering | 3 |
| Figure 1.3 | Fractal geometry in nature: fern leaves and cauliflower | 5 |
| Figure 2.1 | Effect of flow resistance on mass flow rate of a fluid that leaks through a hole or a valve | 7 |
| Figure 2.2 | (a) optimal geometry of dendritic trees when \( n_p = 2 \) and \( n_0 = 3 \)  
(b) optimal geometry of dendritic trees when \( n_p = 2 \) and \( n_0 = 3 \)  
(c) optimal geometry of dendritic trees when \( n_p = 4 \) and \( n_0 = 3 \)  
(d) optimal geometry of dendritic trees when \( n_p = 1 \) and \( n_0 = 6 \) | 9 |
| Figure 2.3 | The average velocity of unidirectional flow through a square cross-sectional shape duct | 12 |
| Figure 2.4 | The envelop of global flow resistance \( (R_f) \) when the number of points on the circle \( (N) \) is fixed, and the radius \( (b) \) and the total volume \( (V_C) \) are fixed | 15 |
| Figure 2.5 | Geometry of the optimal Y-shaped assemblies at specific aspect ratios of the occupied area \( (x/y) \) and specific mass fractions \( (\gamma) \) | 17 |
| Figure 2.6 | The optimal angles of Y-shaped assembly corresponding to the aspect ratio of the occupied area \( (x/y) \) and mass fraction \( (\gamma) \) | 19 |
| Figure 2.7 | The optimal length of Y-shaped assembly corresponding to the aspect ratio of the occupied area \( (x/y) \) and mass fraction \( (\gamma) \) | 19 |
| Figure 2.8 | The optimal length ratio of Y-shaped assembly corresponding to the aspect ratio of the occupied area \( (x/y) \) and mass fraction \( (\gamma) \) | 20 |
| Figure 2.9 | The optimal size step of Y-shaped assembly corresponding to the aspect ratio of the occupied area \( (x/y) \) and mass fraction \( (\gamma) \) | 20 |
| Figure 2.10 | The optimal duct size ratio of V-shaped assemblies corresponding to mass fraction \( (\gamma) \) when the aspect ratio \( (x/y) \) is larger than 1.5 | 20 |
| Figure 2.11 | Flow resistance of Y- and V-shaped assemblies. The constant \( C \) depends on the shape of channel cross sections, i.e., \( C = 8\pi \) for round shape cross section and \( C = 29.39 \) for square cross section | 22 |
Figure 2.12  Dichotomy of the trees that distribute flow at the same mass flow rate constructed according to Figures 2.6–2.9 when the aspect ratio of the area occupied by each Y-shape (x/y) = 0.7:
(a) from one point source to ten point sinks and
(b) from one point source to 30 point sinks around the source

Figure 3.1  T-shaped flow channel with symmetric geometry and mass bifurcation

Figure 3.2  T-shaped channel with asymmetric mass bifurcation

Figure 3.3  Variation in $R_{f,\text{min}}$ with mass fraction $\gamma$

Figure 3.4  Variation in $(L_1/L_0)_{\text{opt}}$ with mass fraction $\gamma$

Figure 3.5  Variation in $(D_1/D_0)_{\text{opt}}$ and $(D_2/D_0)_{\text{opt}}$ with mass fraction $\gamma$

Figure 3.6  Optimal T-shaped channels with mass induced asymmetry

Figure 3.7  Three points connected using asymmetric T-shaped channels

Figure 3.8  T-shaped channel with mass and geometry induced asymmetry

Figure 3.9  Minimum non-dimensional flow resistance $R_{f,\text{min}}$ at different aspect ratios and mass fractions

Figure 3.10  Variation in optimal diameter ratios and non-dimensional flow resistance $R_{f,\text{min}}$ with aspect ratio, geometric deviation and mass fractions

Figure 3.11  Optimal T-shaped channels with mass and geometry induced asymmetry

Figure 3.12  Symmetric T-shaped channel networks in 2D (64 outlets) and 3D (128 outlets)

Figure 3.13  Asymmetric T-shaped channel network

Figure 4.1  Y-shaped channel with mass and geometry induced asymmetry

Figure 4.2  Variation in non-dimensional flow resistance $R_{f,\text{min}}$ and bifurcation angle $(\theta_1)_{\text{opt}}$ with aspect ratio

Figure 4.3  Variation in $(L_1/L_0)_{\text{opt}}$ and $(D_1/D_0)_{\text{opt}}$ with aspect ratio $F$
| Figure 4.4 | Optimal Y-shape configurations symmetric in mass and geometry at different aspect ratios | 49 |
| Figure 4.5 | Variation in $R_{f,min}$ with aspect ratio $F$ at different mass fractions | 51 |
| Figure 4.6 | (a) Variation in optimal $\theta_0$ with aspect ratio $F$ at different mass fractions  
(b) Variation in optimal $\theta_1$ with aspect ratio $F$ at different mass fractions  
(c) Variation in optimal $\theta_2$ with aspect ratio $F$ at different mass fractions | 52, 53 |
| Figure 4.7 | (a) Variation in $(L_1/L_0)_{opt}$ with aspect ratio $F$ at different mass fractions  
(b) Variation in $(L_2/L_0)_{opt}$ with aspect ratio $F$ at different mass fractions | 54 |
| Figure 4.8 | (a) Variation in $(D_1/D_0)_{opt}$ with aspect ratio $F$ at different mass fractions  
(b) Variation in $(D_2/D_0)_{opt}$ with aspect ratio $F$ at different mass fractions | 55, 56 |
| Figure 4.9 | Optimal Y-shape configurations with mass induced asymmetries at different aspect ratios | 57 |
| Figure 4.10 | Variation in $R_{f,min}$ with aspect ratio $F$ at different geometric deviations | 58 |
| Figure 4.11 | (a) Variation in optimal $\theta_0$ with aspect ratio $F$ at different geometric deviations  
(b) Variation in optimal $\theta_1$ with aspect ratio $F$ at different geometric deviations  
(c) Variation in optimal $\theta_2$ with aspect ratio $F$ at different geometric deviations | 59, 60 |
| Figure 4.12 | (a) Variation in $(L_1/L_0)_{opt}$ with aspect ratio $F$ at different geometric deviations  
(b) Variation in $(L_2/L_0)_{opt}$ with aspect ratio $F$ at different geometric deviations | 61 |
| Figure 4.13 | (a) Variation in $(D_1/D_0)_{opt}$ with aspect ratio $F$ at different geometric deviations  
Variation in $(D_2/D_0)_{opt}$ with aspect ratio $F$ at different geometric deviations | 62 |
| Figure 4.14 | Optimal Y-shape configurations with geometry induced asymmetries at different aspect ratios | 64 |
Figure 4.15  Comparison between rectangular areas between configurations with \( g = 0.25 \) and \( g = 0.75 \)  

Figure 4.16  Symmetric Y-shaped channel network on a circular disc  

Figure 4.17  Asymmetric Y-shaped channel network  

Figure 5.1  T-shaped channels in 3D space connecting multiple outlets  

Figure 5.2  Symmetric 3D T-shaped channels  

Figure 5.3  Variation in minimum non-dimensional flow resistance \( R_{f,min} \) with aspect ratio \( F \)  

Figure 5.4  Variation in Optimal diameter ratio \( (D_1/D_0)_{opt} \) with number of outlets \( N \)  

Figure 5.5  Three of the possible orientations of T-shaped channels with gravity is assumed acting in negative \( z \)-direction  

Figure 5.6  Variation in function \( k \) with \( B \)  

Figure 5.7  3D T-shaped channel with mass induced asymmetry  

Figure 5.8  Variation in non-dimensional flow resistance \( R_{f,min} \) with aspect ratio and mass fraction  

Figure 5.9  Variation in optimal diameter ratios with mass fractions  

Figure 5.10  Optimal configurations for 3D T-shaped channels with mass induced asymmetries  

Figure 5.11  3D T-shaped channel with geometry induced asymmetry  

Figure 5.12  Variation in non-dimensional flow resistance \( R_{f,min} \) with aspect ratio and geometric deviation  

Figure 5.13  Variation in optimal diameter ratios at different geometric deviations  

Figure 5.14  Optimal configurations for 3D T-shaped channels with geometry induced asymmetry  

Figure A.1  T-shaped channel with asymmetric geometry and mass bifurcation
Figure A.2  Variation in $R_{f,\text{min}}$ with mass flow parameter  

Figure A.3  Variation in optimal length ratio $(L_2/L_1)_{\text{opt}}$ with mass flow parameter  

Figure A.4  Variation in optimal diameter ratio $(D_2/D_1)_{\text{opt}}$ with mass flow parameter
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 1</td>
<td>Optimized geometric features of symmetrical trees with two levels of pairing</td>
<td>106</td>
</tr>
<tr>
<td>Table 2</td>
<td>Optimized geometric features of symmetrical trees with three levels of pairing</td>
<td>107</td>
</tr>
<tr>
<td>Table 3</td>
<td>Optimized geometric features of symmetrical trees with four levels of pairing</td>
<td>108</td>
</tr>
<tr>
<td>Table 4</td>
<td>Optimized geometric features of symmetrical trees with five levels of pairing</td>
<td>109</td>
</tr>
<tr>
<td>Table 5</td>
<td>Optimized geometric features of symmetrical trees with six levels of pairing</td>
<td>110</td>
</tr>
<tr>
<td>Table 6</td>
<td>Optimized geometric features of symmetrical trees with seven levels of pairing</td>
<td>111</td>
</tr>
</tbody>
</table>
ABSTRACT

This dissertation investigates the optimal geometric characteristics of elemental flow channels with asymmetries due to geometry and mass distribution. The optimization techniques used follow the constructal approach [26]. The elemental flow channels considered in this dissertation include channels on circular discs, T-shaped channels in 2D and 3D and Y-shaped channels in 2D. Symmetric channel configurations have flow dividing into equal proportions at the junction and the source placed equidistantly between the outlets. Mass asymmetry is introduced by creating imbalance in the division of flow at the junction. Geometric asymmetry is introduced by moving the source away from the equidistant point between the outlets. During the optimization the local junction losses are assumed to be negligible and the inner walls of the channels are assumed to be smooth. The pressure drop across the channels is non-dimensionalized into flow resistance with respect to mass flow rate, the volume occupied by the channels and the area/volume influenced by the channels. The non-dimensional flow resistance is minimized to obtain optimal geometric characteristics in the form of length and diameter ratios of the channels sections. The dimensionless results can be applied in a broad range of length scales without losing generality of the optimization. The optimization results brought up important observations in both 2D and 3D channels. The optimal diameter ratio of symmetric channel sections is observed to depend only upon the number of branch sections. The optimal bifurcating angle in symmetric 2D Y-shaped channels is found to be a constant. Mass induced asymmetry is observed to reduce the flow resistance in all channel configurations where as geometric asymmetry tend to increase the flow resistance.
CHAPTER 1

INTRODUCTION

1.1 Introduction

In this dissertation we intend to optimize elemental flow channels with mass and geometry induced asymmetries. The optimization techniques are based upon constructal principles where the flow resistance across the channels is minimized. The optimization results are in the form of geometrical characteristics (e.g. architecture) that can be maintained in a wide range of length scales. The research is intended to develop fundamental results of optimization that serve as a basis for the development of asymmetric flow networks.

1.2 Dissertation Objectives

The primary objective of this research is to develop optimal geometric characteristics for elemental T and Y-shaped channels with mass and geometrical asymmetries. The resulting constructal geometries will be used to connect multiple outlets to a source using optimal flow networks.

1.3 Intellectual Merit

Optimal T and Y-shaped channels that are symmetric in design are proposed for many applications including chemical diffusers, fuel cell bipolar plates and heat exchangers. Flow networks created using symmetric T and Y-shaped channels help increase the overall efficiency of the system by reducing the pumping power required to transport the fluid. However, the importance of asymmetry in elemental shapes has been overlooked in literature. Optimization results of asymmetric T and Y-shaped channels help create complex flow networks that are impossible using symmetric designs. This work focuses primarily on optimizing asymmetric T and Y-shaped channels and obtaining the optimal geometric parameters that define them. These fundamental results will assist in the engineering of large scale flow networks with multiple outlets. Moreover, the optimal geometric characteristics obtained in this research can be used to make comparisons between symmetric and asymmetric designs when subjected to similar constraints and find out whether optimal in nature is symmetric or asymmetric.
Figure 1.1 Flow systems in nature

River Delta         Leaf Venation
Human Bronchia      Human Vascular System
1.4 Broader Impact

The impact of this work will be demonstrated in two fronts. First, it helps understand the principles of formation behind the natural flow systems that we observe in everyday life, such as river deltas, venation in leaves, human bronchial and vascular systems, etc (Figure 1.1). Under similar constraints, the flow networks we optimize in this research replicate designs we find in nature. Secondly, the research results can be used to improve existing engineering systems for improved efficiency or create new systems that are optimal for flow distribution. Examples are irrigation systems, petrochemical refineries, natural gas pipeline, air conditioning ducts, etc (Figure 1.2). Engineering systems where pressure drop in the channels and pumping power are of primary concern can always use optimal geometric parameters evaluated in this research. Optimal parameters of symmetric designs published in literature become part of a broader spectrum of asymmetric designs optimized in this research.
1.5 Literature Review

The anatomical designs observed in advanced life forms enumerate the attempts of sophistication nature endeavored through evolutionary techniques. Natural systems such as plant root systems, human bronchia, and blood vessels developed in such a way that they expend least amount of energy to transport fluids across the body. The design knowledge of such systems, however, was not completely understood in the early times, primarily due to unavailability of precision equipment of observation. The invention of microscope, however, revolutionized the exploration of the design in nature.

Particular interest in the fields of medicine and bioengineering is in the structure of human bronchia and vascular systems, shown in Figure 1.1. They are systems of flow channels with the function of supplying fluids for physiological purposes. The design parameters and fluid dynamics in these channels have been studied by many researchers for more than a century. One of the important contributions made towards understanding vascular systems is by Cecil D. Murray [1, 2]. He was able to calculate the pumping power required in the blood vessels and explain the differences in design parameters between two types of vascular channels: macroscopic arteries and microscopic capillaries. Another important contribution in understanding vascular systems came from Cohn [3]. In his publication, Cohn reported that optimization of flow channels yield an important design parameter in the form of bifurcation diameter ratio \( D_{n+1}/D_n = 1/2^{1/3} \). He was able to make comparisons with published physiological data of vascular systems in dogs and found that the derived parameters are in good agreement with observed parameters. Existence of engineering phenomenon such as fluid mechanics in human body led Horsfield [4-11] and Wilson [12] to understand the geometrical parameters in human bronchia. They also found good accord in comparisons made between theoretical and observed parameters such as fluid channel lengths and diameters and angles of bifurcation in the bronchial system.

In the second half of 20\(^{th}\) century a new concept of deriving order in nature known as fractal theory was initiated by Mandelbrot [13-15]. According fractal theory, a repeating property in natural systems arises because of an inert fractal dimension. Some natural systems indeed are observed to posses such dimension as shown in Figure 1.3. Fractal theory led researchers to create engineering systems that also possess the repeating dimension. Chen and Cheng [16,17] created microchannels using fractal networks for cooling a heat generating
volume that find application in electronics and MEMS. Ghodoossi [18] conducted the hydrodynamic and thermal analysis on similar fractal microchannel networks.

Though fractal theory became prominent in the second half of the last century, it is also observed that fractal theory cannot be applied to all physical systems with absolute agreement, particularly when we approach systems at microscale [19]. In 1996, A. Bejan initiated a new hypothesis called constructal theory to explain the natural systems. According to the constructal principle – “for a finite-size flow system to persist in time to survive its configuration must evolve in such a way that it provides an easier access to the currents that flow through it”. It is, at its foundation, a theory of optimization. Any system with a flow medium can be optimized using constructal theory. Using constructal theory, it became possible to optimize the fluid systems [20-34] where the familiar diameter ratio $D_{n+1}/D_n = 1/2^{1/3}$ was derived under design constraints and optimization. This theory proved to be very effective in optimizing engineering system that involve flow such as in fluid dynamics, heat transfer, electrical networks, etc. Many researchers have used constructal theory for optimizing practical applications. Arion et al. [35] used constructal networks for distribution of electrical power. Errera et al. [36] used constructal flow networks in composite porous media. Escher et al. [37] optimized constructal microchannel networks for applications in electronics cooling. Fan et al. [38-40] conducted numerical and
experimental investigation on microchannel heat exchangers designed using constructal approach. Luo et al. [41, 42, 47] studied constructal distributor in a mini cross-flow heat exchanger. Morega et al. [43] optimized photovoltaic array using constructal theory. Ordonez et al. [44] conducted system level optimization on heat and flow systems. Senn et al. [45, 46] used constructal channels in cooling fuel cell bipolar plates. Vargas et al. [49, 50] conducted a complete optimization of PEM fuel cell using constructal theory. Gosselin et al. [51] considered the optimization of channels with geometrical asymmetries. Lorente et al. [52] and Queirosconde et al. [53] conducted parabolic scaling of constructal networks and defined the effects of junction losses in terms of geometry svelteness. Wechsatol et al. [54-58] optimized flow channels while considering local junction losses and obtained limits where the junction losses become negligible and can be ignored.

So far, a great portion of the literature on optimization of flow channels using constructal theory concentrated primarily on symmetric designs. Natural systems and many engineering systems, however, utilize channels that bifurcate in asymmetric proportions. Optimization of such channel designs is studied in this research. Chapter 2 deals with the constructal dendritic geometry and the existence of asymmetric bifurcation, as well as optimization of channels and flow networks on circular disc-like surfaces. The relation between mass flow rate and the corresponding flow resistance in a channel has been derived in this chapter. Also, the elemental Y-shape with asymmetry due to mass imbalance is optimized.

Chapter 3 deals with the optimization of elemental T-shaped channels. Optimization of symmetric T-shaped channels as well as channel that are asymmetric in geometry and mass distribution are discussed in this chapter. Chapter 4 considers optimization of Y-shaped channels. Along with optimizing asymmetric Y-shaped channels, this chapter also determine the limits where optimal Y-shaped channels become V-shaped channels.

Chapter 5 is dedicated to optimization of T-shaped channels in 3D. Channels with three, four and five outlets are considered for optimization for symmetric geometries. In asymmetric channel optimization, however, only T-shaped channels with 3 outlets are considered. The effects of gravity on optimal geometrical characteristics are discussed for three different orientations. Chapter 6 is the summary of the results of chapters 2 -5 and a brief discussion of the future work.
2.1 Effect of resistance on mass flow rate in a pipe

In this section we examine the effect of flow resistance on mass flow rate of fluid flowing through a pipe. Let us consider a simple model shown in Figure 2.1. Fluid flows through a straight pipe with round cross-section of length \( L \) and diameter \( D \) at mass flow rate \( \dot{m}_0 \). We assume that the flow is fully developed in laminar regime. In Hagen-Poiseuille flow through round cross-sectional shape duct, the flow resistance can be calculated from

\[
\frac{\Delta P_i}{\dot{m}_i} = \frac{128\nu L_i}{\pi D_i^4}
\]  

(2.1)

where \( \Delta P_i \), \( \dot{m}_i \), \( L_i \) and \( D_i \) are the pressure drop, mass flow rate, length, and diameter of each duct, respectively.

Figure 2.1 Effect of flow resistance on mass flow rate of a fluid that leaks through a hole or a valve
In Figure 2.1 we alter the flow situation by punching a hole at length $L_0$ from the entrance in order to change the total flow resistance of the pipe. When the hole is small compared to the diameter of the pipe: the pressure drop across the hole is high, there is small leak though the hole ($\dot{m}_1 > \dot{m}_2$). When the hole is large: the pressure drop across the hole is small compared to the straight pipe, all fluid then leak though the hole ($\dot{m}_2 = \dot{m}_0$). In the last figure of Figure 2.1, the hole is replaced by a valve. The mass flow rate through both outlets can be controlled by adjusting the flow resistance of the valve. Let us define the fraction of mass flow through the valve $\gamma$ as,

$$\gamma = \frac{\dot{m}_2}{\dot{m}_0}$$

The pressure drop along the pipe can be calculated from

$$\frac{\Delta P_1}{\dot{m}_0} = \frac{128\nu L_0}{\pi D^4} \left[ 1 + \left( 1 - \gamma \right) \left( \frac{L}{L_0} - 1 \right) \right]$$

while the pressure drop between the entrance and outlet at the valve is

$$\frac{\Delta P_2}{\dot{m}_0} = \frac{128\nu L_0}{\pi D^4} \gamma R_v$$

where $R_v$ is the flow resistance at the valve. The terms on the right side of Equations (2.3) and (2.4) are the flow resistance of the pipe. If both outlets connect to the same reservoir, they both have the same outlet pressure ($\Delta P_1 = \Delta P_2$). Therefore the mass fraction $\gamma$ can be calculated from

$$\gamma = \frac{1}{\frac{1}{R_v} + \frac{128\nu (L - L_0)}{\pi D^4}}$$

Equation (2.5) and Figure 2.1 demonstrate the effect of resistance to the fraction of mass flow rate. Equation (2.5) can also be used to calculate the flow resistance of a hole or valve when the fraction of the leaking fluid ($\gamma = \dot{m}_2 / \dot{m}_0$) is known. The mass fraction $\gamma$ is equal to one-half (or $\dot{m}_1 = \dot{m}_2$) when

$$R_v = \frac{128\nu (L - L_0)}{\pi D^4}$$
in other words, the flow resistance of the valve $R_v$ is the same as the flow resistance in the second part of the pipe (between the valve and the end of the pipe). If $R_v > 128\nu(L - L_0)/\left(\pi D^4\right)$, then $\dot{m}_2 < \dot{m}_1$. Note that $R_v \to \infty$ when the valve is fully closed.

Figure 2.2 (a) asymmetric geometry of dendritic trees when $n_p = 2$ and $n_0 = 3$, (b) optimal geometry of dendritic trees when $n_p = 2$ and $n_0 = 3$, (c) optimal geometry of dendritic trees when $n_p = 4$ and $n_0 = 3$, and (d) optimal geometry of dendritic trees when $n_p = 1$ and $n_0 = 6$

2.2 Symmetrical dendritic networks when the flow is divided in half at every bifurcation node

In this section we consider the optimal geometry of dendritic trees that connect one point source to multiple points situated equidistantly around the source. Wechsatol et al. [33] obtained the optimal dendritic configurations assembled with round cross-sectional shape ducts by
minimizing the flow resistance applying the optimal size step in diameter proposed by Cohn [3] in the optimization algorithm. In previous studies [26-30] the flow of fluid is divided into half at every bifurcation node and the symmetrical geometry of the tree-networks is presumed. Whether the optimal dendritic trees are symmetric or asymmetric forms the scope of this section.

Let us consider the asymmetric dendritic networks with two levels of pairing \((n_p = 2)\) that connect the disc center with 12 points along the perimeter in Figure 2.2a. Fluid is pumped through the disc center and divided in half at each bifurcation node in order to deliver flow to the outlets along the perimeter at the same mass flow rate. Unlike in previous studies [26-30], the symmetrical condition of dendritic trees is not presumed, in other words the two branches at the second level of pairing in Figure 2.2a (ducts that touch the rim) do not need to have the same length. The position of the second bifurcation (or the angle \(\alpha\)) is allowed to vary. The size step in diameter at each node of bifurcation is also free to vary. For simplicity the effect of junction losses is assumed negligible. The fluid properties are assumed constant. By introducing the length scale,

\[
\hat{L}_i = L_i / b
\]

where \(b\) is the disc radius, the total duct volume of networks in Figure 2.2a can be calculated from

\[
V_c = n_0 \frac{\pi}{4} b \left( \hat{L}_0 D_0^2 + 2 \hat{L}_1 D_1^2 + 2 \hat{L}_2 D_2^2 + 2 \hat{L}'_2 D'_2^2 \right) \tag{2.8}
\]

where \(n_0\) is the number of ducts that reach the disc center, i.e. \(n_0 = 3\) in Figure 2.2a. Combining Eq. (2.1) with (2.8), the flow resistance between the disc center and the outlet of duct \(L_2\) can be calculated from

\[
\frac{R_f}{n_0} = \left( \hat{L}_0 + \frac{1}{2} \left( \frac{D_0}{D_1} \right)^4 \hat{L}_1 + \frac{1}{4} \left( \frac{D_0}{D_2} \right)^4 \hat{L}_2 \right) \left( \hat{L}_0 + 2 \left( \frac{D_1}{D_0} \right)^2 \hat{L}_1 + 2 \left( \frac{D_2}{D_0} \right)^2 \hat{L}_2 + 2 \left( \frac{D'_2}{D_0} \right)^2 \hat{L}'_2 \right)^2 \tag{2.9}
\]

and the flow resistance between the disc center and the outlet of duct \(L'_2\) is

\[
\frac{R_f}{n_0} = \left( \hat{L}_0 + \frac{1}{2} \left( \frac{D_0}{D_1} \right)^4 \hat{L}_1 + \frac{1}{4} \left( \frac{D_0}{D'_2} \right)^4 \hat{L}'_2 \right) \left( \hat{L}_0 + 2 \left( \frac{D_1}{D_0} \right)^2 \hat{L}_1 + 2 \left( \frac{D_2}{D_0} \right)^2 \hat{L}_2 + 2 \left( \frac{D'_2}{D_0} \right)^2 \hat{L}'_2 \right)^2 \tag{2.10}
\]

where the flow resistances of dendritic networks are defined by
\[ R_f = \frac{\Delta P}{8\pi \nu m} \frac{V_c^2}{b^3} \quad \text{and} \quad R'_f = \frac{\Delta P'}{8\pi \nu m} \frac{V_c^2}{b^3} \quad (2.11) \]

where \( m \) is the total mass flow rate. In case that all outlets along the perimeter in Figure 2.2a have the same pressure, \( R_f \) must be equal to \( R'_f \). The configuration of dendritic trees in Figure 2.2a was optimized in such the way that both \( R_f \) and \( R'_f \) were minimized. The constraints are the total duct volume \( V_c \) and the disc size \( b \). The optimal configuration appears to be symmetric as shown in Fig. 2b (\( L_{-2,\text{opt}} = L'_{-2,\text{opt}} \)). Figure 2.2c shows the symmetrical geometry of the optimal dendritic trees with four levels of pairing (\( n_p = 4 \)) and \( n_0 = 3 \). The optimal size step in diameter at every node of bifurcation (\( D_{1+1}/D_1 \))\text{opt} agrees with the optimal value suggested by Cohn [3].

The optimal geometries of dendritic trees in a disc up-to seven levels of pairing are reported in Tables I-VI. Tables I-VI show that there are certain optimal angles at every node of bifurcation, \( \beta_{i,\text{opt}} \).

For the dendritic trees with only one level of pairing (\( n_p = 1 \)) in Figure 2.2d, the optimal angle \( \beta_1 \) is always equal to 37.47°. Figure 2.2d shows the dendritic trees with one level of pairing connecting the center with 12 points situated equidistantly along the circular perimeter. The optimal length and the minimum flow resistance of dendritic trees with one level of pairing depend only on \( n_0 \) and can be calculated from

\[
\hat{L}_{0,\text{opt}} = \cos \left( \frac{\pi}{2n_0} \right) \frac{\sin(\pi/2n_0)}{\tan(37.47^\circ)}
\]

\[
\hat{L}_{1,\text{opt}} = \frac{\sin(\pi/2n_0)}{\sin(37.47^\circ)}
\]

and

\[
R_{f,\text{min}} = n_0 \left[ \cos \frac{\pi}{2n_0} + \sin \frac{\pi}{2n_0} \left( \frac{2^{1/3}}{\sin(37.47^\circ)} - \frac{1}{\tan(37.47^\circ)} \right) \right]^3 \quad (2.12)
\]

Equation (2.12) and Tables I-VI shows that the optimal geometry is robust. The symmetrical trees serve as the optimal dendritic geometry to distribute flow at the same mass flow rate and pressure to all outlets situated equidistantly around the source when the flow is divided into half at every node of bifurcation.
2.3 Assembling symmetrical dendritic trees with square cross-sectional shape channels

Microchannels generally defined as channels that have hydraulic diameter in the range of 1 μm - 1mm, provide high heat transfer rate due to their large surface to volume ratio. Square cross-sectional shape flow channels are easier to manufacture than round cross-sectional shape channels. Most flow channels in fuel cells have square cross-section [45]. Wechsato et al. [33] showed that square cross-sectional shape channels have the smallest pressure drop among any rectangular channels of the same cross-sectional area and length.

The Hagen-Poiseuille relation, Eq. (2.1) is valid only for fully developed laminar flow through round cross-sectional shape ducts [59]. In this section we start with developing the formula to calculate the pressure drop of unidirectional laminar flow through the square cross-sectional shape duct of length L and side a, in Figure 2.3. Begin with the Poisson equation

\[
\frac{\partial P}{\partial x} = \mu \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)
\]  

(2.13)
where \( u \) is the velocity component of flow along x direction. The directions x, y and z are shown in Figure 2.3. The pressure gradient along flow direction x in Eq. (2.13) can be calculated from \( \frac{\partial P}{\partial x} = -\Delta P/L \). Introducing the dimensionless terms,

\[
\tilde{y}, \tilde{z} = \frac{(y, z)}{a/2}
\]

into Equation (2.13), we obtain

\[
\frac{\partial^2 u}{\partial \tilde{y}^2} + \frac{\partial^2 u}{\partial \tilde{z}^2} = -\frac{\Delta P a^2}{4 \mu L}
\]

(2.15)

The velocity component \( u \) on the wall of the channels is equal to zero due to the no-slip condition.

A second-order finite different method was used to find the velocity profile and average velocity across the square cross-section. The uniform grid was generated and refined by half until the step change in the Frobenius norm of velocity component \( u \) was less than 0.2 percent. The approximated velocity profile is within 0.2 percents of the exact solution [59],

\[
u(\tilde{y}, \tilde{z}) = \frac{16}{\pi^3} \left( \frac{\Delta Pa^2}{4 \mu L} \right) \sum_{j=1, 3, 5, \ldots}^{\infty} \frac{(-1)^{(j-1)/2}}{j^3} \left[ 1 - \frac{\cosh(j \pi \tilde{y}/2)}{\cosh(j \pi/2)} \right] \cos\left( \frac{j \pi \tilde{z}}{2} \right)
\]

(2.16)

The average velocity across the cross-section \( \bar{u} \) is linearly proportional to \( \Delta Pa^2/4 \mu L \) as shown in Figure 2.3,

\[
\bar{u} = 0.1361 \frac{\Delta Pa^2}{4 \mu L}
\]

(2.17)

From Eq. (2.17) the pressure drop across a square cross-sectional shape duct can be calculated from

\[
\frac{\Delta P}{\bar{m}} = 29.39 \nu \frac{L}{a^4}
\]

(2.18)

Equation (2.18) is different from Eq. (2.1). Equation (2.18) shows that the pressure drop of flow through a square cross-sectional shape duct is about 16% greater than the pressure drop across a round cross-sectional shape duct of the same cross-section area and length. The increase of pressure drop is the consequence of larger perimeter and losses at the corners. Geometrically the perimeter of a square cross-section is about 13% greater than the one of a round cross-section of the same cross-sectional area. If we apply the concept of hydraulic diameter (\( D_h = a \)) into Eq.
(2.1), we can see that Eq. (2.1) does not give an accurate approximation of the pressure drop and flow resistance of square cross-sectional shape ducts. In this section we employ Eq. (2.18) to calculate the pressure drop and flow resistance of dendritic tree networks assembled with square cross-sectional shape ducts.

Let us consider the symmetrical dendritic trees in a disc with square cross-sectional shape ducts by starting with one pairing trees similar to Figure 2.2d. The total volume of all square ducts in one pairing trees can be calculated by

$$V_c = n_0 \left( L_0 a_0^2 + 2 L_1 a_1^2 \right)$$  \hspace{1cm} (2.19)

From Eq. (2.18) and (2.19), the flow resistance of one pairing trees assembled with square cross-sectional shape ducts can be calculated from

$$R_f = \frac{\Delta P}{29.39 \nu m} \frac{V_c^2}{b^3} = n_0 \left( \hat{L}_0 + \frac{1}{2} \left( \frac{a_0}{a_1} \right)^4 \hat{L}_1 \right) \left( \hat{L}_0 + 2 \left( \frac{a_1}{a_0} \right)^2 \hat{L}_1 \right)$$  \hspace{1cm} (2.20)

The lengths of each duct in Eq. (2.20) are geometrically related to the pairing angle $\beta_1$ by

$$\hat{L}_0 = \cos \left( \frac{\pi}{2n_0} \right) - \sin \left( \frac{\pi}{2n_0} \right) \tan(\beta_1)$$

$$\hat{L}_1 = \frac{\sin \left( \frac{\pi}{2n_0} \right)}{\sin(\beta_1)}$$  \hspace{1cm} (2.21)

By varying the angle of pairing ($\beta_1$) and the size step of ducts ($a_1/a_0$), the optimal geometry of one pairing trees assembled with square cross-sectional shape ducts is obtained when $\beta_1, \text{opt} = 37.47^\circ$ and $\left( a_1/a_0 \right), \text{opt} = 2^{-1/3}$. The result agrees with the optimal geometry of one pairing trees assembled with round cross-sectional shape ducts in section 2.2.
We continue the search for the optimal configurations of trees with two levels of pairing similar to Figure 2.2b. The flow resistance of the two pairing trees assembled of square cross-sectional shape ducts can be calculated by

\[
R_f = n_0 \left( \hat{L}_0 + \frac{1}{2} \left( \frac{a_0}{a_1} \right)^4 \hat{L}_1 + \frac{1}{4} \left( \frac{a_0}{a_2} \right)^4 \hat{L}_2 \right) \left( \hat{L}_0 + 2 \left( \frac{a_1}{a_0} \right)^2 \hat{L}_1 + 4 \left( \frac{a_2}{a_0} \right)^2 \hat{L}_2 \right)^2
\]  

(2.22)

The minimal flow resistance in Eq. (2.22) is obtained when \( \left( a_1/a_0 \right)_{\text{opt}} = \left( a_2/a_1 \right)_{\text{opt}} = 2^{-1/3} \). The optimal lengths of each square cross-sectional shape duct in two pairing trees are the same as the optimal geometry of two pairing trees assembled with round cross-sectional shape ducts reported in Table I. We further investigate the optimal geometry of dendritic trees assembled with square cross-sectional shape channels up-to seven levels of pairing. The results suggest that the optimal size step at each node of bifurcation recommended by Cohn [3] and the optimal configurations shown in Tables I-VI are valid for both square and round cross-sectional shape dendritic tree...
networks. Figure 2.4 shows the minimal flow resistance of both square and round cross-sectional shape dendritic tree networks. The flow resistance in Figure 2.4 is defined by

\[
R_f = \frac{\Delta P}{C_{vm}} \frac{V_c^2}{b^4} \quad \text{for round shape cross-section}
\]

\[
C = 8\pi \quad \text{for square shape cross-section}
\]

(2.23)

Figure 2.4 shows that the geometry of dendritic trees becomes more complex (higher level of pairing \( n_p \)) to decrease the flow resistance as the number of points on the perimeter \( N \) increases.

Figure 2.4 also shows the envelope in performance of symmetrical dendritic flow networks to deliver flow. Wechsatol et al. [33] reported a similar set of performance curves for symmetrical dendritic trees assembled with only round cross-sectional shaped channels.

2.4 Asymmetry of bifurcation (the elemental Y and V shapes)

In this section we consider how to connect one point source to two point sinks situated equidistantly from the source. The three points form a rectangular shape area \( S \) with aspect ratio \( x/y \), shown in Figures 2.5 and 2.10. Whether the optimal shape of the networks that connect the three points is Y or V shapes depends on the aspect ratio of the occupied area \( S \) as will be shown in this section. Figure 2.5 shows the effect of the mass fraction \( \gamma \) and the aspect ratio \( x/y \) of the occupied area \( S \) on the optimal shape of Y-shaped assemblies. The mass fraction \( \gamma \) is defined by the ratio between the mass flow rate in the duct \( L_2 \) and the total mass flow rate \( \dot{m} \) through the Y-shaped assemblies (\( \gamma = \dot{m}_2/\dot{m} \)). The flow resistance between the flow entrance and the two exits can be calculated from

\[
R_{f_1} = \frac{\Delta P_1}{C_{vm}} \frac{V_c^2}{S^{3/2}} = \left( \hat{L}_0 + (1-\gamma)\hat{L}_1 \left( \frac{D_0}{D_1} \right)^4 \right) \left( \hat{L}_0 + \hat{L}_1 \left( \frac{D_1}{D_0} \right)^2 + \hat{L}_2 \left( \frac{D_2}{D_0} \right)^2 \right)^2
\]

and

\[
R_{f_2} = \frac{\Delta P_2}{C_{vm}} \frac{V_c^2}{S^{3/2}} = \left( \hat{L}_0 + \gamma\hat{L}_2 \left( \frac{D_0}{D_2} \right)^4 \right) \left( \hat{L}_0 + \hat{L}_1 \left( \frac{D_1}{D_0} \right)^2 + \hat{L}_2 \left( \frac{D_2}{D_0} \right)^2 \right)^2
\]

(2.24)

where the constant \( C \) is reported in Eq. (2.23). The length scale of each duct in this section (for both Y and V shapes) is defined by
\[ \hat{L}_i = \frac{L_i}{S^{1/2}} \] (2.25)

The length of each duct in Y shaped assemblies in Figure 2.5 relates to the bifurcation angles \( \theta_i \) and aspect ratio \( x/y \) by

\[ \hat{L}_0 \sin \theta_0 + \hat{L}_1 \sin \theta_1 = \frac{1}{2} (x/y)^{1/2} \]

\[ \hat{L}_0 \cos \theta_0 + \hat{L}_1 \cos \theta_1 = (x/y)^{-1/2} \]

and

\[ \hat{L}_2 \cos \theta_2 = \hat{L}_1 \cos \theta_1 \] (2.26)

Figure 2.5 Geometry of the optimal Y-shaped assemblies at specific aspect ratios of the occupied area \( (x/y) \) and specific mass fractions \( (\gamma) \)
The geometry of Y assemblies was optimized in such a way that both $R_{f_1}$ and $R_{f_2}$ were minimized. The constraints are the total duct volume $V_c$ and the size of occupied area $S$. The effect of flow fraction ($\gamma$) on the optimal shape of Y assemblies was taken into account. The optimization shows that the optimal geometry not only provides the lowest flow resistance but also balances the resistance ($R_{f_1} = R_{f_2}$).

Figure 2.5 shows that the duct $L_0$ shortens as the aspect ratio ($x/y$) increases. The duct $L_1$ becomes shorter and the duct $L_2$ longer as the flow fraction $\gamma$ decreases. The symmetry of Y-shaped assemblies appears when mass flow rates in both ducts $L_1$ and $L_2$ are the same ($\gamma = 0.5$). The corresponding optimal bifurcation angles $\theta_i$ are reported in Figure 2.6. Figure 2.7 reports the optimal lengths of each duct in Y-shaped assemblies when the aspect ratio ($x/y$) is between 0.5 and 1.5. The stem duct $L_0$ vanishes when the aspect ratio is 1.5. The V-shaped assemblies perform better than Y-shaped assemblies, when the aspect ratio $x/y$ is greater than 1.5. Figure 2.8 reports the optimal length ratios corresponding to the optimal lengths in Figure 2.7.

The optimal size step of ducts in Y-shaped assemblies ($D_i/D_0$)$_{opt}$ is reported in Figure 2.9. The optimal size step is almost insensitive to the aspect ratio ($x/y$). The size of the duct $D_2$ becomes smaller as the fraction of flow through the duct ($\gamma$) decreases. Figure 2.9 shows that the optimal size step at each bifurcation node recommended by Cohn [3] is valid only when the flow fraction ($\gamma$) at the bifurcation nodes is equal to one half.
Figure 2.6 The optimal angles of Y-shaped assembly corresponding to the aspect ratio of the occupied area $(x/y)$ and mass fraction $(\gamma)$

Figure 2.7 The optimal length of Y-shaped assembly corresponding to the aspect ratio of the occupied area $(x/y)$ and mass fraction $(\gamma)$
Figure 2.10 reports the optimal ratio of duct sizes in V-shaped assemblies when the aspect ratio (x/y) of the occupied area (S) is larger than 1.5. The flow resistance of the V-shaped assemblies can be defined as
\[
R_{f, V_1} = \frac{\Delta P_{L_1}}{C_{vm}} \frac{V_C}{S^{3/2}} = (1 - \gamma) \hat{L}_1 \left( \hat{L}_1 + \hat{L}_2 \left( \frac{D_2}{D_1} \right)^2 \right)^2
\]
and
\[
R_{f, V_2} = \frac{\Delta P_{L_2}}{C_{vm}} \frac{V_C}{S^{3/2}} = \gamma \hat{L}_2 \left( \hat{L}_1 + \hat{L}_2 \left( \frac{D_2}{D_1} \right)^2 \right)^2
\]
(2.27)

where the constant C is defined in Eq. (2.23). The lengths of both ducts in Figure 2.10 are the same. The optimal geometry of V-shaped assemblies is obtained when both resistances in Eq. (2.27) are minimized. The optimization algorithm shows that the minimum values of \( R_{f, V_1} \) are equal to the minimum values of \( R_{f, V_2} \). The optimal size of each duct in the V shaped assemblies depends only on the flow fraction (\( \gamma \)). The optimal duct size is independent of the aspect ratio (\( x/y \)) of the occupied area. The two ducts in V-shaped assemblies in Figure 2.10 have the same size when the mass flow rates in both ducts are the same (\( \gamma = 0.5 \)). The duct \( D_2 \) becomes smaller as the fraction of flow through the duct (\( \gamma \)) decreases.

Figure 2.11 shows the limit in performance to deliver flow between one point source to the other two points situated equidistantly from the source. Figure 2.11 shows the minimal flow resistance of both Y and V shaped assemblies. The optimal Y shaped assemblies exist when the aspect ratio (\( x/y \)) is lower than 1.5. The V-shaped assemblies provide lower flow resistance than Y-shaped assemblies when the aspect ratio (\( x/y \)) of the occupied area is greater than 1.5. Asymmetry of mass flow rate and geometry of Y and V-shaped assemblies provides lower flow resistance than symmetric Y and V-shaped assemblies as observed in Figures 2.4 and 2.11. The flow resistance in Figure 2.11 reaches the minimum point in the domain of V-shaped assemblies when the aspect ratio (\( x/y \)) is equal to 2. Both ducts in V-shaped assemblies are shortest when the aspect ratio (\( x/y \)) is equal to 2.
Throughout this paper the effect of junction losses on the optimal geometry of bifurcation is assumed negligible. Wechsatol et al. [56] studied the effect of junction losses on the optimal geometry of bifurcation. They concluded that in case of laminar flow, the junction losses have sizable effects on the optimal diameter ratio at each node of bifurcation only when the svelteness $S_v$, which is defined by the ratio between the external and internal length scales is lower than the square root of 10, and that the junction losses have no effects on the optimal length ratio.

### 2.5 Asymmetry of dendritic trees networks

This section examines the relation between the geometry and the flow resistance of tree-like networks. The fraction of flow throughout networks can be controlled by controlling the resistance of each flow path. Even though the pressure drops across round and square cross-sectional shape ducts are different; the flow resistance of networks constructed by both duct shapes can be defined by a similar relation.

Section 2.4 presented the criteria to select between Y- and V- shaped assemblies in the simplest networks that connect one point source to other two points (sinks) situated equidistantly from the source. The three points in the Y- and V-shaped assemblies form a rectangular shape.
area. The Y-shaped assembly performs better than the V-shaped assembly when the aspect ratio of the occupied area is lower than 1.5.

Whether the optimal flow networks are symmetric or asymmetric depends on the fraction of flow \((\gamma)\) at each bifurcation node in the networks. Asymmetry of bifurcation (asymmetry of Y-shaped assemblies) appears when the flow fraction at bifurcation nodes is not equal to one half. In sections 2.2 and 2.3 we discussed the symmetrical geometry of dendritic trees. However asymmetric bifurcations have lower flow resistance than symmetric bifurcations as shown in Figure 2.11. When the geometry of flow networks does not need to be symmetric (or the flow fraction at each bifurcation node \(\gamma\) does not need to be one half), the flow structures would become asymmetric in order to lower the flow resistance.

Consider the asymmetric tree that distributes flow from one point source to 10 sinks as shown in Figure 2.12a. The geometry of each bifurcation in Figure 2.12a follows the rules of asymmetric bifurcation proposed in section 2.4 (Figures 2.6-2.9). The flow fractions \((\gamma)\) at nodes a, b, c, and d are equal to 0.2, 0.25, 0.333 and 0.5, respectively. It is, however, not possible to
simply connect Y-shaped assemblies optimized in section 2.4 to obtain the required mass fractions. It is because of the difference in pressure drop between different levels of branching. The optimal geometrical characteristics can only be calculated successively beginning from the farthest outlet and match the pressure drop at all outlets. Optimization procedure for such network is given in Chapter 4. Figure 2.12b shows the network connected in such a way that 30 outlets can be connected to a source and mass is divided into equal proportions at all outlets.

Asymmetry is employed by nature and man-made flow networks in order to lower flow resistance. By designing the flow geometry we control the resistance and the flow fraction in each path of the flow network.
CHAPTER 3

ASYMMETRIC BIFURCATION OF T-SHAPED CHANNELS

3.1 Introduction

A T-shaped flow channel is a simple form of Y-shaped construct where the branch sections bifurcate at right angles from the stem section as shown in Figure 3.1. T-shaped channels are widely used in engineering applications such as utility water supply, chemical transport and HVAC ducts. Though Figure 3.1 depicts a T-shaped channel with symmetric geometry and mass bifurcation, it is not unusual to find channels with mass imbalance or geometrical asymmetries. The optimization of such T-shaped designs with mass and geometrical asymmetry is discussed in this chapter.

![Figure 3.1 T-shaped flow channel with symmetric geometry and mass bifurcation](image)

The constructal optimization of T-shaped flow channel is similar to the procedure discussed in the previous chapter. Assuming the channels in Figure 3.1 are circular in cross section, we can write the expression for pressure drop in the T-shape using Equation 2.1 as
The non-dimensional flow resistance number $R_f$ can be defined as

$$R_f = \frac{\Delta P}{8\pi \nu m^2} \frac{V_c^2}{S^{3/2}}$$

where $S$ is area of the rectangle that bounds the channels and $V_c$ is the volume occupied by the channels.

$$S = 2L_0L_1$$

$$V_c = \frac{\pi}{4} \left[ L_0D_0^2 + 2L_1D_1^2 \right]$$

Using Equations 3.1, 3.3 and 3.4, we can $R_f$ such that

$$R_f = \left[ \frac{L_0}{D_0^4} + \frac{1}{2} \frac{L_1}{D_1^4} \right] \left( \frac{L_0D_0^2 + 2L_1D_1^2}{S^{3/2}} \right)^2$$

By introducing non-dimensional length scales $\hat{L}_1 = L_1/S^{1/2}$, we can further modify $R_f$ such that

$$R_f = \hat{L}_1^3 \left[ 1 + \frac{x}{2y^4} \right] \left[ 1 + 2x y^2 \right]^2$$

where $x = \hat{L}_1/\hat{L}_0 = L_1/L_0$ is the ratio of lengths and $y = D_1/D_0$ is the ratio of diameters.

Using the expression $x = \hat{L}_1/\hat{L}_0$ and the fact that $2\hat{L}_0\hat{L}_1 = 1$, we can eliminate $\hat{L}_1$ from Equation 3.6 and rewrite $R_f$ such that

$$R_f = \frac{1}{(2x)^{3/2}} \left[ 1 + \frac{x}{2y^4} \right] \left[ 1 + 2x y^2 \right]^2$$

At a constant mass flow rate and constrained area of rectangle $S$ and channel volume $V_c$, we can minimize the pressure drop by minimizing the $R_f$ in Equation 3.7. Here, the length and diameter ratios, $x$ and $y$ respectively, are the degrees of freedom in the optimization, and we can compute the combination of $x$ and $y$ where $R_f$ has its minimum value ($R_{f,\text{min}}$) for T-shaped channels.
Bejan et al. [26] reported that a symmetric T-shaped channel with equal mass bifurcation has $R_{f,\text{min}}$ value of 4 when $x = 1/2^{1/3}$ and $y = 1/2^{1/3}$. It creates a T-shaped channel bound by a rectangle of aspect ratio $2^{2/3} \sim 1.58$.

### 3.2 Mass induced asymmetry in T-shaped channels

Figure 3.2 T-shaped channel with asymmetric mass bifurcation

![Figure 3.2 T-shaped channel with asymmetric mass bifurcation](image)

Figure 3.2 shows a symmetric T-shaped channel with asymmetric mass bifurcation. The diameters of the branch sections A and B are $D_1$ and $D_2$, respectively. Of the total mass flow rate $\dot{m}$, only a fraction $\gamma \dot{m}$ flows through branch A and the rest $(1-\gamma)\dot{m}$ flows through branch B. Because of the geometrical symmetry the length of the branch sections are equal to each other and equal to $L_1$. The pressure drop in the branch sections have to be equal to each other to maintain a continuous and constant (but not necessarily equal) flow rate. The total pressure drop in the T-shape can be written as

$$
\Delta P = \frac{128}{\pi} \dot{m} v \left[ \frac{L_0}{D_0^4} + \gamma \frac{L_1}{D_1^4} \right] = \frac{128}{\pi} \dot{m} v \left[ \frac{L_0}{D_0^4} + (1-\gamma) \frac{L_1}{D_1^4} \right] = \frac{128}{\pi} \dot{m} v \left[ \frac{L_0}{D_0^4} + (1-\gamma) \frac{L_1}{D_1^4} \right]$$

(3.8)

The volume $V$ occupied by the channels is
The non-dimensional flow resistance number \( R_f \) can be written as

\[
R_f = \frac{\Delta P}{8\pi \mu v S^{3/2}} V^2 = \left[ \frac{L_0}{D_0^4} + \gamma \frac{L_1}{D_1^4} \right] \left( \frac{L_0 D_0^2 + L_1 D_1^2 + L_1 D_2^2}{S^{3/2}} \right) \tag{3.10}
\]

By including the geometric parameters of section B, we can rewrite the Equation for \( R_f \) as

\[
R_f = \left[ \frac{L_0}{D_0^4} + \gamma \frac{L_1}{2 D_1^4} + \frac{(1-\gamma)}{2} \frac{L_1}{D_2^4} \right] \left( \frac{L_0 D_0^2 + L_1 D_1^2 + L_1 D_2^2}{S^{3/2}} \right) \tag{3.11}
\]

This presumption is valid because the pressure drop in both the branch sections is equal to each other. Equation 3.11 can be reduced further by using non-dimensional length scales \( \hat{L}_i = L_i / S^{1/2} \) such that

\[
R_f = \frac{1}{(2x)^{3/2}} \left[ 1 + \frac{\gamma x}{2 y_1^4} + \frac{(1-\gamma)x}{2 y_2^4} \right] \left( 1 + xy_1^2 + xy_2^2 \right)^2 \tag{3.12}
\]

where \( x = \hat{L}_1 / \hat{L}_0 = L_1 / L_0 \) is the ratio of lengths and \( y_1 = D_1 / D_0 \) and \( y_2 = D_2 / D_0 \) are the ratios of diameters. At a constant mass flow rate and constrained area \( S \) and volume \( V \), we can minimize the pressure drop by minimizing the flow resistance number \( R_f \). The degrees of freedom would be the length ratio, \( x \), and diameter ratios, \( y_1 \), and, \( y_2 \).

Figure 3.3 shows the variation in \( R_{f,\text{min}} \) with the mass fraction \( \gamma \). It can be observed that mass asymmetry reduces the flow resistance in the channels. Also, the flow resistance values are symmetric about \( y \) – axis when \( \gamma = 0.5 \). The designs with mass fractions \( \gamma \to 0 \) or \( \gamma \to 1 \) are found to exert least amount of flow resistance. The \( R_{f,\text{min}} \) increases with mass fraction and reaches a maximum values of 4 when \( \gamma = 0.5 \) where the flow divides into equal halves.
Figure 3.3 Variation in $R_{f,\text{min}}$ with mass fraction $\gamma$

Figure 3.4 Variation in $(L_1/L_0)_{\text{opt}}$ with mass fraction $\gamma$
Figure 3.4 shows the variation in optimal length ratio \( \left( \frac{L_1}{L_0} \right)_{opt} \) with mass fraction \( \gamma \). It can be observed that optimal length ratio has its maximum values when the mass fraction \( \gamma \) tend to reach 0 or 1. The optimal length ratio reaches a minimum value of \( 1/2^{1/3} \) when \( \gamma = 0.5 \). The graph is symmetric about \( \gamma = 0.5 \).

Figure 3.5 shows the variation in diameter ratios \( \left( \frac{D_1}{D_0} \right)_{opt} \) and \( \left( \frac{D_2}{D_0} \right)_{opt} \) with mass fraction \( \gamma \). It can be observed that, with increasing mass fraction, \( \left( \frac{D_1}{D_0} \right)_{opt} \) increases whereas \( \left( \frac{D_2}{D_0} \right)_{opt} \) decreases. The graphs are mirror images of each other about mass fraction \( \gamma = 0.5 \). The lines intersect at \( \gamma = 0.5 \) and are equal to \( \left( \frac{D_{1,2}}{D_0} \right)_{opt} = 1/2^{1/3} \). The optimal length and diameter ratio values at \( \gamma = 0.5 \) are in agreement with the values obtained for symmetric T-shaped channels by Bejan et al. [26].

Figure 3.6 shows optimal T-shaped channel designs, drawn to scale, at different mass fractions and corresponding \( R_{f,\text{min}} \), x, y_1 and y_2 values.
3.3 Mass and geometry induced asymmetry in T-shaped channels

In the previous section, T-shaped channels with mass induced asymmetry were optimized for least flow resistance. The geometry is assumed to be symmetric about vertical axis as shown in Figure 3.1. However, it is possible to have T-shaped channels that are asymmetric in both mass distribution as well as geometry. The situation is similar to the problem of connecting three points (one source and two outlets) arbitrarily placed in space using optimal T-shaped channels. Figure 3.7 different designs where a source is connected to outlets using asymmetric T-shaped channels.
Flow optimization of a T-shaped channel with combined mass and geometric asymmetry, as shown in Figure 3.8, is relatively simple compared to designs with only mass induced asymmetry. This is because, in Figure 3.8 we can observe that once the position of source and outlets are defined, the ratio of channel lengths, $L_1/L_0$ and $L_2/L_0$, between stem and branch sections can be readily calculated without requiring an optimization program. The only degrees of freedom in optimization are the diameter ratios $D_1/D_0$ and $D_2/D_0$. From Figure 3.8, the geometric deviation $g$ can be defined as

$$ g = \frac{L_1 - L_2}{2(L_1 + L_2)} \quad (3.13) $$

Values of $g$ varies from 0 to 0.5 such that $g = 0$ represents a symmetric geometry with stem section at the center and $g = 0.5$ represents stem section at one of the ends. We can also define another variable $p$ such that

$$ p = \frac{L_1}{L_1 + L_2} \quad (3.14) $$

We can define the aspect ratio of the rectangle from Figure 3.8 such as

$$ F = \frac{L_1 + L_2}{L_0} \quad (3.15) $$

Once the aspect ratio $F$, geometric deviation $g$ and variable $p$ are calculated, we can calculate the length ratios $x_1$ and $x_2$ in terms of $p$ and $F$. 

---

Figure 3.7 Three points connected using asymmetric T-shaped channels
Figure 3.8 T-shaped channel with mass and geometry induced asymmetry

The pressure drop in the T-shape is equal to the sum of pressure drops in stem and branch sections. The two branch sections have to exert equal flow resistance to maintain a continuous and constant (but not necessarily equal) mass flow rates. Such a way, the total pressure drop in the T-shape can be expressed as

$$
\Delta P = \frac{128}{\pi} \left[ \frac{L_0}{D_0^4} + \frac{\gamma L_1}{D_1^4} \right] = \frac{128}{\pi} \left[ \frac{L_0}{D_0^4} + \left(1 - \gamma \right) \frac{L_2}{D_2^4} \right] \quad (3.17)
$$

Because the pressure drop in branch sections is equal to each other we can modify Equation 3.17 such that

$$
\Delta P = \frac{128}{\pi} \left[ \frac{L_0}{D_0^4} + \frac{\gamma L_1}{D_1^4} + \frac{(1 - \gamma) L_2}{2 D_2^4} \right] \quad (3.18)
$$

The non-dimensional flow resistance number $R_f$ can be defined as

$$
R_f = \frac{\Delta P}{8\pi\mu V C^2 S^{3/2}} \quad (3.19)
$$

where $S$ is the rectangular area under the channels and $V$ is the volume occupied by the channels.
Using Equations 3.18, 3.20 and 3.21, we can write \( R_f \) such that

\[
R_f = \left[ \frac{L_0 + \gamma L_1 + \frac{(1-\gamma) L_2}{2}}{D_0^4 + 2D_1^4 + 2D_2^4} \right] \left( \frac{(L_0D_0^2 + L_1D_1^2 + L_2D_2^2)^{3/2}}{S^{3/2}} \right)^2
\]

Equation 3.22 can further be modified by using non-dimensional length scales \( \hat{L}_i = L_i/S^{1/2} \) such that \( R_f \) becomes

\[
R_f = \frac{1}{(x_1 + x_2)^{3/2}} \left[ 1 + \frac{\gamma x_1^4}{2 y_1^4} + \frac{(1-\gamma) x_2^4}{2 y_2^4} \right] \left( 1 + x_1 y_1^2 + x_2 y_2^2 \right)^2
\]

Here, \( x_1 = \hat{L}_1/\hat{L}_0 = L_1/L_0 \) and \( x_2 = \hat{L}_2/\hat{L}_0 = L_2/L_0 \) are the ratios of the lengths and \( y_1 = D_1/D_0 \) and \( y_2 = D_2/D_0 \) are the ratio of diameters of the channel sections. Using Equation 3.16, we can write the final form of \( R_f \) such that

\[
R_f = \frac{1}{F^{3/2}} \left[ 1 + \frac{\gamma}{2 y_1^4} + \frac{(1-\gamma)(1-p)F}{2 y_2^4} \right] \left( 1 + pFy_1^2 + (1-p)Fy_2^2 \right)^2
\]

When the mass flow rate is constant and the rectangular area \( S \) and the channel volume \( V_c \) are constrained, the pressure drop can be minimized by minimizing \( R_f \). Here, only the diameter ratios \( y_1 \) and \( y_2 \) are the degrees of freedom.
Figure 3.9 Minimum non-dimensional flow resistance $R_{f,\text{min}}$ at different aspect ratios and mass fractions

Figure 3.9 shows the $R_{f,\text{min}}$ values calculated at different mass fractions and aspect ratios when the geometric deviation is varied between 0 and 0.5. It can be observed that variation in flow resistance with mass fraction is very high in designs with very large geometric deviation ($g = 0.499$). When the geometry is symmetric about vertical axis ($g = 0$), the flow resistance values become relatively equal at all mass fractions. It is also noticed that small aspect ratios exert high flow resistance and the flow resistance drops with increasing aspect ratio until it reaches a minimum value and increases again. In general, the flow resistance is observed to be minimum for designs with high mass fraction ($\gamma = 0.9$), large geometric deviation ($g = 0.499$) and aspect ratios between 2 and 3.
Figure 3.10 Variation in optimal diameter ratios and non-dimensional flow resistance $R_{f,min}$ with aspect ratio, geometric deviation and mass fractions

Figure 3.10 shows the optimal diameter ratios at different combinations of aspect ratios, mass fractions and geometric deviations. It can be observed from Figure 3.10a that optimal diameter ratios $(D_1/D_0)_{opt}$ (and subsequently $(D_2/D_0)_{opt}$) are independent of aspect ratio. They only depend upon the mass fraction $\gamma$ and geometric deviation $g$. Figure 3.10b shows the variation in $R_{f,min}$ with geometric deviation at different mass fractions when the aspect ratio is fixed at $F = 2$. We can observe that smaller mass fractions ($\gamma \sim 0.1$) have high flow resistance when the geometric deviation is large ($g \sim -0.499$) and the flow resistance reduces with increasing geometric deviation. For larger mass fractions ($\gamma \sim 0.9$) the flow resistance is minimum when the geometric deviation is large ($g \sim -0.499$) and starts increasing with increasing geometric deviation. The case with equal mass distribution ($\gamma = 0.5$) has flow
resistance reduced with increasing geometric deviation until it reaches a minimum value of 4 when \( g = 0 \) and starts increasing again.

Figures 3.10c and 3.10d show the variation in optimal diameter ratios \( (D_1/D_0)_{opt} \) and \( (D_2/D_0)_{opt} \) at different geometric deviations and mass fractions. We can observe that \( (D_1/D_0)_{opt} \) increases with increasing geometric deviation. Also, larger mass fractions have higher value of \( (D_1/D_0)_{opt} \) compared to smaller mass fractions. \( (D_2/D_0)_{opt} \), on the other hand reduces with increasing geometric deviation. Here, smaller mass fractions have higher values of \( (D_2/D_0)_{opt} \) compared to larger mass fractions.

Figure 3.11 shows optimal T-shaped channels, drawn to scale, designed for multiple combinations of aspect ratios, mass fractions and geometric deviations.

![Figure 3.11 Optimal T-shaped channels with mass and geometry induced asymmetry](image)

\[
\begin{align*}
F &= 2.5, \ g = 0.4, \\
\gamma &= 0.9, \ R_f = 2.0867
\end{align*}
\]

\[
\begin{align*}
F &= 2.25, \ g = 0.3, \\
\gamma &= 0.9, \ R_f = 2.3988
\end{align*}
\]

\[
\begin{align*}
F &= 2, \ g = 0.2, \\
\gamma &= 0.9, \ R_f = 2.726
\end{align*}
\]

\[
\begin{align*}
F &= 2, \ g = 0.1, \\
\gamma &= 0.9, \ R_f = 3.0688
\end{align*}
\]

\[
\begin{align*}
F &= 1.75, \ g = 0, \\
\gamma &= 0.9, \ R_f = 3.4189
\end{align*}
\]

3.4 T-shaped Flow Channel Networks

In the previous section we have optimized elementary T-shaped channels connecting a source to two outlets with mass and geometrical asymmetries. However, in practical
applications, we may require to connect 3 or more outlets to a source using T-shaped channels. Such networks can be designed by connecting multiple elementary channels in an organized fashion.

Two kinds of networks can be designed using T-shaped channels. The first kind of networks is symmetric in geometry and mass distribution. They can be used to supply fluid mass to N number of outlets where N is in multiples of 2, i.e., N = 2, 4, 8, 16 --. Figure 3.12a shows symmetric T-shaped channels with aspect ratio of $2^{1/2}$ connecting 64 outlets in a 2D plane and Figure 3.12b shows symmetric T-shaped channels with aspect ratio of $2^{2/3}$ connecting 128 outlets in 3D.

![Figure 3.12 Symmetric T-shaped channel networks in 2D (64 outlets) and 3D (128 outlets)](image)

The second kind of networks is asymmetric, either due to geometry or mass distribution or both. Asymmetric networks usually are employed when the number of outlets is different than the multiples of 2 and the outlets are placed arbitrarily in 2D or 3D. Figure 3.13 shows an example of such network.

In Figure 3.13a, four outlets are placed in a 2D plane and are to be connected to a source also placed in the same plane. Every outlet is to be supplied with a different fraction of mass flow rate. Easiest way of connecting the outlets to the source would be using individual channels. However, the pumping power required in such configuration would be very high. Another way of connecting the outlets is through a single channel with multiple branching as shown in Figure
3.13b. From Figure 3.13b, we can observe that the channel lengths are predefined by the nature of the problem. Such a way, the only parameters to be optimized are the diameter ratios of the channels.

From Figure 3.13b, we can observe that channel sections F and G are identical in length and mass fractions. If we are able to fix the diameter of channel section F or G to \( d \), using optimal configurations calculated in section 3.2 we can calculate the diameter of channel section E such that it becomes approximately 1.25\( d \). The total pressure drop in channel sections E and F would be

\[
\Delta P_{EF} = \Delta P_E + \Delta P_F = 128 \frac{\pi}{\nu} \cdot \text{mv} \left[ 0.4 \times \frac{0.5L}{(1.25d)^{\frac{1}{4}}} + 0.2 \times \frac{0.5L}{d^{\frac{3}{4}}} \right] = 128 \frac{\pi}{\nu} \cdot 0.18192 \text{mv} \cdot \frac{L}{d^{\frac{5}{4}}}
\]

(3.25)

The total pressure drop in channel sections E and F must be equal to pressure drop in channel section D because they are connected to same junction point. Such a way, the pressure drop in channel section D is

\[
\Delta P_D = 128 \frac{\pi}{\nu} \cdot \text{mv} \left[ 0.4 \times \frac{0.5L}{d_D^{\frac{1}{4}}} \right] = 128 \frac{\pi}{\nu} \cdot 0.18192 \text{mv} \cdot \frac{L}{d^{\frac{5}{4}}}
\]

(3.26)
From Equation 3.26, we get the diameter of channel section D as 
\[ d_D = 1.024d \] (3.27)

Now, we know that the mass fraction in channel section C is 0.8 of which 0.4 is entering channel section D and 0.4 is entering channel section C. From optimal flow configurations, we can obtain the diameter of channel section C such that 
\[ d_C = 1.25d_D = 1.28d \] (3.28)

The total pressure drop in channel sections C and D would be 
\[ \Delta P_D = \frac{128}{\pi} \frac{\nu}{L} \left[ 0.8 \times \frac{0.5L}{d_C^4} + 0.4 \times \frac{0.5L}{d_D^4} \right] = \frac{128}{\pi} \frac{0.331\nu L}{d^4} \] (3.29)

Now, the total pressure drop in channel sections C and D (or C, E and F) must be equal to the pressure drop in channel section B, because they share one bifurcating junction.

\[ \Delta P_D = \frac{128}{\pi} \frac{\nu}{L} \left[ 0.2 \times \frac{0.5L}{d_B^4} \right] = \frac{128}{\pi} \frac{0.331\nu L}{d^4} \] (3.30)

From Equation 3.30, we get the diameter of channel section B as 
\[ d_B = 0.741d \] (3.31)

Now, we observe the mass is bifurcating from channel section A into 0.8 and 0.2 fractions into channel sections C and B, respectively. From the optimal flow configurations of section 3.3, we can write the expression for the diameter of channel section A such that 
\[ d_A = 1.165d_C = 1.49d \] (3.32)

Here, we have obtained diameters of all channel sections in terms of diameter d of channel section F (or G). Such a way, we can optimize the flow channel network with asymmetric geometries and mass fractions using optimal geometric configurations of elementary channels obtained in this chapter.

3.5 Conclusions

In this chapter we have made the following conclusions: (1) The optimal aspect ratio of 2D T-shaped channels that are symmetric in geometry and mass distribution is \( 2^{2/3} \) and the corresponding optimal diameter ratio is \( 2^{4/3} \), assuming the junction losses and wall roughness
are negligible. (2) Mass induced asymmetry reduces the flow resistance in 2D T-shaped channels. (3) Optimal diameter ratio in mass asymmetric T-shaped channels is only a function of mass fraction and is independent of the aspect ratio of the geometry. (4) Geometric asymmetry increases the flow resistance when compared to symmetric T-shaped channels. (5) Optimal diameter ratio in channels with geometric asymmetry is a function of mass fraction and geometric deviation. These conclusions are complimented by the ones in [56] and appendix A.
CHAPTER 4

ASYMMETRIC BIFURCATION OF Y-SHAPED CHANNELS

4.1 Introduction

In Chapter 3, we have discussed ways of connecting a source to two outlets arbitrarily placed in space using optimal T-shaped flow channels with mass and geometry induced asymmetries. However, given considerable design flexibility, Y-shaped channels can also be used as a means to connect three points in space. Bejan [26] discusses optimization of Y-shaped channels that are symmetric in design and mass distribution to connect a source to two outlets. The results confirmed that Y-shaped channels perform better for fluid flow when compared to T-shaped channels designed under identical constraints. In this chapter, we consider Y-shaped channels with mass and geometry induced asymmetries and optimize them for minimum pressure drop across the channels.

4.2 Y-shaped channel with mass and geometry induced asymmetry

Figure 4.1 shows a typical Y-shaped channel with mass and geometry induced asymmetry. It has a stem section of length $L_0$ and diameter $D_0$ and two branch sections $A$ and $B$, of lengths $L_1$ and $L_2$ and diameters $D_1$ and $D_2$, respectively. The imaginary rectangle enclosing the channels has an aspect ratio $F$ and area $S$. The channels make different angles $\alpha$, $\theta_0$, $\theta_1$, and $\theta_2$, with respect to horizontal and vertical planes, as shown in Figure 4.1. We can write the expressions for aspect ratio $F$ and surface area $S$ in terms of the lengths and angles of the channel sections, such that
The mass induced asymmetry in the Y-shape is caused by changing the mass fraction $\gamma$ of the total mass $\bar{m}$ entering the branch section A. From Figure 4.1, $\gamma \bar{m}$ enters the branch section A, whereas $(1-\gamma) \bar{m}$ enters the branch section B. The values of $\gamma$ vary from 0 to 0.5, $\gamma = 0$ describing the scenario where no mass is entering branch section A and $\gamma = 0.5$ describing the case where mass is equally dividing into the branch sections. The optimization results for cases with $\gamma > 0.5$ can be obtained by mirror imaging the results obtained for $(1-\gamma)$.

\[
F = \frac{L_1 \sin \theta_1 + L_2 \sin \theta_2}{L_0 \sin(\alpha + \theta_0) + L_1 \cos \theta_1}
\] 

\[
S = (L_1 \sin \theta_1 + L_2 \sin \theta_2)(L_0 \sin(\alpha + \theta_0) + L_1 \cos \theta_1)
\] 

The volume $V_C$ occupied by the channels will be

\[
V_C = \frac{\pi}{4} \left( L_0 D_0^2 + L_1 D_1^2 + L_2 D_2^2 \right)
\]
The geometry induced asymmetry can be introduced by varying the position of the source along the horizontal plane. This variation is described using a parameter that we call geometric deviation, and denoted by \( g \). From Figure 4.1, \( g \) is defined as

\[
g = \frac{2(L_0 \cos(\alpha + \theta_0) + L_1 \sin \theta_1)}{(L_1 \sin \theta_1 + L_2 \sin \theta_2)} - 1 \tag{4.4}
\]

The values of \( g \) vary from 0 to 0.5, \( g = 0 \) describes the scenario where the source is positioned equidistantly from the outlets, whereas \( g = 0.5 \) describes the scenario where the source is placed right beneath one of the outlets as illustrated in the Figure 4.14. The scenarios with \( g > 0.5 \) cannot be compared with \( g < 0.5 \) for specific reasons that are discussed in the latter sections.

4.3 Pressure drop and non-dimensional flow resistance

The total pressure drop across the Y-shaped channels will be the sum of individual pressure drops in stem section and one of the branch sections and the pressure drop in the junction. If we assume the channels are long and slender, then the pressure drop in the junction becomes negligible compared to the pressure drop in channel sections and can be neglected. Using the Equation (2.1) for pressure drop in circular channels, we can write the expression for total pressure drop \( \Delta P \) such that

\[
\Delta P = \frac{128}{\pi} \frac{m}{v} \left[ \frac{L_0}{D_0^4} + \frac{\gamma L_1}{D_1^4} \right] = \frac{128}{\pi} \frac{m}{v} \left[ \frac{L_0}{D_0^4} + \frac{(1 - \gamma)L_2}{D_2^4} \right] \tag{4.5}
\]

The pressure drops in the branch sections A and B have to be equal to maintain a continuous and constant mass flow rate in the channels. Such that, we can modify the expression for total pressure drop into

\[
\Delta P = \frac{128}{\pi} \frac{m}{v} \left[ \frac{L_0}{D_0^4} + \frac{\gamma L_1}{2D_1^4} + \frac{(1 - \gamma) L_2}{2D_2^4} \right] \tag{4.6}
\]

without losing any generality. The pressure drop can be non-dimensionalized by defining a flow resistance number \( R_f \) such that

\[
R_f = \frac{\Delta P}{8\pi m v \frac{V_c}{S^{3/2}}} \tag{4.7}
\]
where \( V_C \) is the volume occupied by the channels and \( S \) is the area of the rectangle enclosing the channels.

Using Equations (4.3) and (4.6), we can redefine \( R_f \) such that

\[
R_f = \left[ \frac{L_0}{D_0^4} + \frac{\gamma}{2} \frac{L_1}{D_1^4} + \frac{(1-\gamma)}{2} \frac{L_2}{D_2^4} \right] \left( \frac{L_0D_0^2 + L_1D_1^2 + L_2D_2^2}{S^{3/2}} \right)
\]  (4.8)

Using length and diameter ratios \( x_1 = L_1/L_0 \), \( x_2 = L_2/L_0 \) and \( y_1 = D_1/D_0 \), \( y_2 = D_2/D_0 \), we can modify \( R_f \) into

\[
R_f = L_0^3 \left[ 1 + \frac{\gamma}{2} \frac{x_1}{y_1^4} + \frac{(1-\gamma)}{2} \frac{x_2}{y_2^4} \right] \frac{(1 + x_1y_1^2 + x_2y_2^2)}{S^{3/2}}
\]  (4.9)

The area \( S \) can be replaced with aspect ratio \( F \) when the length and width of the rectangle are scaled to \( F \) and 1, respectively. It further changes the expression for \( R_f \) into

\[
R_f = L_0^3 \left[ 1 + \frac{\gamma}{2} \frac{x_1}{y_1^4} + \frac{(1-\gamma)}{2} \frac{x_2}{y_2^4} \right] \frac{(1 + x_1y_1^2 + x_2y_2^2)}{F^{3/2}}
\]  (4.10)

In the above expression the length ratios \( x_1 \) and \( x_2 \), however, depend upon the lengths \( L_1 \) and \( L_2 \), which in turn depend on \( L_0 \) and angles \( \alpha \) and \( \theta_0 \). The length ratios \( x_1 \) and \( x_2 \) can be calculated from \( F \), \( g \), \( L_0 \) and \( \theta_0 \) using the following expressions.

\[
\alpha = \tan^{-1} \left( \frac{1}{0.5 + gF} \right)
\]  (4.11)

\[
\theta_1 = \tan^{-1} \left[ \frac{(0.5 + g)F - L_0 \cos(\alpha + \theta_0)}{1 - L_0 \sin(\alpha + \theta_0)} \right]
\]  (4.12)

\[
\theta_1 = \tan^{-1} \left[ \frac{(0.5 - g)F + L_0 \cos(\alpha + \theta_0)}{1 - L_0 \sin(\alpha + \theta_0)} \right]
\]  (4.13)

\[
L_1 = \frac{1 - L_0 \sin(\alpha + \theta_0)}{\cos\theta_1}
\]  (4.14)

\[
L_2 = \frac{\cos\theta_1}{\cos\theta_2}
\]  (4.15)

\[
x_1 = L_1/L_0
\]  (4.16)

\[
x_2 = L_2/L_0
\]  (4.17)
The pressure drop in the two branch sections have to be equal in order to maintain and continuous and constant mass flow rate. This property requires that

\[ y_2 = \left[ y_1^4 \frac{1 - \gamma}{\gamma} x_2 \right]^{1/4} \]  

(4.18)

For every combination of aspect ratio \( F \), mass fraction \( \gamma \) and geometric deviation \( g \), we can minimize \( R_f \) in Equation (4.10) by varying the values of \( L_0 \) and \( \theta_0 \) and obtain optimal length and diameter ratios \( x_1 \), \( x_2 \) and \( y_1 \), \( y_2 \). During the optimization the total mass flow rate \( \dot{m} \), area \( S \) of the rectangle and volume \( V_C \) of the channels are constrained for all \( F \), \( \gamma \) and \( g \) values such that comparisons can be made between different optimal designs.

### 4.4 Symmetric Y-shaped channel

A symmetric Y-shaped channel has the source placed symmetrically in between the outlets \((g = 0)\) and the mass bifurcates equally into the branch sections \((\gamma = 0.5)\). The optimization results of such designs are discussed in [26]. The following results are found to be in good accord with the published results and can be used for further comparison between symmetric and asymmetric designs.

Figure 4.2 shows the variation in minimized non-dimensional flow resistance \( R_{f,\text{min}} \) between aspect ratios \( F = 0.5 \) and \( F = 3 \) when the mass fraction \( \gamma = 0.5 \) and geometric deviation \( g = 0 \) (symmetric in both geometry and mass distribution). From Figure 4.2 it can be observed that \( R_{f,\text{min}} \) has a very high value at lower aspect ratios \((F \sim 0.5)\) and tends to reach infinity as the aspect ratio approaches 0. With increasing aspect ratio, the \( R_{f,\text{min}} \) decreases and reaches a minimum value of \( R_{f,\text{min}} = 2 \) at aspect ratio \( F = 2 \). The \( R_{f,\text{min}} \) again increases for aspect ratios \( F > 2 \). It means, of all possible shapes of symmetric Y-shaped channels, the optimal Y-shape for fluid flow occurs when the aspect ratio \( F = 2 \).
Figure 4.2 Variation in non-dimensional flow resistance $R_{f,min}$ and bifurcation angle $(\theta_1)_{opt}$ with aspect ratio $F$

Figure 4.2 also shows the variation in optimal angle $(\theta_1)_{opt}$ within aspect ratios $F = 0.5$ and $F = 3$. It is also equal to optimal angle $(\theta_2)_{opt}$ when $\gamma = 0.5$. It can be observed that $(\theta_1)_{opt}$ has a constant value approximately equal to $37.5^\circ$ until the aspect ratio becomes $F = 1.5$. Beyond that $(\theta_1)_{opt}$ increased with aspect ratio. This observation leads to conclusion that optimal angle for bifurcation is always constant and is approximately equal to $37.5^\circ$ for aspect ratios $F < 1.5$. The variation in $(\theta_1)_{opt}$ beyond $F = 1.5$ is explained using the optimal length and diameter ratios shown in Figure 4.3.

Figure 4.3 shows the variation in optimal length ratio $x_{opt} = (L_1/L_0)_{opt}$ between aspect ratios $F = 0.5$ and $F = 1.5$. The length ratio $x_{opt}$ is relatively small at lower aspect ratios $(F \sim 0.5)$ but increases rapidly with increasing aspect ratio. It indicates that increasing the aspect ratio increases the length of the branch sections and they become
more dominant compared to the stem section. The stem section gradually become small and disappears completely at about aspect ratio $F = 1.5$. Beyond $F = 1.5$, it is not possible to calculate optimal length ratio because no discernable stem section exists. From Figure 4.1, we learned that the optimal bifurcation angle $(\theta_1)_{opt}$ has a constant value until $F = 1.5$ and increases beyond that. Combining the results of Figures 4.1 and 4.2, we can conclude that optimal Y-shape configuration changes from Y-shape to V-shape at an aspect ratio of $F = 1.5$. Also, the optimal aspect ratio $F = 2$ where $R_{f,\text{min}} = 2$ falls in the V-shape, implying that the optimal Y-shape of all symmetric configurations is in fact a V-shape.

![Figure 4.3 Variation in $(L_1/L_0)_{opt}$ and $(D_1/D_0)_{opt}$ with aspect ratio $F$](image)

Figure 4.3 also shows optimal diameter ratio $y_{opt} = (D_1/D_0)_{opt}$ between aspect ratios $F = 0.5$ and $F = 1.5$. It is a constant value for all aspect ratios and is equal to $1/2^{1/3}$. This value is of high significance because the optimal diameter ratio for symmetric geometry and mass distribution is always constant and is independent of aspect ratio.
This value has been observed in symmetric geometries of natural systems such as human blood vessels [3]. These results are also in good accord with the results published in [26].

Figure 4.4 Optimal Y-shape configurations symmetric in mass and geometry at different aspect ratios

Figure 4.4 shows optimal Y-shaped configurations drawn to scale at different aspect ratios, which are symmetric in geometry and mass distribution. It can be observed that the $R_{f,\text{min}}$ has reduced with increasing aspect ratio $F$ and reaches a minimum of $R_{f,\text{min}} = 2$ when the aspect ratio becomes $F = 2$. The shape also changes from Y-shape
to V-shape at aspect ratio $F = 1.5$. The diameter ratio for all the designs is same and is equal to $1/\sqrt[3]{2} \approx 0.794$.

### 4.5 Y-shaped channel with mass induced asymmetry

In section 2.4 of Chapter 2, Y-shaped channels with mass induced asymmetries are discussed as an introduction to asymmetric bifurcation of channels. Some of those results will be repeated here in order to make comparisons between symmetric configurations and configurations with geometry induced asymmetry. In the Y-shaped configurations with only mass induced asymmetry, the geometric deviation $g$ remains equal to 0, however, mass fraction $\gamma$ varies from 0 to 0.5. Comparisons between non-dimensional flow resistance $R_{f,\text{min}}$, optimal angles $(\theta_0)_{\text{opt}}$, $(\theta_1)_{\text{opt}}$ and $(\theta_2)_{\text{opt}}$ (Figure 4.1) and optimal length and diameter ratios are made between designs with different mass fractions.

Figure 4.5 shows the variation in $R_{f,\text{min}}$ at different mass fractions between aspect ratios $F = 0.5$ and $F = 3$ when the geometric deviation $g = 0$. It can be observed that the $R_{f,\text{min}}$ values decrease with decreasing mass fractions. The difference between successive $R_{f,\text{min}}$ values increase with decreasing mass fraction, i.e., the difference in $R_{f,\text{min}}$ between $\gamma = 0.2$ and $\gamma = 0.1$ is greater when compared with the $R_{f,\text{min}}$ between $\gamma = 0.5$ and $\gamma = 0.4$. The least value of $R_{f,\text{min}}$ at any mass fraction, however, is always observed at aspect ratio $F = 2$. At aspect ratio $F = 2$, the $R_{f,\text{min}}$ for $\gamma = 0.1$ is observed to be 23% smaller compared to $R_{f,\text{min}}$ at $\gamma = 0.5$. These observations lead to conclude that mass induced asymmetry reduces the flow resistance in Y-shaped channels.
Figures 4.6a, 4.6b and 4.6c show the variation in optimal angles $(\theta_0)_{opt}$, $(\theta_1)_{opt}$ and $(\theta_2)_{opt}$ at different mass fractions between aspect ratios 0.5 and 1.5 for a geometric deviation of $g = 0$. The angle $(\theta_0)_{opt}$ increases with increasing aspect ratio and mass fractions. The $\gamma = 0.1$ configuration has lesser optimal $(\theta_0)_{opt}$ value because the stem section of the optimal Y-shape leans towards branch section A instead of aligning to vertical axis. The inclination reduces with increasing mass fraction and at $\gamma = 0.5$, the stem section perfectly aligns with the vertical axis. The angle $(\theta_1)_{opt}$ remains a constant
Figure 4.6a Variation in optimal $\theta_0$ with aspect ratio $F$ at different mass fractions

Figure 4.6b Variation in optimal $\theta_1$ with aspect ratio $F$ at different mass fractions
value when $\gamma = 0.5$ which is approximately equal to 37.5°. However, reducing the mass fraction increases the $(\theta_2)_{opt}$ above 37.5° as shown in Figure 4.6b. This increase is observed to be negligible at higher aspect ratios ($F \sim 1.5$). At aspect ratio $F = 1.5$, $(\theta_1)_{opt}$ become constant and approximately equal to 37.5° for all mass fractions. A similar effect can be observed in the case of $(\theta_2)_{opt}$, however, reducing the mass fraction $\gamma$ reduces the $(\theta_2)_{opt}$ at lower aspect ratios as shown in Figure 4.6c. At higher aspect ratios ($F \sim 1.5$) $(\theta_2)_{opt}$ tends to become a constant value and matches the $(\theta_2)_{opt}$ of $\gamma = 0.5$ (approximately 37.5°) at $F = 1.5$.

Figures 4.7a and 4.7b show the variation in optimal channel length ratios $(x_1)_{opt} = (L_1/L_0)_{opt}$ and $(x_2)_{opt} = (L_2/L_0)_{opt}$ with mass fraction between aspect ratios 0.5 and 1.4. At $F = 1.5$, the Y-shape effectively changes to V-shape and no observable stem
section exists beyond that aspect ratio. Thus the calculation of length ratios is not possible. From Figures 4.7a and 4.7b, we can observe that optimal length ratios increase

![Figure 4.7a Variation in \((L_1/L_0)_{opt}\) with aspect ratio F at different mass fractions](image)

![Figure 4.7b Variation in \((L_2/L_0)_{opt}\) with aspect ratio F at different mass fractions](image)
rapidly with aspect ratio. At lower aspect ratios \((F < 0.7)\), the branch sections are smaller than the stem section in length and the optimal length ratios is slightly larger in channels with mass fractions \(\gamma = 0.1\) compared to channels with symmetric mass bifurcation \((\gamma = 0.5)\). A similar effect can be observed in both length ratios \((x_1)_{opt}\) and \((x_2)_{opt}\).

Figures 4.8a and 4.8b show the variation in optimal diameter ratios \((y_1)_{opt} = (D_1/D_0)_{opt}\) and \((y_2)_{opt} = (D_2/D_0)_{opt}\) at different mass fraction between aspect ratios 0.5 and 1.5. The diameter ratio \((y_1)_{opt}\) has a constant value of \(1/2^{1/3}\) when \(\gamma = 0.5\). However, reducing the mass fraction reduced the \((y_1)_{opt}\) values. The reduction in \((y_1)_{opt}\) increased with increasing aspect ratio. The optimal diameter ratio \((y_2)_{opt}\), on the other hand increased with reducing mass fraction and reached approximately 0.94 for a configuration with \(\gamma = 0.1\) at \(F = 1.5\). This is to accommodate 90% of the flow rate diverted into channel section B and 10% into channel section A while maintaining the pressure drop in the two branch sections equal.

![Figure 4.8a Variation in \((D_1/D_0)_{opt}\) with aspect ratio \(F\) at different mass fractions](image)
Figure 4.8b Variation in \((D_2/D_0)_{opt}\) with aspect ratio \(F\) at different mass fractions

Figure 4.9 shows optimal Y-shape configurations for a set of aspect ratios and mass fraction combinations. It can be observed that the stem section of the channel leans towards branch section A for mass fractions below 0.5 and Y-shape turns into a V-shape for aspect ratios above 1.5.
Figure 4.9 Optimal Y-shape configurations with mass induced asymmetries at different aspect ratios

$F = 0.5, R_{c_{\text{min}}} = 4.1663, \gamma = 0.1$
$x_1 = 0.7229, x_2 = 0.5633$
$y_1 = 0.565, y_2 = 0.9194$

$F = 0.5, R_{c_{\text{min}}} = 4.6609, \gamma = 0.3$
$x_1 = 0.654, x_2 = 0.5833$
$y_1 = 0.715, y_2 = 0.8588$

$F = 0.5, R_{c_{\text{min}}} = 4.7858, \gamma = 0.5$
$x_1 = 0.6179, x_2 = 0.6179$
$y_1 = 0.794, y_2 = 0.794$

$F = 1, R_{c_{\text{min}}} = 2.1611, \gamma = 0.1$
$x_1 = 2.7095, x_2 = 2.4169$
$y_1 = 0.55, y_2 = 0.9258$

$F = 1, R_{c_{\text{min}}} = 2.5464, \gamma = 0.3$
$x_1 = 2.4889, x_2 = 2.3549$
$y_1 = 0.705, y_2 = 0.8594$

$F = 1, R_{c_{\text{min}}} = 2.6465, \gamma = 0.5$
$x_1 = 2.343, x_2 = 2.343$
$y_1 = 0.794, y_2 = 0.794$

$F = 1.5, R_{c_{\text{min}}} = 1.7011, \gamma = 0.1$
$y_1 = 0.54, y_2 = 0.9345$

$F = 1.5, R_{c_{\text{min}}} = 2.0374, \gamma = 0.3$
$y_1 = 0.7, y_2 = 0.8645$

$F = 1.5, R_{c_{\text{min}}} = 2.1259, \gamma = 0.5$
$y_1 = 0.794, y_2 = 0.794$
4.6 Y-shaped channel with geometry induced asymmetry

In this section we assume that the mass flow is equally bifurcating between the branch sections at the junction ($\gamma = 0.5$), however, the position of the source is flexible to move along the bottom edge of the rectangle. That is, the geometric deviation $g$ can vary from 0 to 0.5. Comparisons are made for $R_{f,\text{min}}$, optimal $\theta_0$, $\theta_1$ and $\theta_2$ and optimal length and diameter ratios between configurations with different aspect ratios and geometric deviations.

Figure 4.10 Variation in $R_{f,\text{min}}$ with aspect ratio $F$ at different geometric deviations

Figure 4.10 shows the variation in $R_{f,\text{min}}$ between aspect ratios $F = 0.5$ and $F = 1.5$ at different geometric deviations when the mass fraction $\gamma = 0.5$ (symmetric mass distribution). It can be observed from figure that geometric deviation has indeed increased the flow resistance. The increase is very significant at higher aspect ratios ($F \sim 1.5$). Also, the rate of increase in flow resistance increased with increasing geometric deviation. The optimal aspect ratio is observed to reduce with increasing
geometric deviation, $F_{\text{opt}}$ for $g = 0$ is observed at $F = 2$, where as $F_{\text{opt}}$ for $g = 0.5$ is observed at $F = 1.4$.

Figure 4.11a Variation in optimal $\theta_0$ with aspect ratio $F$ at different geometric deviations

Figure 4.11b Variation in optimal $\theta_1$ with aspect ratio $F$ at different geometric deviations
Figures 4.11a, b and c show the variation in optimal angles \( (\theta_0)_{opt} \), \( (\theta_1)_{opt} \) and \( (\theta_2)_{opt} \), respectively, between aspect ratios \( F = 0.5 \) and \( F = 1.5 \) at different geometric deviations when the mass fraction \( \gamma = 0.5 \). We can observe that the rate of increase in \( (\theta_0)_{opt} \) has reduced with increasing geometric deviation. Also the difference in \( (\theta_0)_{opt} \) is more dominant at higher aspect ratios \( (F \sim 1.5) \). At lower aspect ratios \( (F \sim 0.5) \), the \( (\theta_0)_{opt} \) is almost same for all geometric deviations. \( (\theta_1)_{opt} \) and \( (\theta_2)_{opt} \) has a constant value (~37.5°) when \( g = 0 \). With increasing geometric deviation, however, \( (\theta_1)_{opt} \) has increased with aspect ratio where as \( (\theta_2)_{opt} \) reduced. The rate of increase in \( (\theta_1)_{opt} \) or decrease in \( (\theta_2)_{opt} \) is almost linear with increasing geometric deviation \( g \).
Figures 4.12a and 4.12b show the variation in optimal length ratios $(x_1)_{opt} = (L_1/L_0)_{opt}$ and $(x_2)_{opt} = (L_2/L_0)_{opt}$ between aspect ratios $F = 0.5$ and $F = 1.5$ at different geometric deviations when the mass fraction $\gamma = 0.5$. We can observe that at lower aspect ratios ($F \sim 0.5$), the optimal length ratios are almost equal to each other for all geometric deviations. However, the increase in aspect ratio has significant effect in
optimal length ratios at different geometric deviations. From the previous section we observed that the stem section of the Y-shaped channel virtually disappears beyond $F = 1.5$ at $\gamma = 0.5$ and $g = 0$. However, as $g$ increased from 0 to 0.5, the stem section

![Figure 4.13a](image1.png)

Figure 4.13a Variation in $(D_1/D_0)_{opt}$ with aspect ratio $F$ at different geometric deviations

![Figure 4.13b](image2.png)

Figure 4.13b Variation in $(D_2/D_0)_{opt}$ with aspect ratio $F$ at different geometric deviations
doesn’t disappear and continues to exists beyond F = 1.5. The transformation from Y-shape to V-shape in symmetric channels at aspect ratio F = 1.5 is not valid for Y-shaped channels with higher geometric deviations.

Figures 4.13a and 4.13b show the variation in optimal diameter ratios \((y_1)_{opt} = (D_1/D_0)_{opt}\) and \((y_2)_{opt} = (D_2/D_0)_{opt}\) between aspect ratios F = 0.5 and F = 1.5 at different geometric deviations when the mass fraction \(\gamma = 0.5\). It can be observed that both \((y_1)_{opt}\) and \((y_2)_{opt}\) are equal to \(1/2^{1/3}\) when \(g = 0\). However, increasing geometric deviation increases \((y_1)_{opt}\) whereas \((y_2)_{opt}\) decreased. In Y-shaped channels, both mass fraction and geometric deviation affects the optimal length and diameter ratios.

Figure 4.14 shows optimal Y-shaped channel configurations at different aspect ratios and geometric deviations. It can be observed that the optimal configuration still retains the Y-shape at F=1.5 when the geometric deviation is sufficiently increased beyond g=0.25.

4.7 Y-shaped configurations with g > 0.5

In the previous section only the configurations with g < 0.5 were considered. Y-shaped configurations with geometric deviation g > 0.5 cannot be compared with configurations where g < 0.5 because the area of the rectangle enclosing the channels is no longer the same. Figure 4.15 shows the difference in area between two scenarios where g changes from 0.25 to 0.75. In the figure, the rectangular area \(S_2\) of the configuration with g = 0.75 is greater than the area \(S_1\) of the configuration with g = 0.25. The optimal geometric parameters for geometric deviations g > 0.5 can still be obtained using the approach followed in section 4.6.
Figure 4.14 Optimal Y-shape configurations with geometry induced asymmetries at different aspect ratios
Figure 4.15 Comparison between rectangular areas between configurations with \( g = 0.25 \) and \( g = 0.75 \)

4.8 Y-shaped Channel Networks

Figure 4.16a illustrates symmetric Y-shaped channel networks on a disc-like structure with a source placed at the center and the outlets distributed equidistantly over the perimeter. If the number of outlets is 2, 3, 4 and 5 radial channels, as shown in Figure 4.16a, are optimal for flow distribution. However, if the outlets are more than 5, then elemental Y-shaped channels optimized for flow distribution in section 4.2 can be used to create flow networks. The symmetric networks, however, can only be produced for outlets that are multiples of 3, 4 and 5, such as 6, 8, 10 (one level of branching), 12, 16, and 20 (two levels of branching). Figures 4.16b and 4.16c show symmetric Y-shaped channel networks on a circular disc with one and two levels of branching.

The assembly of asymmetric flow networks, on the other hand, is more complicated. Here we consider optimization of flow networks to connect 3 outlets to a source. Figure 4.17a shows a configuration where three outlets are placed equidistantly from the source. The amount of mass to be delivered is same for all outlets.
Figure 4.16 Symmetric Y-shaped channel networks on a circular disc

Figure 4.17 Asymmetric Y-shaped channel network
The simplest possible network in this case would be radial channels as shown in Figure 4.17a. The respective pressure drops and pumping powers can be calculated as

$$\Delta P_a = \frac{128}{\pi} m \frac{R}{D^4}$$  \hspace{1cm} (4.19)$$

$$\dot{W}_a = \frac{3}{\rho} \Delta P_a = \frac{128}{\pi} \frac{3 m^2 \nu R}{\rho D^4}$$  \hspace{1cm} (4.20)$$

And the volume of the channels is

$$V_c = \frac{3\pi}{4} R D^2$$  \hspace{1cm} (4.21)$$

The volume of the channels will be constrained so that comparisons can be made between different channel networks. In Figure 4.17a, two of the radial channels can be replaced with an optimal Y-shaped channel as shown in Figure 4.17b. The respecting pressure drops and pumping power would be

$$\Delta P_{b1} = \frac{128}{\pi} m \frac{R}{D^4} = \Delta P_a$$  \hspace{1cm} (4.22)$$

$$\Delta P_{b2} = \frac{128}{\pi} 2 m \left[ \frac{0.6286 R}{(1.3D)^4} + \frac{0.2125 R}{(1.04D)^4} \right] = 0.8 \left[ \frac{128}{\pi} m \frac{R}{D^4} \right] = 0.8\Delta P_a$$  \hspace{1cm} (4.23)$$

$$\dot{W}_b = \frac{m}{\rho} [\Delta P_{b1} + 2\Delta P_{b2}] = 0.867 \left[ \frac{128}{\pi} \frac{3 m^2 \nu R}{\rho D^4} \right] = 0.867 \dot{W}_a$$  \hspace{1cm} (4.24)$$

We observe from Equations 4.23 and 4.24 that pressure drop and pumping power has reduced because of networking of channels when compared to radial channels.

We can further modify the channels in Figure 4.17b using optimal Y-shaped channels with mass asymmetry. The resulting network is shown in Figure 4.17c. We can calculate the pressure drop and pumping power for the network as

$$\Delta P_c = \frac{128}{\pi} 3 m \left[ \frac{0.2921 R}{(1.52D)^4} + \frac{2}{3} \frac{0.3713 R}{(1.2856D)^4} + \frac{1}{3} \frac{0.425 R}{(0.9424D)^4} \right]$$  \hspace{1cm} (4.25)$$

$$\Delta P_c = 0.776 \left[ \frac{128}{\pi} m \frac{R}{D^4} \right] = 0.776 \Delta P_a$$  \hspace{1cm} (4.26)$$

$$\dot{W}_b = \Delta P_c \frac{3 m}{\rho} = 0.77 \left[ \frac{128}{\pi} \frac{3 m^2 \nu R}{\rho D^4} \right] = 0.77 \dot{W}_a$$
From Equations 4.25 and 4.26 we observe that the total pressure drop across the channels and the pumping power required to deliver the same mass flow rate are further reduced when compared to configurations 4.17a and 4.17b. This analysis shows the application of optimal flow channels in reducing the pressure drop and pumping power in fluid mass distribution systems.

4.8 Conclusion

The following conclusions can be made from the optimization of 2D Y-shaped channels: (1) Optimal 2D Y-shaped channels that are symmetric in geometry and mass distribution bifurcate at a constant angle approximately equal to 37.5°. (2) Symmetric 2D Y-shaped channels exert lesser flow resistance compared to symmetric 2D T-shaped channels when the area influenced by the channels and the volume occupied by the channels are constrained. (3) Optimal Y-shaped channels transform into V-shaped channels for aspect ratios above 1.5 for symmetric configurations. (4) The optimal symmetric 2D Y-shaped channels is a V-shaped configuration at aspect ratio 2. (5) Mass induced asymmetry in 2D Y-shaped channels reduces the flow resistance compared to the case with symmetric mass distribution. (6) Geometric asymmetry in 2D Y-shaped channels increases flow resistance compared to the case with source placed equidistantly between the outlets. (7) 2D Y-shaped channels with geometric deviation $g > 0.5$ can not be compared with cases where $g < 0.5$ because the area influenced by the channels is more in the former case.
CHAPTER 5

T-SHAPED FLOW CHANNELS IN 3D

5.1 Introduction

In the previous two chapters we dealt with optimization of flow channels between a source and two outlets. That was a three point problem and can be solved in a two dimensional plane. However, if the number of outlets is more than two, then it is possible to have source and outlets positioned in 3D. This multi-point flow configuration can be connected using channels that are either in T-shape or in Y-shape. In this chapter we consider the 3D optimization of T-shaped channels connecting a source to three or more number of outlets.

We start with the optimization of channels in 3D that are symmetric in geometry and mass distribution. In the later sections, the procedure will be extended to configurations with mass and geometry induced asymmetries. The number of outlets is limited to three in most cases, however, four and five outlets are considered in symmetric configurations for the purpose of comparison between different outlets.

5.2 Symmetric T-shaped channels in 3D

It is possible to connect multiple outlets located in a plane to a source located in the third dimension using T-shaped channels. The branch sections essentially bifurcate at right angles from the stem section. Figure 5.1 shows the T-shaped channels symmetric in geometry (equal length and diameter of branch sections and source positioned equidistantly between the outlets) connecting three, four and five number of outlets to a source. The volumes enclosing the source and outlets are pyramids (tetrahedrons in the case of 3 outlets). For comparison between different configurations, we assume that the volume enclosing the points is always constant.
Figure 5.1 T-shaped channels in 3D connecting multiple outlets

Figure 5.2 Symmetric 3D T-shaped channels
Figure 5.2 shows a T-shaped configuration connecting three outlets to a source using circular channels. The stem section has a length $L_0$ and diameter $D_0$ and three identical branch sections of length $L_1$ and diameter $D_1$. The volume of the tetrahedron enclosing the points is given by

$$V = \frac{\sqrt{3}}{4} L_0 L_1^2$$  \hspace{1cm} (5.1)

The aspect ratio $F$ in this case is defined as

$$F = \frac{L_1}{L_0}$$  \hspace{1cm} (5.2)

A similar definition of aspect ratio is used for configurations with four and five number of outlets. The volume $V_C$ occupied by the channels will be

$$V_C = \frac{\pi}{4} \left[L_0 D_0^2 + 3L_1 D_1^2\right]$$  \hspace{1cm} (5.3)

The total pressure drop in the channels will be the sum of the individual pressure drops in stem section and one of the branch sections and the pressure drop in the junction. If we ignore the pressure drop in the junction, the total pressure drop in the T-shape can be written as

$$\Delta P = \frac{128}{\pi} \bar{m} \nu \left[ \frac{L_0}{D_0^4} + \frac{L_1}{3D_1^4} \right]$$  \hspace{1cm} (5.4)

where $\bar{m}$ is the mass flow rate entering the stem section and $\nu$ is the kinematic viscosity of the fluid.

The pressure drop can be non-dimensionalized using volume $V$ enclosing the channels and the volume $V_C$ occupied by the channels. The non-dimensional flow resistance number $R_f$ can be defined as

$$R_f = \frac{\Delta P}{8\pi\bar{m} \nu} \frac{V_C^2}{V}$$  \hspace{1cm} (5.5)

From Equations 5.1, 5.4 and 5.5 we can rewrite the expression for $R_f$ such that

$$R_f = \left[ \frac{L_0}{D_0^4} + \frac{L_1}{3D_1^4} \right] \sqrt{\frac{L_0 D_0^2 + 3L_1 D_1^2}{\sqrt{3L_0 L_1^2 / 4}}}$$  \hspace{1cm} (5.6)
We can further reduce this by introducing length and diameter ratios, \( x = L_1/L_0 \) and \( y = D_1/D_0 \), such that \( R_f \) becomes

\[
R_f = \left[ 1 + \frac{x}{3y^4} \right] \frac{(1 + 3xy^2)^2}{\sqrt{3x^2/4}} \quad (5.7)
\]

We can replace the length ratio \( x \) with aspect ratio \( F \).

\[
R_f = \left[ 1 + \frac{F}{3y^4} \right] \frac{(1 + 3Fy^2)^2}{\sqrt{3F^2/4}} \quad (5.8)
\]

At a constant mass flow rate and constrained volumes \( V \) and \( V_c \), the pressure drop can be minimized by minimizing \( R_f \). From Equation 5.8 we can observe that \( R_f \) can be minimized by varying the value of the diameter ratio \( y \) at different aspect ratios.

In the case of four or five number of outlets, the formulation will be the same except for the volume of pyramids enclosing the channels:

\[
V = \frac{2}{3} L_0 L_{-1}^2 \quad \text{(four outlets)} \quad (5.9)
\]

\[
V = \frac{5}{6} L_0 L_{-1}^2 \sin \left( \frac{2\pi}{5} \right) \quad \text{(five outlets)} \quad (5.10)
\]

Figure 5.3 shows the comparison for minimum flow resistance number \( R_{f,\text{min}} \) three, four and five outlets at different aspect ratios. The comparisons are made while keeping the volume enclosing the channels constant in all cases. We can observe that T-shaped configurations with three branch section exert higher flow resistance compared to configurations with four and five branch sections. This is due to extra length of branch sections that is required in three outlet configuration to fill the same pyramidal volume influenced by otherwise four or five number of branch sections and subsequent reduction in the diameter to constrain the total volume occupied by the channels. We can conclude that more number of branch sections reduce the flow resistance when the volume enclosing the channels and the volume occupied by the channels are constrained.
The optimal diameter ratio \( \left( \frac{D_1}{D_0} \right)_{\text{opt}} \) is observed to be independent of aspect ratio for all symmetric 3D T-shaped configurations. However it depends upon the number of branch sections \( N \) such a way that

\[
\left( \frac{D_1}{D_0} \right)_{\text{opt}} = \left( \frac{1}{N} \right)^{\frac{1}{3}} \quad (5.11)
\]

Figure 5.4 shows the variation in optimal diameter ratio with the number of outlets \( N \). We can observe that when \( N = 2 \), the optimal diameter ratio is \(-0.794\), an observation we made in the previous two chapters. From this result we can conclude that in symmetric geometries, whether in 2D or 3D, the optimal diameter ratio depends only up on the number of branch sections as long as the mass is dividing in equal proportions at the junction.
5.3 Effect of gravity on the optimal T-shaped configuration

Applications of T-shaped channels in 3D are exposed to gravitational forces which affect mass flow rates and pressure drops. The gravity might increase or reduce the total pressure drop depending upon the orientation of the channels and can alter the total amount of mass flow rate delivered to the outlets. In this section we consider different orientations of symmetric T-shaped channels with three outlets and derive dimensional parameters to account for the gravitational effects. Figures 5.5a-5.5c show three particular orientations that are effected by gravity. In all the cases, the gravity is assumed to act in the negative $z$-direction.
Effect of gravity on stem section

Figure 5.5a has stem section aligned to z-axis with three branch sections in xy-plane. In this case, the gravitational effect is extended only on the stem section. The increase in the pressure drop in the stem section due to gravity is equal to the weight of the fluid in the stem section. The total pressure drop in the stem section becomes

\[
\Delta P_s = \frac{128}{\pi} \frac{\bar{m}v}{D_0^4} L_0 + L_0 \rho g
\]  

(5.12)

It can be written as increased length of stem section such that the pressure drop in the stem section becomes

\[
\Delta P_s = \frac{128}{\pi} \frac{\bar{m}v}{D_0^4} L_0 \left[ 1 + \frac{\pi D_0^2 \rho g}{128 \bar{m} v} \right]
\]  

(5.13)

This increase in pressure drop, however, does not modify the optimal diameter ratios of branch sections, because, in the previous section we have observed that optimal diameter ratio in the symmetric T-shaped channels is independent of aspect ratio (ratio between lengths of branch and stem sections). The ratio \((D_1/D_0)_{opt} = 1/3^{1/3}\) is still valid when the stem section is aligned in the direction of gravity.
Effect of gravity on branch sections

Figure 5.5b shows another orientation of the T-shaped configuration where the effect of gravity is extended on the branch sections. The stem section is in the xy-plane and experiences no gravitational effects. Here we assume that the geometry is still symmetric, i.e., the length of all branch sections is same and the junction is equidistantly placed between the outlets. The diameters, however, vary to counter the gravitational effects and deliver equal mass flow rate at the outlets. From Figure 5.5b we can observe that, while branch sections A and B experience reduced pressure drop due to gravity, branch section C has pressure drop increased because the flow has to move against gravitational force. From previous chapters we understood that in order to maintain a continuous and constant mass flow rate the pressure drop in all the branch section must be equal. This property enables us to write an expression for pressure drops in the branch sections.

\[
\Delta P_A = \Delta P_C \Rightarrow \frac{128 \bar{m}}{\pi} \frac{L_{1}}{D_{1}^3} - L_1 \rho g \cos \left( \frac{\pi}{3} \right) = \frac{128 \bar{m}}{\pi} \frac{L_{1}}{D_{2}^4} + L_1 \rho g
\]  

(5.14)

If we express \(D_1\) as a function of \(D_2\) such that \(D_1 = kD_2\), we modify Equation 5.14 to

\[
k = \left[ 1 \left( \frac{9\pi}{256} B + 1 \right) \right]^{\frac{1}{4}}
\]  

(5.15)

where \(B\) is a function of \(\rho, g, \bar{m}, \nu\) and \(D_2\).

\[
B = \frac{\rho g D_2^4}{\bar{m} \nu}
\]  

(5.16)

The maximum possible value for \(k\) is 1, where all branch sections have equal diameters.

With in the effects of gravity, \(k\) always assumes value below 1. Figure 5.6 shows the
variation in $k$ at different $B$ values. Once the fluid material properties, mass flow rate and the value of $D_2$ are defined, we can calculate $B$ using Equation 5.16 and the corresponding $k$ value from Equation 5.15. Using $k$, we can calculate the value of $D_1$ corresponding to $D_2$, which delivers equal mass flow rate in all outlets.

We can conclude that gravity in effect increases the flow resistance of channels. The optimal configuration (length and diameter ratios) is independent of gravity effects when the gravitational force is aligned with stem section. However, if gravity acts on of the one branch sections, the optimal diameter ratios have to modified using Equation 5.15 such that equal mass flow rate is delivered at all outlets.
In the previous section we have assumed that mass divides into equal proportions at the junction. The resulting configuration has all the branch sections identical with equal lengths and diameters. However, if the mass is not dividing into equal proportions at the junction, then the optimal diameters of individual branch sections would not be identical anymore. In this section we optimize a T-shaped channel configuration connecting a source to three outlets that are symmetric in geometry but have asymmetry induced due to mass distribution.

Figure 5.7 shows schematic of the T-shaped channel configuration with mass induced asymmetry. The stem section is of length $L_0$ and diameter $D_0$. The three branch sections A, B and C have equal length $L_1$ but different diameters $D_1$, $D_2$ and $D_3$. Of the total mass $m$ entering the stem section, $\gamma_1m$ is entering branch section A, $\gamma_2m$ is entering branch section B and the rest $\gamma_3m = (1 - \gamma_1 - \gamma_2)m$ is entering branch section C.
The volume \( V \) of the tetrahedral enclosing the source and outlets is

\[
V = \frac{\sqrt{3}}{4} L_0 L_1^2
\]

and the expression for volume \( V_c \) occupied by the channels is

\[
V_c = \frac{\pi}{4} \left[ L_0 D_0^2 + L_1 \left( D_1^2 + D_2^2 + D_3^2 \right) \right]
\]

Although the mass flow rates in individual branch sections are different, the pressure drop has to be same to maintain a continuous and constant mass flow rate. Such that

\[
\Delta P_A = \Delta P_B = \Delta P_C
\]

This enables us to obtain relation between the diameters of the branch sections and the respective mass flow rate fractions.

\[
\frac{\gamma_1}{D_1^4} = \frac{\gamma_2}{D_2^4} = \frac{\gamma_3}{D_3^4}
\]

As for the total pressure drop in the T-shape, it is equal to the sum of individual pressure drops in stem section and one of the branch sections and the pressure drop in the junction. Ignoring the pressure drop in the junction, we can write the expression for total pressure drop

\[
\Delta P = \Delta P_s + \Delta P_A = \Delta P_s + \Delta P_B = \Delta P_s + \Delta P_C
\]

\[
\Delta P = \frac{128}{\pi} \frac{\dot{m} \gamma}{V} \left[ \frac{L_0}{D_0^4} + \frac{\gamma_1 L_1}{D_1^4} \right]
\]

The non-dimensional flow resistance number \( R_f \) can be written as

\[
R_f = \frac{\Delta P}{8\pi \dot{m} V} \frac{V_c^2}{V} = \left[ \frac{L_0}{D_0^4} + \frac{\gamma_1 L_1}{D_1^4} \right] \left[ \frac{L_0 D_0^2 + L_1 \left( D_1^2 + D_2^2 + D_3^2 \right)}{\sqrt{3} L_0 L_1^2 / 4} \right]^{\frac{3}{2}}
\]

We can modify the expression for \( R_f \) by introducing length and diameter ratios, \( x = L_1/L_0 \), \( y_1 = D_1/D_0 \), \( y_2 = D_2/D_0 \) and \( y_3 = D_3/D_0 \) such that

\[
R_f = \left[ 1 + \frac{\gamma_1 x}{y_1^4} \right] \left[ 1 + x \left( \frac{y_1^2 + y_2^2 + y_3^2}{\sqrt{3} x^2 / 4} \right) \right]^{\frac{3}{2}}
\]

The length ratio \( x \) can be replaced with aspect ratio \( F \). Also, the diameter ratios \( y_2 \) and \( y_3 \) can be written as functions of \( y_1 \) using Equation 5.20.
\[
R_f = \left[ 1 + \frac{\gamma_1 F}{y_1^4} \right] \frac{1 + F(y_1^2 + y_2^2 + y_3^2)^{\frac{3}{2}}}{\sqrt{3F^2/4}} \tag{5.25}
\]

where \( y_2 = \left( \frac{\gamma_2}{\gamma_1} \right)^{\frac{1}{4}} y_1 \) and \( y_3 = \left( \frac{\gamma_3}{\gamma_1} \right)^{\frac{1}{4}} y_1 \). \tag{5.26}

The non-dimensional flow resistance number \( R_f \) in Equation 5.26 can be minimized by varying the values of \( y_1 \) at different aspect ratios and mass fractions. Optimal diameter ratios of \( y_1 \), \( y_2 \) and \( y_3 \) can be calculated with in a range of aspect ratios and \( \gamma_1 \) and \( \gamma_2 \) values.

Figures 5.8a-5.8h show the variation in minimum non-dimensional flow resistance \( R_{f,\text{min}} \) for aspect ratios in the range \([0.4, 2.6]\). Figures only show the variation in \( \gamma_2 \) below 0.4 because the rest of the results can be obtained as mirror images of the existing results by replacing \( \gamma_2 \) with \((1 - \gamma_1 - \gamma_2)\).

One of the important observations we can make from Figure 5.8 is that asymmetry due to mass imbalance reduces the flow resistance. Among the different cases shown in Figure 5.8, the combination \( \gamma_1 = 0.1 \) and \( \gamma_2 = 0.1 \) has the least possible flow resistance at all aspect ratios. Also, lower aspect ratios have significantly higher flow resistance compared to higher aspect ratios. There is always an optimal aspect ratio for a set of \( \gamma_1 \) and \( \gamma_2 \). This aspect ratio, however, is not same for all configurations. Within the observed range, the geometry with minimum flow resistance \((\gamma_1 = 0.1, \gamma_2 = 0.1)\) has the optimal aspect ratio at \( F \sim 2.3 \), which is slightly higher than the optimal aspect ratio for geometry with equal mass proportions \((\gamma_1 = \gamma_2 = 1/3)\), which is \( F \sim 2.1 \). The rate of change in \( R_{f,\text{min}} \) is observed to decrease with increasing mass fraction. That is, the difference in \( R_{f,\text{min}} \) between configurations with \( \gamma_2 = 0.1 \) and \( \gamma_2 = 0.2 \) is significantly higher than the difference between configurations with \( \gamma_2 = 0.3 \) and \( \gamma_2 = 0.4 \).
Figure 5.8 Variation in non-dimensional flow resistance $R_{f,\text{min}}$ with aspect ratio and mass fraction
During the optimization, we have also obtained the optimal diameter ratios \((D_1/D_0)_{opt}\) within a range of aspect ratios and mass fractions. The optimal diameter ratios are observed to be independent of aspect ratios. However, mass fractions in the individual branch section greatly control the optimal diameter ratios. Figures 5.9a, b and c show the variation in optimal diameter ratios at different mass fractions.

From Figure 5.9a we can observe that optimal diameter ratio of branch section A is almost independent of mass fraction \(\gamma_2\) and only depends on the mass fraction \(\gamma_1\). It is important to recognize that mass fractions in the other two branch sections have little effect on the optimal diameter ratio of branch section A. A similar effect can be observed in Figure 5.9b where the optimal diameter ratio of branch section B is shown at different mass fractions. Change in \(\gamma_1\) has negligible effect on \((D_2/D_0)_{opt}\) and varies only with \(\gamma_2\). Figure 5.9c is similar to Figure 5.9b, because optimal diameter ratio \((D_3/D_0)_{opt}\) depends only up on mass fraction \(\gamma_3\). Figure 5.10 shows different T-shaped configurations that are symmetric in geometry but have mass induced asymmetries with optimal diameter ratios.
Figure 5.9 Variation in optimal diameter ratios with mass fractions
Figure 5.10 Optimal configurations for 3D T-shaped channels with mass induced asymmetries
5.5 Geometry induced asymmetry in 3D T-shaped channels

In section 5.2, we studied T-shaped channels that exhibit following symmetric properties: the branch sections are of equal lengths and diameters and the source is placed equidistantly from the outlets. However, it is not always possible to have a geometrically symmetric configuration. In this section, we consider the optimization of T-shaped channel configurations with geometrical asymmetries induced due to the position of the source. Figure 5.11 shows the schematic of such configuration. The position of source is defined with respect to the circumcenter of the equilateral triangle formed by connecting the outlets as shown in Figure 5.11. It is possible to draw a circle of radius R passing through the outlets with the center of the circle coinciding with the circumcenter of the triangle. The geometric deviations $g_1$ and $g_2$ are defined to represent the position of the source such that, the source is $g_1R$ units away from the circumcenter in $x$-direction and $g_2R$ units away in the $y$-direction.

We define the aspect ratio $F$ with respect to the radius $R$ of the circle and length of the stem section $L_0$, such that
\[ F = \frac{R}{L_0} \]  

(5.27)

The branch sections \( A, B \) and \( C \) of the T-shaped configuration make angles \( \alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1 \) and \( \delta_2 \) with the sides of the triangle as shown in Figure 5.11. The lengths and angles of branch sections are given by

\[
\alpha_2 = \tan^{-1} \left[ \frac{1 - g_2}{\frac{\sqrt{3}}{2} - g_1} \right] \quad \alpha_1 = \frac{\pi}{3} - \alpha_2
\]

(5.28)

\[
L_1 = \left( \frac{\sqrt{3}}{2} - g_1 \right) \frac{R}{\cos \alpha_2}
\]

(5.29)

\[
\beta_1 = \tan^{-1} \left[ \frac{1 - g_2}{\frac{\sqrt{3}}{2} + g_1} \right] \quad \beta_2 = \frac{\pi}{3} - \beta_1
\]

(5.30)

\[
L_2 = \left( \frac{\sqrt{3}}{2} + g_1 \right) \frac{R}{\cos \beta_1}
\]

(5.31)

\[
\delta_2 = \tan^{-1} \left[ \frac{L_1 \sin \alpha_1}{\sqrt{3}R - L_1 \cos \alpha_1} \right] \quad \delta_1 = \frac{\pi}{3} - \delta_2
\]

(5.32)

\[
L_3 = \frac{L_1 \sin \alpha_1}{\sin \delta_2}
\]

(5.33)

The volume \( V \) of the tetrahedral enclosing the channels would be

\[
V = \frac{\sqrt{3}}{4} L_0 R^2
\]

(5.34)

and the volume \( V_c \) occupied by the channels is

\[
V_c = \frac{\pi}{4} \left[ L_0 D_0^2 + L_1 D_1^2 + L_2 D_2^2 + L_3 D_3^2 \right]
\]

(5.35)

Now we define the pressure drop in the T-shaped channels. The total pressure drop will be the sum of individual pressure drops in the stem section and one of the branch sections and the pressure drop in the junction. Ignoring the pressured drop in the junction we can write the expression for total pressure drop such that
\[ \Delta P = \Delta P_s + \Delta P_A = \Delta P_s + \Delta P_B = \Delta P_s + \Delta P_c \] (5.36)

The pressure drop in the branch sections have to be equal to maintain a continuous and constant mass flow rate.

\[ \frac{L_1}{D_1^4} = \frac{L_2}{D_2^4} = \frac{L_3}{D_3^4} \] (5.37)

Assuming that the mass is dividing into equal proportions at the junction, we can write

\[ \Delta P = \frac{128}{\pi} \left( \frac{L_0}{D_0^4} + \frac{L_1}{3D_1^4} \right) \] (5.38)

The pressure drop is non-dimensionalized by defining the non-dimensional flow resistance number \( R_f \) such that

\[ R_f = \frac{\Delta P}{V_c^2} \] (5.39)

Using Equations 5.35, 5.38 and 5.39, we can write the expression for \( R_f \) such that

\[ R_f = \frac{L_0}{D_0^4} + \frac{L_1}{3D_1^4} \left( \frac{L_0D_0^2 + L_1D_1^2 + L_2D_2^2 + L_3D_3^2}{\sqrt{3L_0R^2/4}} \right) \] (5.40)

We can modify Equation 5.40 by defining length and diameter ratios, \( x_1 = L_1/L_0 \), \( x_2 = L_2/L_0 \), \( x_3 = L_3/L_0 \), \( y_1 = D_1/D_0 \), \( y_2 = D_2/D_0 \) and \( y_3 = D_3/D_0 \), such that \( R_f \) becomes

\[ R_f = \left[ 1 + \frac{x_1}{3y_1^4} \right] \left[ 1 + x_1y_1^2 + x_2y_2^2 + x_3y_3^2 \right]^{1/2} \] (5.41)

In the above expression \( x_1 \), \( x_2 \) and \( x_3 \) in turn can be written as functions of aspect ratio \( F \).

\[ x_1 = \left( \frac{\sqrt{3}}{2} - g_1 \right) \frac{F}{\cos \alpha_2} \] (5.42)

\[ x_2 = \left( \frac{\sqrt{3}}{2} + g_1 \right) \frac{F}{\cos \beta_1} \] (5.43)

\[ x_3 = \frac{x_1 \sin \alpha_1}{\sin \delta_2} \] (5.44)
Also, we can define $y_2$ and $y_3$ as functions of $y_1$ using Equation 5.37.

\[
y_2 = \left( \frac{x_2}{x_1} \right)^{1/4} y_1
\]

\[
y_3 = \left( \frac{x_3}{x_1} \right)^{1/4} y_1
\]  

We can observe that using Equations 5.41-5.46 we can express $R_f$ as a function that depends only on $F$, $g_1$, $g_2$ and $y_1$. For a set of geometric deviation $g_1$ and $g_2$, we can calculate optimal value for $y_1$ (diameter ratio) at different aspect ratios that minimizes the non-dimensional flow resistance number $R_f$. From the optimal $y_1$, we can proceed to calculate other optimal diameter ratios $y_2$ and $y_3$ using Equations 5.45 and 5.46.
Figures 5.12a-5.12f show the variation in $R_{f_{_{\text{min}}}}$ with aspect ratio at different geometric deviations $g_1$ and $g_2$. While $g_2$ increases from 0.1 to 0.5 the variation in

Figure 5.12 Variation in non-dimensional flow resistance $R_{f_{_{\text{min}}}}$ with aspect ratio and geometric deviation
\( R_{f,\text{min}} \) is observed at different \( g_1 \) values. Figures 5.12a-5.12f also show \( R_{f,\text{min}} \) values for a T-shaped configuration (plotted in dotted line) that is symmetric both in geometry \((g_1 = g_2 = 0)\) and mass distribution, for comparison purposes. It is observed that flow resistance of asymmetric configurations is larger than the flow resistance of symmetric configurations. Also, the flow resistance increases with increasing \( g_1 \) at any particular \( g_2 \) value. The variation in \( R_{f,\text{min}} \) with \( g_1 \) is considerably large when \( g_2 = 0.1 \) compared to the cases when \( g_2 = 0.5 \). This can be attributed to the varying position of the source with respect to the outlets at different geometric deviations. The aspect ratio where minimum \( R_{f,\text{min}} \) is observed decreases with increasing geometric deviation. For example, in the symmetric T-shaped configuration the least value for \( R_{f,\text{min}} \) is observed at an aspect ratio of \( F \sim 1.4 \) where as in the asymmetric case with \( g_1 = 0.5 \) and \( g_2 = 0.1 \), the least value for \( R_{f,\text{min}} \) is observed at an aspect ratio of \( F \sim 1.1 \).

The length ratios of the branch sections \((x_1, x_2 \text{ and } x_3)\) are defined by the aspect ratio \( F \) and geometric deviations, \( g_1 \) and \( g_2 \). The diameter ratios \((y_1, y_2 \text{ and } y_3)\), however, can be optimized to minimize flow resistance. We have observed that the optimal diameter ratios are independent of aspect ratio and depend only upon mass fraction in the channels and the geometric deviations. While keeping the mass fraction in the individual branch sections constant \((\gamma_1 = \gamma_2 = 1/3)\), we have obtained optimal diameter ratios at different geometric deviations. Figure 5.13a, b and c illustrate the
Figure 5.13 Variation in optimal diameter ratios at different geometric deviations
variation in optimal diameter ratios \((D_1/D_0)_{opt}\), \((D_2/D_0)_{opt}\), and \((D_3/D_0)_{opt}\) within a range of \(g_1\) and \(g_2\) values. From Figure 5.13a, we can observe that \((D_1/D_0)_{opt}\) decreases with increasing \(g_1\) as well as \(g_2\) values. At any particular \(g_1\), the difference in \((D_1/D_0)_{opt}\) between \(g_2 = 0.1\) and \(g_2 = 0.2\) is larger than the difference between \(g_2 = 0.4\) and \(g_2 = 0.5\). That is, \((D_1/D_0)_{opt}\) decreases more rapidly within the initial increase in geometric deviation \(g_2\) and does not show significant variation at higher \(g_2\) values. We can also observe that the \((D_1/D_0)_{opt}\) reaches the smallest value when \(g_1 = 0.5\). This is due to the reduction the length of the branch section A, which results in decrease in pressure drop. The decrease is pressure drop is compensated by reducing the diameter of the branch section A and effectively reducing the optimal diameter ratio.

Figures 5.13b shows the variation in optimal diameter ratio \((D_2/D_0)_{opt}\) at different geometric deviations. \((D_2/D_0)_{opt}\) increases with increasing \(g_1\) values, however the variation in \(g_2\) plays an important role. At lower \(g_1\) values (e.g. \(g_1 = 0.01\)), increasing \(g_2\) decreases the \((D_2/D_0)_{opt}\). However, at higher \(g_1\) values (e.g. \(g_1 = 0.5\)), increasing \(g_2\) increases the \((D_2/D_0)_{opt}\). Figure 5.11c shows the optimal diameter ratio \((D_3/D_0)_{opt}\) at different geometric deviations. We can observe that \((D_3/D_0)_{opt}\) increased with increasing \(g_1\) when \(g_2 = 0.01\), however, no significant increase is observed when \(g_2 = 0.5\). Figure 5.14 shows different T-shaped configurations optimal for flow distribution with asymmetry in the position of source.
Figure 5.14 Optimal configurations for 3D T-shaped channels with geometry induced asymmetry

\[ g_1 = 0.1 \]
\[ y_1 = 0.661, \ y_2 = 0.693 \]
\[ y_3 = 0.721 \]

\[ g_2 = 0.2 \]
\[ y_1 = 0.653, \ y_2 = 0.687 \]
\[ y_3 = 0.729 \]

\[ g_3 = 0.3 \]
\[ y_1 = 0.646, \ y_2 = 0.682 \]
\[ y_3 = 0.737 \]
5.6 Conclusion

The optimization of 3D T-shaped flow channels yield the following conclusions: (1) the optimal diameter ratio in 3D T-shaped channels that are symmetric in geometry and mass distribution are only governed by the number of branch sections as long as the junction losses and wall roughness are ignored, (2) when the mass flow rate, volume occupied by the channels and the pyramidal volume influenced by the channels are constrained, increasing the number of branch sections reduces the flow resistance, (3) mass induced asymmetry in the branch sections reduces the flow resistance compared to symmetric mass distribution, (4) the optimal diameter ratio of a branch section depends only upon the fraction of mass flow rate passing through that particular branch section, (5) geometrical asymmetry in 3D T-shaped channels accounts for additional flow resistance.
CHAPTER 6
SUMMARY AND FUTURE WORK

6.1 Summary

The dissertation on ‘geometry optimization of elemental flow constructs with asymmetric bifurcation’ is aimed at developing fundamental results for the design of optimal fluid flow channels. Different fluid channel configurations are optimized by reducing overall pressure drop. Comparisons between different configurations are made by constraining the volume occupied by the channels and the area (in 2D channels) or the volume (in 3D channels) influenced by the channels. The Pressure drop in the channels is non-dimensionalized using fluid properties, inlet mass flow rate, the volume occupied by the channels, and the area/volume influenced by the channels. The optimization results in most cases are in the form of ratios between length and diameters of stem and branch sections. In certain cases such as Y-shaped channels, the optimal angles of bifurcation are also reported.

One of the important assumptions made during the optimization is neglecting the local junction losses. Though junction losses contribute to significant pressure drop in certain channel configurations, we are able to obtain design limits where junction losses can be ignored in laminar flow regime. Also, it is assumed that flow channels are circular in cross section. The optimization results of circular channels can be extended to channels of other cross sectional shapes without losing any authority.

As the results are non-dimensional, they can be applied in a broad range of length scales. One of the limitations on the applicable scale region is when the channel lengths and diameters are in the orders of millimeters or microns where capillary forces become mode dominant compared to viscous forces. The pressure drop calculations and optimization results discussed in this work are no longer valid when that is the case.

Many researchers have performed optimization of flow channels that are symmetric in both geometry and mass distribution. However, most of the flow distribution applications demand channels with mass imbalance or asymmetries in
geometry. Optimization of such designs is the novelty of the current work. We have obtained optimal T-shaped and Y-shaped channel configurations with mass and geometry induced asymmetries in 2D and T-shaped channels in 3D. Some of the results are fundamental and become building blocks of flow channel design.

In Chapter 2, we have calculated the effects of mass flow rate on pressure drop in the channel by introducing a valve and increasing the mass flow rate across the valve. We used the formulation obtained in optimizing radial flow channels networks on a circular disk-like surface. It is found that channels with multiple branching exert lesser flow resistance compared to radial channels. We have also calculated flow resistance and optimal geometric configurations of Y-shaped channels with mass induced asymmetries.

In Chapter 3, we have optimized T-shaped channels in 2D. We started from a simple T-shaped channels symmetric in geometry and mass distribution. We optimized the aspect ratio of the rectangle enclosing the channels. It was found that optimal aspect ratio for symmetric T-shaped channels is \( \sqrt{2} \). Also, the optimal diameter ratio (ratio between the diameters of branch and stem sections) is equal to a constant value \( 1/2^{1/3} \). This configuration yields minimum pressure drop of any T-shaped channel configuration that is symmetric in geometry and mass distribution.

We have continued on T-shaped channels by introducing mass imbalance in the branch sections. It was found that the fraction of mass flow rate in the branch sections govern the diameter ratio of the channels. Also, the flow resistance is reduced with mass imbalance in the branch sections when compared to the case with symmetric mass distribution. The optimal aspect ratio has reduced with increasing mass fraction to reach a constant value of \( 2^{2/3} \) when the mass fraction in both branch sections is equal to 0.5 and starts increasing again for mass fractions above 0.5. The diameter ratio, on the other hand, increases with increasing mass fraction until it reaches 1 when the mass fraction is 1. When the mass fraction is 0.5, the diameter ratio is a constant value of \( 1/2^{1/3} \), a result we obtained in the symmetric T-shaped channels.

We continued optimizing T-shaped channels in 2D by introducing geometry induced asymmetries. The position of source is varied along the base of the rectangle enclosing the channels. It is found that geometrical asymmetries increase the flow resistance when compared to the symmetric channels. When the mass bifurcates into
equal proportions, the optimal diameter ratio is found to depend only upon the geometric deviation of the source position.

In Chapter 4, we have optimized Y-shaped channels in 2D connecting two outlets to a source. We started out optimizing symmetric Y-shaped channels. It is found that Y-shaped channels are more optimal than T-shaped channels under identical constraints. The channels are found to be in Y-shape until the aspect ratio becomes 1.5 and the channels transform into V-shape for aspect ratios above 1.5. Also, the optimal Y-shape of all aspect ratios is in fact a V-shape which forms at aspect ratio 2. The angle of bifurcation in the optimal Y-shaped channels is always a constant and is approximately equal to 37.5°.

In the later section we have introduced mass imbalance in the branch sections of the Y-shaped channel. Similar to T-shape, the flow resistance reduces with mass imbalance in Y-shaped channels. The geometry asymmetry was introduced by varying the position of the source. It is found that asymmetry in geometry increases the flow resistance in Y-shaped channels as well.

In Chapter 5, we focus on the optimization of T-shaped channels in 3D. In 3D T-shaped channels, the outlets are assumed to be in a plane which is perpendicular to the stem section of the T-shape. The number of outlets can be three or more than three. During optimization, we assumed that outlets form a regular shape such as equilateral triangle (in the case of 3 outlets), square (4 outlets) or a pentagon (5 outlets). In the symmetric T-shaped channels, the source is placed equidistantly from all the outlets in the third dimension. The volume occupied by the channels and the volume enclosing the channels are constrained during the optimization.

The results of symmetric T-shaped channels show that the optimal diameter ratio depends upon the number of outlets N such that \( (D_{i+1}/D_i) = 1/\sqrt[3]{N} \). Also, the flow resistance reduces with increasing number of outlets. We have introduced mass asymmetry in T-shaped channels with 3 outlets. It was found that optimal diameter ratio of a branch section depends only upon the mass fraction of the fluid entering that branch section. Similar to the observations made in 2D channels, the mass imbalance in the branch sections reduces the flow resistance in 3D T-shaped channels. In the following section, geometry induced asymmetry is introduced by varying the position of the source.
Geometrical asymmetries are found to increase the flow resistance and have significant effect on the optimal diameter ratios of the branch sections.

6.2 Future Work

As discussed in section 6.1, further investigation is need in obtaining optimal geometric configurations for 3D Y-shaped channels with mass and geometry induced asymmetries. Considering the number of degrees of freedom and related variables, the computational power required is estimated to be $10^2$ to $10^4$ times more than the T-shaped channels in 3D. Advanced workstations and optimizations techniques are expected to help in obtaining optimal configurations for 3D Y-shaped channels.

Another important extension would be asymmetric flow networks designed using the elemental flow constructs optimized in the current work. Flow networks serving multiple outlets in 2D and 3D can be designed by proper utilization of current results to obtain optimal channel configuration. Examples of such applications include utility water distribution, heat exchangers, natural gas pipelines and HVAC ducts.

Some of the results obtained in the current work are observed to depend only upon one variable (eg. optimal diameter ratios in 2D T-shaped channels with mass asymmetry) or two variables (eg. 2D T-shaped channels with mass and geometry induced asymmetry, 2D Y-shaped channels with mass or geometry asymmetry). Reporting those results in the form of analytical expressions rather than in graphs would be desirable.
APPENDIX

EFFECT OF LOCAL JUNCTION LOSSES IN T-SHAPED CHANNELS IN LAMINAR FLOW REGIME

A T-shaped channel essentially has a stem section with two branch sections bifurcating at rectangles as shown in Figure A.1. The rectangular area under the branch sections is known as area of influence and is represented by $A_c$. From Figure A.1,

$$A_c = 2L_1L_2$$

(A.1)

The volume occupied by the channels $V_c$ is the sum of volumes occupied by stem section and two branch sections.

$$V_c = \frac{\pi}{4} \left( L_1 D_1^2 + 2L_2 D_2^2 \right)$$

(A.2)

We define a geometrical parameter called ‘svelteness’ to relate the area of influence and the volume occupied by the channels. It can be written as
During the optimization, both \( A_C \) and \( V_C \) are constrained for all configurations, which make the svelteness \( S_V \) of the design also constrained. The only degrees of freedom are the length and diameters of stem and branch sections. The mass is assumed to be bifurcating into equal proportions at the junction and the source is placed equidistantly between the outlets as shown in Figure A.1.

**Optimization Procedure**

In a T-shaped channel, the total pressure drop will be equal to the sum of individual pressure drops in stem and branch sections and the pressure drop in T-junction.

\[
\Delta P = \Delta P_S + \Delta P_B + \Delta P_T \tag{A.4}
\]

The stem and branch sections behave like individual channels whose pressure drop can be expressed using the expression

\[
\Delta P_i = \frac{1}{\rho} \frac{f_i}{2} \frac{L_i}{D_i^5} \frac{m_i^2}{(\pi/4)^2} \tag{A.5}
\]

where \( L_i \) is the length of the channel, \( D_i \) is the diameter, \( m_i \) is the mass flow rate, \( \rho \) is density of the fluid and \( f_i \) is friction coefficient.

The pressure drop in the T-junction can be approximated as

\[
\Delta P_T = \frac{f_T}{2} \frac{m^2}{(\pi/4)^2 D_1^4} \tag{A.6}
\]

where \( A_C = 2L_1L_2 \) is the friction coefficient for T-junction.

Using Equations (A.4), (A.5) and (A.6), we can write the expression for total pressure drop as

\[
\Delta P = \frac{1}{\rho} \left[ \frac{f_1}{2} \frac{L_1}{D_1^5} \frac{m^2}{(\pi/4)^2} + \frac{f_2}{8} \frac{L_2}{D_2^5} \frac{m^2}{(\pi/4)^2} + \frac{f_T}{2} \frac{m^2}{(\pi/4)^2 D_1^4} \right] \tag{A.7}
\]

Equation (A.7) can be modified by eliminating geometrical parameters using non-dimensional length scales \( \hat{L}_i = L_i/A^{1/2} \) and \( \hat{L}_2 = L_2/A^{1/2} \) such that
\[
\dot{m}^2 = \frac{\rho \Delta P}{\left(\pi/4\right)^{1/2}} \frac{V_c^{3/2}}{R_f \ A_c^{3/4}}
\]

(A.8)

where \( R_f \) is a non-dimensional flow resistance number and is equal to

\[
R_f = \left[ \frac{f_1 \hat{L}_1}{2} + \frac{f_2}{8} \hat{L}_2 \left( \frac{D_1}{D_2} \right)^{1/2} \right] \left[ \hat{L}_1 + 2 \hat{L}_2 \left( \frac{D_2}{D_1} \right)^{2} \right]^{3/2} + \frac{f_T}{2} \left[ \hat{L}_1 + 2 \hat{L}_2 \left( \frac{D_2}{D_1} \right)^{2} \right] \frac{S_v^{3/2}}{\left(\pi/4\right)^{1/2}}
\]

(A.9)

From Equation (A.8), it can be observed that for a fixed pressure drop \( \Delta P \), mass flow rate \( \dot{m} \) can be increased by reducing \( R_f \). However, to find the \( R_f \), friction coefficients \( f_1, f_2 \) and \( f_T \) are to be calculated. In laminar flows, \( f_1 \) and \( f_2 \) can simply written as

\[
f_{1,2} = \frac{64}{\text{Re}_{1,2}}
\]

(A.10)

where \( \text{Re} \) is the Reynolds number inside the channel. The Reynolds number in the stem section \( \text{Re}_1 \) is related to channel geometry in such a way that

\[
\text{Re}_1 = \frac{\dot{m}}{\mu A^{1/2}} \left(\frac{\pi/4}{\left(\pi/4\right)^{1/2}}\right)^{1/2} \left[ \hat{L}_1 + 2 \hat{L}_2 \left( \frac{D_2}{D_1} \right)^{2} \right]^{1/2}
\]

(A.11)

The Reynolds number in branch section \( \text{Re}_2 \) can be calculated from \( \text{Re}_1 \) such that

\[
\text{Re}_2 = \text{Re}_1 \frac{D_1}{2D_2}
\]

(A.12)

Using Equations (A.10), (A.11) and (A.12), friction coefficients \( f_1 \) and \( f_2 \) can be calculated.

For the friction coefficient at the T-junction \( f_T \), Idelchik [60] approximates it as

\[
f_T = 1 + K f_1 \left( \frac{D_1}{4D_2} \right)^{4}
\]

(A.13)

where \( K = 1 \) for merging flow and \( K = 1.5 \) for splitting flow.

Once the friction coefficients are available it is now possible to calculate \( R_f \) at different geometrical parameters. The expression for \( R_f \) can be modified further by introducing length and diameter ratios of stem and branch sections.
\[ R_t = \left[ \frac{(1 + 2xy^2)^2(\pi/4)^{1/2}}{2xS_v^{3/2}} \right] \left[ \frac{1}{\dot{m}/\mu A_c^{1/2}} \left( \frac{1}{(2x)^{1/2}} \left( 32 + 16 \frac{x}{y^4} \right) + \frac{2}{\pi} \left( 1 + \frac{K}{4y^4} \right) \right) \right] \]  \hspace{1cm} (A.14)

where \( x = \hat{L}_2/\hat{L}_1 = L_2/L_1 \) and \( y = D_2/D_1 \) are the ratios to be optimized.

**Results and Discussion**

A close observation of Equation (A.14) shows that minimization of \( R_t \) is independent of svelteness \( S_v \) and depends only on mass flow parameter \( \dot{m}/\mu A_c^{1/2} \). We can calculate optimal \( x \) and \( y \) where \( R_t \) is minimum at different \( \dot{m}/\mu A_c^{1/2} \) values. Calculations can be made separately for splitting and merging flow by modifying the K value in Equation (A.14). Figure A.2 shows the variation in \( R_{t, \text{min}} \) values against mass flow parameter \( \dot{m}/\mu A_c^{1/2} \) in splitting and merging flow.
merging flows. It can be observed that as $\dot{m}/\mu A_c^{1/2}$ increases, the flow resistance number $R_{f,\text{min}}$ decreases. At large values of $\dot{m}/\mu A_c^{1/2}$, the splitting flow is observed to exert lesser flow resistance compared to merging flows.

Figure A.3 shows the variation in optimal length ratio $(L_2/L_1)_{\text{opt}}$ against mass flow parameter $\dot{m}/\mu A_c^{1/2}$. The change in optimal length ratio is observed to be more rapid in merging flows compared to splitting flows at higher $\dot{m}/\mu A_c^{1/2}$ values. It can also be observed that for $\dot{m}/\mu A_c^{1/2} < 1$, there is no considerable variation in the optimal length ratio. The value reached a constant and is equal to $1/2^{1/3}$ which is also an optimal length ratio when the junction losses are ignored [26].
Figure A.4 Variation in optimal diameter ratio $(D_2/D_1)_{opt}$ with mass flow parameter $\dot{m}/\mu A_c^{1/2}$

Figure A.4 shows the variation in optimal diameter ratio $y = D_2/D_1$ against mass flow parameter $\dot{m}/\mu A_c^{1/2}$. Here also, the variation in the optimal diameter is observed to be more rapid in merging flows compared to splitting flows at high $\dot{m}/\mu A_c^{1/2}$ values. The optimal diameter ratio became independent of mass flow parameter for $\dot{m}/\mu A_c^{1/2} < 0.1$. The optimal diameter ratio reached a constant value of $1/2^{1/3}$ which is optimal when the junction losses are ignored [26].

**Conclusions**

T-shaped channels are optimized within laminar flow regime while considering local junction losses. Optimal geometric parameters such as channel length and diameter ratios are calculated by minimizing pressure drop across the channels. The area of influence and the volume occupied by the channels are constrained in the optimization. It is found that optimal results are independent of design svelteness and only functions of
mass flow parameter $\dot{m}/\mu A_c^{1/2}$. This is an important difference because in turbulent flows, both svelteness and mass flow parameters are found to influence the optimal ratios [56]. Merging flows have rapid change in optimal ratios at higher $\dot{m}/\mu A_c^{1/2}$ values. At lower $\dot{m}/\mu A_c^{1/2}$ values ($< 0.1$), both length and diameter ratios became independent of mass flow parameter and retained a constant value of $1/2^{1/3}$, suggesting the limits where junction losses can be ignored in laminar flows.
### TABLES

Table 1: Optimized geometric features of symmetrical trees with two levels of pairing

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Table 2: Optimized geometric features of symmetrical trees with three levels of pairing

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Table 3: Optimized geometric features of symmetrical trees with four levels of pairing

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Table 4: Optimized geometric features of symmetrical trees with five levels of pairing

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Table 6: Optimized geometric features of symmetrical trees with seven levels of pairing

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|----|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|-------|
| 3  | 37.5    | 38.1    | 43.1    | 49.8    | 56.6    | 62.8    | 68      | 0.106   | 0.405   | 0.325   | 0.174   | 0.083   | 0.039   | 0.018   | 0.009   | 21.6  |
| 4  | 37.5    | 39.9    | 45.9    | 52.7    | 59.2    | 65      | 69.9    | 0.269   | 0.398   | 0.257   | 0.128   | 0.06    | 0.028   | 0.014   | 0.007   | 21.9  |
| 5  | 37.5    | 41.5    | 48      | 54.8    | 61.2    | 66.7    | 71.3    | 0.393   | 0.364   | 0.209   | 0.101   | 0.047   | 0.022   | 0.011   | 0.005   | 22.2  |
| 6  | 37.5    | 42.8    | 49.7    | 56.5    | 62.6    | 67.9    | 72.1    | 0.486   | 0.329   | 0.175   | 0.083   | 0.039   | 0.018   | 0.009   | 0.004   | 22.6  |
| 7  | 37.5    | 43.8    | 50.9    | 57.7    | 63.7    | 68.7    | 72.9    | 0.556   | 0.297   | 0.15    | 0.07    | 0.033   | 0.016   | 0.008   | 0.004   | 23.1  |
| 8  | 37.5    | 44.6    | 51.9    | 58.7    | 64.7    | 69.6    | 73.6    | 0.61    | 0.269   | 0.131   | 0.061   | 0.029   | 0.014   | 0.007   | 0.003   | 23.6  |
REFERENCES


BIOGRAPHICAL SKETCH

Srinivas Chakravarthi Kosaraju

Srinivas C. Kosaraju was born on June 5th, 1979 in Vijayawada, India. He obtained his Bachelors degree in Mechanical Engineering from Nagarjuna University, Guntur, India in 2001. He attended Rutgers University in New Jersey from Jan 2002 to May 2004 and obtained MS in Mechanical Engineering. During his term at Rutgers, he specialized in analyzing flow dynamics of optical fiber coating process using numerical simulations and experimental techniques such as particle image velocimetry (PIV).

In August 2004 he joined Florida State University, Tallahassee, Florida in PhD program. He initially worked at Fluid Mechanics Research Laboratory (FMRL) under the advisement of Dr. Krothapalli Anjaneyulu. During his term at FMRL, he worked on high density cooling applications of microjets in PEM Fuel Cells. He joined the Thermal Management group at Center for Advanced Power Systems (CAPS) under the advisement of Dr. Juan Ordonez in Jan 2006. At CAPS, he worked on geometry optimization techniques for fluid channels that find applications in flow distribution, heat exchangers and HVAC.

Mr. Srinivas Kosaraju is an active member of American Society for Mechanical Engineers (ASME), American Physical Society (APS) and Indian Society for Technical Education (ISTE).