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Partial Differential Equation Methods to Price Options in the Energy Market

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PARTIAL DIFFERENTIAL EQUATION METHODS TO PRICE OPTIONS IN THE ENERGY MARKET

By

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To my parents...
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ABSTRACT

We develop partial differential equation methods with well-posed boundary conditions to price average strike options and swing options in the energy market. We use the energy method to develop boundary conditions that make a two space variable model of Asian options well-posed on a finite domain. To test the performance of well-posed boundary conditions, we price an average strike call. We also derive new boundary conditions for the average strike option from the put-call parity. Numerical results show that well-posed boundary conditions are working appropriately and solutions with new boundary conditions match the similarity solution significantly better than those provided in the existing literature.

To price swing options, we develop a finite element penalty method on a one factor mean reverting diffusion model. We use the energy method to find well-posed boundary conditions on a finite domain, derive formulas to estimate the size of the numerical domain, develop a priori error estimates for both Dirichlet boundary conditions and Neumann boundary conditions. We verify the results through numerical experiments. Since the optimal exercise price is unknown in advance, which makes the swing option valuation challenging, we use a penalty method to resolve the difficulty caused by the early exercise feature. Numerical results show that the finite element penalty method is thousands times faster than the Binomial tree method at the same level of accuracy. Furthermore, we price a multiple right swing option with different strike prices. We find that a jump discontinuity can occur in the initial condition of a swing right since the exercise of another swing right may force its optimal exercise region to shrink. We develop an algorithm to identify the optimal exercise boundary at each time level, which allows us to record the optimal exercise time. Numerical results are accurate to one cent comparing with the benchmark solutions computed by a Binomial tree method.

We extend applications to multiple right swing options with a waiting period restriction. A waiting period exists between two swing rights to be exercised successively, so we cannot exercise the latter right when we see an optimal exercise opportunity within the waiting period, but have to wait for the first optimal exercise opportunity after the waiting period. Therefore, we keep track of the optimal exercise time when pricing each swing right. We also verify an extreme case numerically. When the waiting time decreases, the value of $M$ right swing option price increases to the value of $M$ times an American option price as expected.
CHAPTER 1

INTRODUCTION

In this dissertation, we develop partial differential equation (PDE) methods with well-posed boundary conditions to price average strike options and swing options in the energy market. If one wants to solve a PDE model numerically, one has to solve it with suitable boundary conditions on a finite numerical domain that make the problem well-posed, which we call well-posed boundary conditions. This is the motivation of this dissertation.

For the classic Black-Scholes model, there are well-developed theories and numerical techniques that provide well-posed boundary conditions. One can easily show that Dirichlet boundary conditions (DBCs) or Neumann boundary conditions (NBCs) would make the option pricing problem well-posed. There is also a theoretical result in [12] to support the common choice of the numerical domain, $[0, 2S]$. However, average strike options and swing options in the energy market are relatively new and a careful study about well-posed boundary conditions is lacking. These options are more complex than options in financial markets and their pricing models are accordingly more complex than the Black-Scholes model. It would be careless to apply the boundary conditions for the Black-Scholes model directly to these new PDE models.

Therefore, we develop PDE methods to solve an average strike option model and a one factor mean reverting diffusion model for swing options, with careful attention paid to well-posed boundary conditions and the size of numerical domain. Furthermore, closed form formulas are not available for these models, so we use PDE approaches to approximate solutions. Since the Black-Scholes model is easy to solve and explicitly models the relationship of the option price to the underlying asset price $S$, we will use the Black-Scholes model as a tool to test the performance of numerical schemes.

1.1 Background and Significance

Here we introduce the financial background of average strike options and swing options, their PDE valuation models and existing work about pertinent boundary conditions. We then reach a conclusion: Well-posed boundary conditions need to be clearly defined. We show that average strike options and swing options are relatively new and their generic features make them more complex than options in financial markets through a brief introduction about definitions, features, even a history of development. Their PDE valuation models are more complex than the Black-Scholes model. To solve their PDE models numer-
ically, one need to define well-posed boundary conditions. After a review of existing work about boundary conditions that have been applied, we find that it is still not clear about how those boundary conditions make the pertinent problem well-posed.

**Background Review of Average Strike Option Valuation.** We first introduce the definition and the history of Asian options in general and then move to average strike options. The payoff of Asian options depends on an average price of the underlying asset during at least some part of the life of the option [10]. Asian options allow investors to eliminate losses (and at the same time, sacrifice profits) from movements in an underlying asset without the need for continuously rehedging. They have a lower volatility and hence they are cheaper than their European counterparts [10]. Asian options are used in part to keep speculators from driving up the gains through manipulating the price of an asset near the maturity date [25]. Asian options were used first in 1987 when Banker’s Trust Tokyo office priced average options on crude oil contracts. They are commonly traded in thin markets, like crude oil markets [21].

Eight basic kinds of Asian options are used [10, 21]. An Asian option could be a call or a put. The average could be defined as a geometric or arithmetic average of the stock price over the contract term. In addition, the average stock price could be used in the payoff either in place of the underlying asset price or in place of the strike price. When the average is in place of the asset price, the option is called an average price option (or a fixed strike option). When the average is in place of the strike price, the option is called an average strike option (or a floating strike option). Four types of arithmetic average Asian options are average price calls, average strike calls, average price puts and average strike puts.

We are interested in the valuation of arithmetic average options. The geometric average stock price is computationally easier, but less common in practice [21]. Pricing formulas exist for options based on the geometric average [10, 21], because if the underlying asset price $S$ is lognormally distributed, the geometric average is also lognormally distributed [10, 21]. The arithmetic average option is harder to price than the geometric average. When Asian options are defined using the arithmetic average, exact analytic pricing formulas are not available because the distribution of the sum of a set of lognormal distributions is unknown [10, 21]. One must use numerical methods to price arithmetic average options. We will only price the arithmetic average option.

Before a review of boundary conditions provided in the existing literature, we present the PDE valuation problem of arithmetic average options. The arithmetic average option pricing problem uses the Black-Scholes valuation framework with some basic assumptions:

- An idealized security market (no arbitrage) offers a constant continuously compounded risk free rate of return $r$.
- No transaction costs or taxes are incurred.
- The present time is zero.
- The underlying asset on which the option is based is equal to a stock with price, $S(t)$, where

\[ dS(t) = rS(t)dt + \sigma S(t)dW(t), \]  \hspace{1cm} (1.1)
and \( \sigma \) are constants, and \( W(t) \) is a Wiener process.

Because of the path dependence, arithmetic average options have a more complex PDE valuation model than the Black-Scholes model, i.e., a three dimensional PDE. The arithmetic average option pricing problem takes the path dependent average as one variable \([1, 32, 33]\) by introducing the integral quantity

\[
I(t) = \int_0^t S(u)du \tag{1.2}
\]

or the average quantity

\[
A(t) = \frac{1}{t} \int_0^t S(u)du, \tag{1.3}
\]

and takes the option value as \( V(I, S, t) \). The PDE for the value of the arithmetic average option is \([32]\)

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial I} - rV = 0. \tag{1.4}
\]

The PDE (1.4) is three dimensional: one time dimension and two space-like dimensions. We will call this type of PDE a two space variable PDE. The two space variable PDE (1.4) is typically posed with a final payoff function \( V(I, S, T) = \Lambda(I, S, T) \) at the maturity \( t = T \) with a specified strike price \( K \), where

- Average price call: \( \Lambda(I, S, T) = \max(I(T) - K, 0) \),
- Average price put: \( \Lambda(I, S, T) = \max(K - I(T), 0) \),
- Average strike call: \( \Lambda(I, S, T) = \max(S(T) - I(T), 0) \),
- Average strike put: \( \Lambda(I, S, T) = \max(I(T) - S(T), 0) \).

Several authors have used an alternative form of the PDE (1.4) \([1, 9, 22, 35]\). From (1.2) and (1.3),

\[
dI(t) = S(t)dt \tag{1.5}
\]

and

\[
dA(t) = (\frac{S(t)}{t} - \frac{I}{t^2})dt = \frac{S - A}{t}dt, \tag{1.6}
\]

which gives an equivalent PDE

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + \frac{S - A}{t} \frac{\partial V}{\partial A} - rV = 0. \tag{1.7}
\]

The PDE (1.7), with the final payoff and boundary conditions, becomes an equivalent two space variable problem.
Since an explicit pricing formula for an arithmetic average option is still unknown, a considerable amount of effort has concentrated on deriving approximate methods to estimate \( V \) or an upper and lower bound on \( V \) [25, 3]. Previous work falls into three groups: Analytical approximation [25, 18, 17, 29, 7], the PDE approach [11, 35, 34, 20], and Monte Carlo simulations [13, 28]. Our particular interests in this dissertation are the PDE approaches, which fall into two groups depending on whether a one space variable problem or a two space variable problem is solved.

In general, the value of an arithmetic average option is a two space variable problem, but can be reduced to a one space variable problem for some special cases [11, 32, 25, 28]. Ingersoll [11] finds that the average strike call permits a similarity reduction where the option value depends only on two variables \( S/I \) and \( t \). Ingersoll does not solve the reduced problem, but provides a Dirichlet boundary condition along \( S/I = 0 \) and a Neumann boundary condition along \( S/I = \infty \). However, Ingersoll does not show how a Neumann boundary condition along \( S/I = \infty \) makes the reduced problem well-posed. It turns out that a Dirichlet boundary condition along \( S/I = \infty \) makes the reduced problem well-posed. Later, Wilmott, Dewynne and Howison [32] provided a PDE method to price the average strike call using a different similarity reduction and obtained similarity solutions. They used \( I/S \) as a similarity variable to reduce the dimensionality of (1.4) and used finite difference methods to price a European average strike call and an American average strike call. They provided boundary conditions without discussing the well-posedness. (To complete their work, we show in Appendix A that the boundary conditions they provide make the reduced problem well-posed.)

Unfortunately, the average price option does not admit a similarity reduction, since the structural form of the payoff is \( \max(I/T - K, 0) \) at expiry does not depend on the similarity variable \( I/S \).

Similarity solutions are still useful since they provide a tool to test the performance of a PDE approach that prices a general arithmetic average option. Rogers and Shi [25] reduce the general two space state problem to a problem depending on \( \frac{K-I/T}{S} \) and \( t \), which can be used to price the European fixed strike option. When \( K = 0 \), it becomes the similarity reduced formulation by Wilmott, Dewynne and Howison but can be used to price only the European average strike option since the probability density is defined when exercise occurs at expiry. Večer [28] developed a so-called “unifying approach” that leads to a one space variable PDE, which explicitly depends on the maturity of the option. Kim [14] proved that the generalized solution of the resulting problem introduced by Večer is indeed a classical solution and pointed out that the regularity of the generalized solution remains unclear.

Two space variable PDE models (1.4) or (1.7) are the most generalized models, and can be used to price any Asian option, but there is a need to study the boundary conditions. Two space variable problems have been solved without showing that boundary conditions make the problem well-posed [20, 32, 34, 35]. Marcozzi [20] approximated the boundary conditions by discounting the payoff at the risk free rate. Zhu and Stokes [34] solved (1.7) with a Galerkin finite element method but did not specify boundary conditions for \( S \to \infty \) and \( A \to \infty \). They claim that if the boundary conditions do not come from a clear financial argument, it is better to “impose no boundary conditions at all” [34]. Zvan, Forsyth, and Vetzal [35] priced both average price and average strike options by solving (1.7) with a flux-limiting technique and obtained solutions free of oscillations. The authors mention that an
“appropriate” boundary condition should be imposed, but they don’t show what boundary conditions they choose. Wilmott, Dewynne and Howison [32] presented the set of boundary conditions

\[
V(I, S, t) \sim S \quad \text{as} \quad S \to \infty \quad (1.8)
\]
\[
V(I, 0, t) = 0 \quad (1.9)
\]
without analysis or numerical evidence.

**Background Review of Swing Option Valuation.** Swing options are complex and versatile, but the literature about the swing option valuation is lacking. Here we introduce the financial background of swing options, including the history, current situations, a definition and features. The financial background implies that swing options have unique features that make the swing option valuation model different from the Black-Scholes model. For the PDE approach, we need to define boundary conditions to make the swing option valuation model well-posed. A review of existing PDE approaches show that well-posed boundary conditions have not been derived.

Before we go through the details of the PDE model of swing options, we will introduce how swing options are developed. Professionals in the industry develop swing options to hedge volumetric risks in the energy market, which have a shorter history than options do in financial markets. Before the 1980’s, energy prices were determined by regulatory authorities controlled by each government and simply reflected the cost of energy production. There were no risks for producers and no drive for them to lower the cost [5]. However, consumers wanted better services and lower prices.

Energy markets were steadily opened to competition at the wholesale level in the 1980s [5]. Consumers in parts of the U.S. were allowed to select a supplier other than a traditional utility. In the early 1990’s, natural gas prices started to be deregulated in the U.S., which resulted in higher productivity and lower energy prices overall [5]. Since then, energy markets were gradually deregulated from one commodity to another and from one country to another [5]. Furthermore, financial derivatives trading in the energy markets around the world has developed dramatically [5].

Today, energy is traded as a commodity in the form of a delivery contract on energy exchanges or over the counter [5, 36]. Energy exchanges refer to the futures markets, which are found on regulated financial exchanges such the NYMEX and London’s ICE Futures (formally called the International Petroleum Exchange or IPE, and owned by the Intercontinental Exchange (ICE)) [36].

Over the counter energy derivatives include swaps and over the counter options, which are usually traded directly between two private parties in the energy markets [5]. Generally, all the key terms of an over the counter derivatives deal are negotiable, which means that the pricing reference, the payment terms, and the volume can be adjusted to suit the counter-parties to the deal [36]. As a result, options in the energy market are versatile.

Furthermore, customized and complex derivative structures that exist only in the energy market have been developed to reduce volumetric risks. Swing options are such structures. Spot prices of electricity are highly dependent on temporal and local supply and demand conditions [19] and demand and supply vary continuously. Even though the average demand for fuels could be estimated from past usage or operational capacities, day-to-day demand
is not known precisely. Therefore, customers would like to hold swing options to cover their daily needs without buying extra power that they don't need.

Then what are swing options?

**Definition 1.** *(Swing Option)* A swing option gives the right to the holder to adjust the volume of some specified commodity within some range at a strike price, denoted by $K$, which is either a fixed value throughout its life or set at the beginning of each time period based on some pre-specified formula. The number of swing rights can be less than or equal to the number of days within the period covered of the contract (typically a month or a quarter). At most one right should be exercised at a time or per time-interval like a day. A swing right could be the right for a swing option buyer to receive the payoff of a call option, a mixture of different payoff functions like calls and puts or calls with different strikes. Penalties are imposed on the buyer if the net volume purchased exceeds or falls below specified upper and lower limits.

Swing options allow flexibility in the delivery time and amount of energy. The specification of a swing option typically includes the time period, the total amount of energy that can be bought or sold and the purchasing or selling price per unit of the commodity. Furthermore, swing contracts include provisions about minimum and maximum volumes or penalty payments in case the volume exceeds established limits.

In practice, a swing contract is negotiated between buyers and sellers. Each swing contract may have different contract components known as swing option constraints. One contract may include some, but not necessarily all of the constraints. As an illustration of contracts, we list three swing contract examples below.

**Example 1.** *(Daily Swing Contract [5])* Power marketer GBH sells a contract with the following specifications: The supplier contracts to deliver 10,000 MMBtu/day of natural gas at Chicago City Gas. The tenor of the contract is the month of January. During the period, the recipient has the right to swing the volume of the gas taken between 5,000 MMBtu and 15,000 MMBtu a total of 10 times (days). The timing of the swings is at the discretion of the recipient. The recipient will pay the first-of-the-month Chicago City Gate index plus $.10 for every MMBtu of gas actually delivered.

To make explanations short, we use $M$ to denote the number of swing rights and $N$ to denote the number of days within the contract period. In Example 1, $M = 10, N = 31$ and the nominated volume is 10,000 MMBtu/day. The minimum daily contract quantity (DCQ) is 5,000 MMBtu and maximum DCQ is 15,000 MMBtu.

**Example 2.** *(15)* The contract runs for one year. Every Friday, the holder decides on which days the following week he wants to buy 1 MWh of electricity at 30 EUR/MWh. At least 50 MWh and at most 100 MWh have to be bought in total during the year. The contract is financially settled with net payments every day.

In Example 2, $E = 30$ EUR/MWh. The minimum annual contract quantity (ACQ) is 50 MWh and the maximum ACQ is 100 MWh.

**Example 3.** Suppose current natural gas prices are 6.00/MMBtu and implied volatility is 0.6. A swing option could grant the option holder the right to take up to 10,000 MMBtu of
natural gas each week in a month (presuming 4 weeks) at a price of 6.00/MMBtu. The total volume purchased during the month must be between 10,000 MMBtu and 30,000 MMBtu.

Example 3 specifies not only the total amount of energy between 10,000 MMBtu and 30,000 MMBtu, but also a weekly amount between zero and 10,000 MMBtu.

Table 1.1: Comparison and contrast of Bermudan, American and swing options

<table>
<thead>
<tr>
<th>Option Name</th>
<th>Characteristics of Exercise Times</th>
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<tbody>
<tr>
<td>Bermudan</td>
<td>May be exercised at a set of predetermined exercise times</td>
</tr>
<tr>
<td>American</td>
<td>Not predetermined, depending on consumers’s decisions and the number of the outstanding unused swing rights.</td>
</tr>
<tr>
<td>Swing</td>
<td>May be exercised at any time before expiry</td>
</tr>
</tbody>
</table>

From Definition 1, swing options are more American than Bermudan in character (Table 1.1). The exercise times of swing rights are not predetermined and rely on the consumers’ decisions as do American options, whereas a Bermudan option gives the holder the right to exercise at a set of predetermined exercise times. Nevertheless, a swing option with $M$ rights differs from $M$ American options. An American option may be exercised at any time before the expiry date, so $M$ American options could be exercised at the same time. However, exercising one swing right at a particular time prevents the exercise of other swing rights at the same time. Fewer opportunities exist to exercise the rest of the swing rights. The first swing right can be regarded as an American option, but the $(j + 1)^{st}$ right has to wait for the next optimal exercise opportunity after the $j^{th}$ right is exercised. Therefore, the value of a swing option with $M$ rights is lower than the value of $M$ identical American options, which gives an upper bound of the value of the swing option with $M$ rights. A lower bound is given by the maximum value of $M$ European options struck at the strike price on the best set of $M$ predetermined exercise times among all possible sets of $M$ distinct exercise times.

Since $M$ swing rights are cheaper than $M$ American options, a swing option would be a perfect instrument when a purchase of the underlying commodity is not needed frequently. For example, suppose a power retailer must consider possible spikes in price in August. If he buys daily European options for the whole month, he is over-protected since the spikes are not likely to occur every day. If he expects at most ten extreme hot days when spikes would occur and buys ten day daily American options, then he is fully protected, since he can exercise American options when spikes occur. However, he still pays more for this protection compared to a swing with ten rights.

Swing options have two special cases that provide important benchmark values for the more general case. The first is where the number of swings is equal to the number of delivery times within the delivery periods ("full-swing"). Full-swing is equivalent to several individual European options with different expiry dates. For example, suppose we have a swing option with five daily rights struck at the fixed price and a five-day period. Each right should be exercised on each day, therefore each right is a European option. The second is where a single swing right is purchased ("one-swing"). One-swing is equivalent to an American option. A one-swing option is more challenging than the valuation of a full-
swing option, since the early exercise feature makes the valuation of an American option is considerably more challenging than the valuation of a European option.

A general case is of more interest to professionals, but also the most complex to value. Like American options, swing options also have the early exercise feature, so the techniques developed to price American options can be modified to handle the more complex challenges of swing options. Besides the early exercise feature, the swing option has another generic feature that any two swing rights cannot be exercised at the same time. These two features make a PDE approach to price a general swing option more challenging than those to price any option in financial markets.

Before reviewing PDE approaches of the swing option valuation, we will present the PDE problem. The PDE problem we use is on the most popular stochastic differential equation (SDE) model for electricity prices that captures seasonality and mean-reversion, first introduced by Lucia and Schwartz in [19]. Lucia and Schwartz assume an Uhlenbeck-Ornstein process for the logarithm of the electricity price, from which one can develop a one factor mean reversion model in the Black-Scholes type. We will call this model “the one factor model” for short in this dissertation. The Black-Scholes model assumes a lognormal distribution of prices, which fails to capture the features in energy markets such as seasonality, mean-reversion and spikes up and down [5]. Especially, “seasonality is one of the most typical characteristics of energy prices and any realistic model must incorporate this characteristic” [5]. Lucia and Schwartz realized the importance of seasonality in modeling the evolution of energy prices. They use $S_t$ to represent for the electricity price and assume the log-price process $\ln S_t$ is written as

$$\ln S_t = f(t) + Y_t, \quad (1.10)$$

where the seasonality in the evolution of the energy prices is given by the time dependent deterministic factor $f(t)$. The stochasticity is introduced by the Uhlenbeck-Ornstein process $Y_t$ evolving in time as

$$dY_t = -K_1 Y_t dt + \sigma(t) dZ_t, \quad (1.11)$$

which describes a mean-reverting process with a mean-reversion level zero and a positive constant mean-reversion speed $K_1$. Time-dependent but deterministic volatility is given by the parameter $\sigma(t)$. Finally, $Z_t$ is a standard Brownian motion.

One can rewrite (1.10) as

$$S_t = e^{f(t) + Y_t}, \quad (1.12)$$

and then

$$d\ln S_t = f'(t) dt + dY_t$$

$$= f'(t) dt - K_1 Y_t dt + \sigma(t) dZ_t. \quad (1.13)$$

Since $Y_t = \ln S - f(t)$, (1.13) becomes

$$d\ln S_t = K_1 (\rho'(t) - \ln S_t) dt + \sigma dZ_t, \quad (1.14)$$
where \[ \rho'(t) = \frac{1}{K_1} f'(t) + f(t). \]

One can apply Itô’s lemma to \( S_t = e^{\ln S_t} \) and obtain
\[
dS_t = K_1(\rho(t) - \ln S_t)S_t dt + \sigma S_t dZ_t,
\]
where
\[
\rho(t) = \frac{1}{K_1} \left( \frac{\sigma^2}{2} + f'(t) \right) + f(t).
\]

Lucia and Schwartz present the SDE model (1.15) for the electricity price. From Eq. (1.15), one can derive the PDE model of the option valuation. The option value \( V \) can be considered to be a function with two independent variables \( S \) and \( t \). By applying Itô’s lemma to (1.15), one can derive the stochastic differential equation (SDE) for \( V \)
\[
dV(S_t, t) = \left( \frac{\partial^2 V}{\partial S^2} S^2 + K_1(\rho(t) - \ln S) \frac{\partial V}{\partial S} S - rV \right) dt + \sigma \frac{\partial V}{\partial S} S dZ_t.
\]

The following PDE model can be derived using the Feynman-Kač theorem. Here we will show a new way, using a financial argument to obtain the PDE model. We introduce the market price risk for sources of unhedged uncertainty [10] and use it to adjust the drift rate by the excess return. Thus,
\[
\frac{\partial V}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + K_1(\rho(t) - \ln S) \frac{\partial V}{\partial S} S - rV = \lambda,
\]
Rearranging (1.17), the PDE for the value of the swing option \( V(S, t) \) is
\[
\frac{\partial V}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (K_1(\rho(t) - \ln S) - \lambda \sigma) S \frac{\partial V}{\partial S} - rV = 0, S \in (0, \infty),
\]
where \( \rho(t) = \frac{1}{K_1} \left( \frac{\sigma^2}{2} + f'(t) \right) + f(t) \) depends on the seasonality \( f(t) \).

To solve a forward time problem, we let \( \tau = T - t \). Then (1.18) becomes
\[
\frac{\partial V}{\partial \tau} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (K_1(\rho(\tau) - \ln S) - \lambda \sigma) S \frac{\partial V}{\partial S} - rV, S \in (0, \infty).
\]

To price the swing rights with (1.19), we make the usual assumptions [2] that a liquid market is in place for these options and for the underlying physical commodity, that players in this market make decisions to maximize their profits, and that transaction costs are negligible. These assumptions provide a good starting point for a general quantitative analysis [5].

It is only very recently that PDE approaches have been developed based on the one factor model in the literature [15, 31], but boundary conditions have not been appropriately applied. In [15], Kjaer “assumed that the solution is linear in \( S \) at both boundaries”, so used
\[
\frac{\partial^2 V}{\partial S^2} = 0
\]
at both boundaries, but the well-posedness of boundary conditions (1.20) is still unproven.

The early exercise feature is the main difficulty when pricing swing options. To avoid a complicated boundary problem, Wilhelm and Winter [31] solved a PDE of the excess to the payoff function where the excess to the payoff function $U$ equals to $V - e^{-r(T-t)} \times payoff$, instead of the option price $V$. However, $U$ is not smooth. Furthermore, they develop a PDE approach to solve for $U$ based on the Black-Scholes model but directly apply it to the one factor model. As we mentioned at the beginning of this chapter, it would be careless to apply the boundary conditions for the Black-Scholes model directly to the factor model.

### 1.2 Contributions

We develop partial differential equation methods with well-posed boundary conditions to price average strike options and swing options in energy markets.

We use the energy method to find sufficient well-posed boundary conditions on a finite domain for Asian options on Eq. (1.4). To test the numerical performance of well-posed boundary conditions, we use the average strike option valuation as an example. We also develop new boundary conditions from financial arguments and the put-call parity for the arithmetic average strike option. Solutions with the new boundary conditions match the similarity solution significantly better than solutions with boundary conditions found in the existing literature. Numerical experiments also confirm that well-posed boundary conditions are working properly.

To price the one factor swing option model, we use the energy method to develop well-posed boundary conditions on a finite domain, derive point-wise interior error formulas from which we estimate the size of the numerical domain, and use a finite element penalty method to price swing options. In [12], Kangro and Nicolaides estimate $S_{\text{max}}$ for Black-Scholes model and we extend their work to the one factor model. We derive formulas for both $S_{\text{max}}/x_{\text{max}}$ and $S_{\text{min}}/x_{\text{min}}$. We also derive a priori error estimates for both Dirichlet boundary conditions and Neumann boundary conditions. We show that the Finite element penalty method we use in this dissertation is thousands times faster than the Binomial tree method at the same level of accuracy. Furthermore, we price a multiple right swing option with different strike prices. We identify that the initial condition of each right is no longer its payoff function. We develop a new algorithm to identify the optimal exercise boundary at each time level which also allows use to record the optimal exercise time. Numerical results are accurate to one cent comparing with the benchmark solutions computed by a Binomial tree method.

We extend applications to multiple right swing options with a waiting period restriction. We keep track of the optimal exercise time when pricing each swing right. When the value of a waiting period goes to zero, we expect the option price of each right approaches the value of an American option. With a waiting period equal to the size of time difference in the numerical scheme, the numerical results show that as the size of the time step decreases the value of $M$ right swing option price increases to the value of $M$ times an American option price as expected.
1.3 Organization

We organize this dissertation in the following way.

In Chapter 2, we use the energy method to develop well-posed boundary conditions to price Asian options. To show the performance of the PDE approach with well-posed boundary conditions, we price an average strike option with new boundary conditions developed from financial arguments, which provide more accurate results than the boundary conditions in the existing literature.

In Chapter 3, we develop a finite element penalty method to price swing options. We start with the energy method to develop well-posed boundary conditions, then develop formulas to determine the size of the numerical domain and derive formulas of a priori error estimates for both Dirichlet boundary conditions and Neumann boundary conditions. We use five numerical experiments to test the performance of the algorithm.

In Chapter 4, we present conclusions.
CHAPTER 2

WELL-POSED BOUNDARY CONDITIONS TO PRICE AN AVERAGE STRIKE OPTION

In Chap. 2, we develop well-posed boundary conditions to price Asian options. There are two equivalent PDE problems of Asian options, two space variable PDE problem (1.4) or an alternative two space variable PDE problem (1.7). The review of PDE methods for the Asian option valuation in Chap. 1 shows that the authors haven’t told how they choose the well-posed boundary conditions for two space variable PDE models in the literature.

To test the performance of well-posed boundary conditions, we price an arithmetic average strike option and use similarity solutions as benchmark solutions. We also develop new boundary conditions from financial arguments and the put-call parity for the arithmetic average strike option. Solutions with new boundary conditions match the similarity solution significantly better than those provided in the existing literature. This also convinces us that the PDE approach with well-posed boundary conditions is working well for the two space variable PDE problem (1.4).

2.1 Well-posed Boundary Conditions

We use the energy method to look for well-posed boundary conditions for both eq. (1.4) and eq. (1.7) and find that Dirichlet boundary conditions make (1.4) a well-posed initial boundary value problem (IBVP). We fail to find well-posed boundary conditions for eq. (1.7) in this dissertation, so we choose the problem with (1.4) as our two space variable problem.

Before we develop well-posed boundary conditions for Asian option valuation models, we will first define well-posedness.

Definition 2 (Well-posedness [8]). Assume a solution exists for the initial BVP (IBVP)

\[
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= Lu(x,t), \quad x \in \Omega, t > 0, \\
u(x,t) &= 0, \quad x \in \partial \Omega, t > 0, \\
u(x,0) &= g(x), \quad x \in \Omega, t = 0.
\end{aligned}
\]
where $\mathcal{L}$ is a linear operator. The problem (2.1) is well-posed for $t \in [0,T]$ in $\| \cdot \|_2$ provided only that there exist constants $C$ and $\alpha$ independent of $g$, such that

$$
\sup_{t \in [0,T]} \| u(t) \|_2 \leq C e^{\alpha t} \| g \|_2.
$$

(2.2)

Generally speaking, an IBVP that is said to be well-posed means that the PDE is solvable for data in a suitable function space and that the solution operator is bounded.

To apply the Def. 2, we change the PDE problem eq. (1.4) to an IBVP by changing $t = T - \tau$, and have

$$
\frac{\partial V}{\partial \tau} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial I} - r V.
$$

(2.3)

Now we use the energy method to develop boundary conditions that make (2.3) well-posed on a finite domain. To make a derivation clear, we introduce the following notations.

Let $\nabla = \left( \frac{\partial}{\partial I}, \frac{\partial}{\partial S} \right)$, $H^T = \left( S \ rS \right)$ and $G = \left( \begin{array}{cc} 0 & 0 \\ 0 & \frac{1}{2} \sigma^2 S^2 \end{array} \right)$. Then to get the weak formulation of eq. (2.3), we multiply (2.3) by a test function $\phi \in H^1$ and integrate it

$$
\int_{\Omega} V_{\tau} \phi d\Omega = \int_{\Omega} (\nabla \cdot G \nabla V) \phi d\Omega + \int_{\Omega} (H^T \nabla V) \phi d\Omega - r \int_{\Omega} V \phi d\Omega,
$$

(2.4)

where “$\cdot$” represents inner product operator. When we apply Green’s first identity to the diffusion term,

$$
\int_{\Omega} (\nabla \cdot G \nabla V) \phi d\Omega = \int_{\partial \Omega} \phi (G \nabla V \cdot \hat{n}) d(\partial \Omega) - \int_{\Omega} (\nabla \phi) \cdot (G \nabla V) d\Omega,
$$

(2.5)

eq. (2.4) becomes

$$
\int_{\Omega} V_{\tau} \phi d\Omega = \int_{\partial \Omega} \phi (G \nabla V \cdot \hat{n}) d(\partial \Omega) - \int_{\Omega} (\nabla \phi) \cdot (G \nabla V) d\Omega \\
+ \int_{\Omega} (H^T \nabla V) \phi d\Omega - r \int_{\Omega} V \phi d\Omega.
$$

(2.6)

Now, choose $\phi = V$ to create $\| V \|_{L_2}$, which represents for the (mathematical) energy. Then

$$
\int_{\Omega} V_{\tau} V d\Omega = \int_{\partial \Omega} V(G \nabla V \cdot \hat{n}) d(\partial \Omega) - \int_{\Omega} (\nabla V) \cdot (G \nabla V) d\Omega \\
+ \int_{\Omega} (H^T \nabla V) V d\Omega - r \int_{\Omega} V^2 d\Omega.
$$

(2.7)

Since $G \geq 0$,

$$
\frac{1}{2} \frac{d}{d\tau} \| V \|_2^2 \leq \int_{\partial \Omega} V(G \nabla V \cdot \hat{n}) d(\partial \Omega) + \int_{\Omega} (H^T \nabla V) V d\Omega \\
= I_1 + I_2.
$$

(2.8)
Numerical methods are applied on a finite domain, \([0, I_m] \times [0, S_m]\), so we look for well-posed boundary conditions on \([0, I_m] \times [0, S_m]\).

\[
I_1 = - \int_0^{I_m} \frac{1}{2} \sigma^2 S^2 V S dI \bigg|_{S=0} + \int_0^{I_m} \frac{1}{2} \sigma^2 S^2 V S dI \bigg|_{S=S_m}
\]

\[
= \int_0^{I_m} \frac{1}{2} \sigma^2 S^2 V S dI \bigg|_{S=S_m} \tag{2.9}
\]

and

\[
I_2 = \int \frac{S}{\Omega} \frac{\partial V}{\partial I} V d\Omega + \int \frac{rS}{\Omega} \frac{\partial V}{\partial S} V d\Omega
\]

\[
= \frac{1}{2} \int \frac{S}{\Omega} (V^2)_{I} d\Omega + \frac{1}{2} \int \frac{rS}{\Omega} (V^2)_{S} d\Omega
\]

\[
= I_3 + I_4, \tag{2.10}
\]

where

\[
I_3 = \int_0^{S_m} SV^2 dS \bigg|_{I=I_m} - \int_0^{S_m} SV^2 dS \bigg|_{I=0} \tag{2.11}
\]

and

\[
I_4 = r \int_0^{I_m} SV^2 dI \bigg|_{S=S_m} - r \int_0^{I_m} \int_0^{S_m} V^2 dS dI. \tag{2.12}
\]

On \([0, I_m] \times [0, S_m]\), with the definition \(\int_0^{I_m} \int_0^{S_m} V^2 dS dI = \|V\|^2\), (2.8) becomes

\[
\frac{d}{d\tau} \|V\|^2 \leq 2 (I_1 + I_3 + I_4)
\]

\[
= \int_0^{I_m} (\sigma^2 S V_S + rV) S V dI \bigg|_{S=S_m} + \int_0^{S_m} SV^2 dS \bigg|_{I=I_m}
\]

\[
- \int_0^{S_m} SV^2 dS \bigg|_{I=0} - r \|V\|_2^2. \tag{2.13}
\]

If the boundary integrals have no or negative contributions to the growth of energy, then \(\frac{d}{d\tau} \|V\|^2 \leq -r \|V\|_2^2\), which sufficiently implies (2.2). And thus the problem is well-posed. Therefore, we infer the boundary conditions:

- **Left Boundary** \((S = 0)\): No boundary condition is necessary since (2.13) has no integral along \(S = 0\).

- **Right Boundary** \((S = S_m)\): If \(V = 0\) or \(\sigma^2 S V_S + rV = 0\) is applied along the entire right boundary, then both relevant integrals vanish.

- **Top Boundary** \((I = I_m)\): The integral vanishes when \(V = 0\).
• Bottom Boundary \((I = 0)\): No boundary condition is necessary. The integral is positive along the bottom boundary.

Now we know that boundary conditions along \(S = S_m\) and \(I = I_m\) are needed. This conclusion is consistent with the argument that the boundary conditions are required where there are inflows, shown in Fig. 2.1. Furthermore, Dirichlet boundary conditions do make the integrals vanish and lead to a well-posed initial boundary value problem (IBVP). Up to here, we have found well-posed boundary conditions for the two space variable model based on (1.4).

There are PDE approaches to solve the alternative two space variable model based on (1.7), but well-posed boundary conditions are not available. We also look for well-posed boundary conditions for eq. (1.7). We change \(t = T - \tau\) to obtain an IBVP

\[
\frac{\partial V}{\partial \tau} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} + \frac{S - A}{T - \tau} \frac{\partial V}{\partial A} - r V. \tag{2.14}
\]

To make an explanation easy, we use the following notations. Let \( \nabla = \begin{pmatrix} \partial A \\ \partial S \end{pmatrix} \), \( H^T = \begin{pmatrix} \frac{S - A}{T - \tau} & rS \end{pmatrix} \), and \( G = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \sigma^2 S^2 \end{pmatrix} \). For any test function \( \phi \in H^1 \),

\[
\int_{\Omega} V_\tau \phi d\Omega = \int_{\Omega} (\nabla \cdot G \nabla V) \phi d\Omega + \int_{\Omega} (H^T \nabla V) \phi d\Omega - r \int_{\Omega} V \phi d\Omega. \tag{2.15}
\]
Repeating the steps as (2.5), (2.6) and (2.7), we reach an equation similar to (2.8). This time

\[ I_1 = - \int_0^{A_m} \frac{1}{2} \sigma^2 S^2 V S dA \bigg|_{S=0} + \int_0^{A_m} \frac{1}{2} \sigma^2 S^2 V S dA \bigg|_{S=S_m} \]

\[ = \int_0^{A_m} \frac{1}{2} \sigma^2 S^2 V S dA \bigg|_{S=S_m} \]  

(2.16)

and

\[ I_2 = \frac{1}{\Omega} \int_0^{S_m} \int_0^{A_m} \frac{S-A}{T-\tau} \partial V d\Omega + \int \frac{r S \partial V}{\partial S} V d\Omega \]

\[ = \frac{1}{2} \int_0^{S_m} \frac{S-A}{T-\tau} (V^2) \bigg|_{A=A_m} d\Omega + \frac{1}{2} \int_0^{S_m} \frac{S}{T-\tau} V^2 d\Omega = I_3 + I_4, \]  

(2.17)

where

\[ I_3 = \frac{1}{2} \int_0^{S_m} \int_0^{A_m} \frac{S-A}{T-\tau} (V^2) \bigg|_{A=A_m} dA dS \]

\[ = \frac{1}{2} \int_0^{S_m} \left( \frac{S-A}{T-\tau} V^2 \bigg|_{A=A_m} + \frac{1}{T-\tau} \int_0^{A_m} V^2 dA \right) dS \]

\[ = \frac{1}{2} \int_0^{S_m} \frac{S-A}{T-\tau} V^2 dS \bigg|_{A=A_m} - \frac{1}{2} \int_0^{S_m} \frac{S}{T-\tau} V^2 dS \bigg|_{A=0} \]

\[ + \frac{1}{2} \int_0^{S_m} \int_0^{A_m} V^2 dA dS \]  

(2.18)

and

\[ I_4 = \frac{1}{2} \int_0^{A_m} \int_0^{S_m} (V^2) S dS dA \]

\[ = \frac{1}{2} \int_0^{A_m} r S V^2 dA \bigg|_{S=S_m} - \frac{1}{2} \int_0^{A_m} \int_0^{S_m} V^2 dS dA. \]  

(2.19)

On \([0, A_m] \times [0, S_m]\), with \( \int_0^{A_m} \int_0^{S_m} V^2 dS dA = \|V\|^2 \), (2.15) becomes

\[ \frac{d}{d\tau} \|V\|^2 \leq 2 (I_1 + I_3 + I_4) \]

\[ = \int_0^{A_m} \sigma^2 S^2 V S dA \bigg|_{S=S_m} + \int_0^{S_m} \frac{S-A}{T-\tau} V^2 dS \bigg|_{A=A_m} - \int_0^{S_m} \frac{S}{T-\tau} V^2 dS \bigg|_{A=0} \]

\[ + \int_0^{A_m} r S V^2 dA \bigg|_{S=S_m} + \left( \frac{1}{T-\tau} - \tau \right) \|V\|^2. \]  

(2.20)

If the boundary integrals have no contribution or negative contributions to the growth of the energy, the problem is well-posed. However, the last term goes to infinity as \( \tau \) goes to \( T \), therefore well-posed boundary conditions are not found.

In summary, the Dirichlet boundary conditions make (1.4) a well-posed initial boundary value problem (IBVP). We fail to find well-posed boundary conditions for eq. (1.7) in this dissertation, so we choose the problem with (1.4) as our two space variable problem.
2.2 Boundary Condition Estimate

Dirichlet boundary conditions along \( S = S_m \) and \( I = I_m \) make (1.4) well-posed, so the value of the average strike call can be solved by the IBVP. For some functions \( F, G \) and \( \Lambda(I, S, T) = \max(S - \frac{I}{T}, 0) \),

\[
\begin{align*}
\frac{\partial V}{\partial \tau} &= \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial I} - rV \\
V_{\text{call}}(I, S, \tau = 0) &= \Lambda(I, S, T) \\
V_{\text{call}}(I, S = S_m, \tau) &= F(I, S_m, \tau) \\
V_{\text{call}}(I = I_m, S, \tau) &= G(I_m, S, \tau).
\end{align*}
\]

(2.21)

To solve IBVP (2.21), we need to find suitable \( F(I, S_m, \tau) \) and \( G(I_m, S, \tau) \).

The boundary condition

\[
G(I_m, S, \tau) \equiv 0
\]

(2.22)

is financially feasible for the European type average strike call. Since when \( I(t) \rightarrow \infty \), \( I(T) \rightarrow \infty \), the accumulated stock price \( I \) increases as time passes. In the numerical implementation, we could set \( I_m \) large enough so that the strike \( \frac{I(T)}{T} \) is larger than the spot price \( S(T) \). This is to say, the payoff is zero. When the payoff is discounted, the option value is zero.

We need another boundary condition, \( F(I, S_m, \tau) \), along \( S = S_m \). Only one choice of the boundary conditions is explicitly mentioned in [32], but no careful analysis or numerical test is given, namely

\[
V(I, S, \tau) \sim S \quad \text{as} \quad S \rightarrow \infty
\]

(2.23)

\[
V(I, 0, t) = 0.
\]

(2.24)

Actually, \( V(I, S \rightarrow \infty, t) \sim S \) is derived from the similarity reduction. However, this set of boundary conditions does not make (2.21) well-posed. According to Fig. 2.1, there are outflows along \( S = 0 \), thus \( V(I, 0, t) = 0 \) is not necessary. On the other hand, we need a boundary value along \( I = I_m \). If we take \( F(I, S_m, \tau) = S_m \) along \( S = S_m \) together with \( G(I_m, S, \tau) = 0 \), we have (BC1) and we will see a discontinuity at \( (S_m, I_m) \), which will lead to a boundary layer.

Marcozzi uses

\[
F(I, S_m, \tau) = e^{-\tau T} \max(S_m - \frac{I}{T}, 0)
\]

(2.25)

to price the fixed strike call in [20]. We can also use

\[
V(I, S, \tau) = e^{-\tau T} E^{I,S,T} \max(S(0) - \frac{I(0)}{T}, 0)]
\]

(2.26)
to price the average strike call, where $F^{I,S,\tau}$ expresses the conditional expectation with respect to $I, S, \tau$. Therefore, eq. (2.25) is an approximation of the option value along the boundary $S = S_m$. Together with $G(I_m, S, \tau) = 0$, we have (BC2). For carefully chosen $I_m$ such that $I_m \geq S_m$, the continuity of the solution at $(I_m, S_m)$ is guaranteed.

We derive a new boundary condition, which we call (BC3). It is easy to develop, but has never been used before. We derive $F(I, S_m, \tau)$ from the put-call parity of the average strike options $[3, 32, 33]$,

$$C - P = S - \frac{S}{rT}(1 - e^{-r\tau}) - \frac{1}{T}e^{-r\tau} I.$$ (2.27)

For the European type average strike put with the same underlying asset, $P(I, S = S_m, \tau) = 0$, then

$$F(I, S_m, \tau) = S_m - \frac{S_m}{rT}(1 - e^{-r\tau}) - \frac{1}{T}e^{-r\tau} I.$$ (2.28)

Furthermore, both values of the call and the put should be nonnegative, so

$$0 \leq C \leq S_m - \frac{S_m}{rT}(1 - e^{-r\tau}) - \frac{1}{T}e^{-r\tau} I.$$ (2.29)

Since we confine ourselves to PDEs with two independent variables, for a fixed value of $S_m$, we would get negative values of the call if we calculate (2.28) directly with a large value of $I$. Therefore, we modify (2.28) to a new form

$$F(I, S_m, \tau) = \max \left( S_m - \frac{S_m}{rT}(1 - e^{-r\tau}) - \frac{1}{T}e^{-r\tau} I, 0 \right).$$

In summary, we have three sets of boundary conditions to test, which we collect in Table 2.1.

<table>
<thead>
<tr>
<th>Table 2.1: Boundary condition choices for the average strike Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(BC1)$</td>
</tr>
<tr>
<td>---------</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$(BC2)$</td>
</tr>
<tr>
<td>$(BC3)$</td>
</tr>
</tbody>
</table>
2.3 Numerical Methods

We use a finite difference approximation to solve (2.21). We start with the approximations
\[
\frac{\partial V}{\partial \tau}_{\tau_n} = \left( \frac{V_{j}^{n+1} - V_{j}^{n}}{\Delta \tau} \right) + O(\Delta \tau)
\]
\[
\frac{\partial V}{\partial S}_{S_j} = \alpha(\frac{V_{j+1}^{n+1} - V_{j-1}^{n+1}}{2\Delta S}) + (1 - \alpha)(\frac{V_{j+1}^{n} - V_{j-1}^{n}}{2\Delta S}) + O(\Delta S^2) \tag{2.30}
\]
\[
\frac{\partial V}{\partial I}_{I_i} = \alpha(\frac{V_{j+1}^{n+1} - V_{j-1}^{n+1}}{2\Delta I}) + (1 - \alpha)(\frac{V_{j+1}^{n} - V_{j-1}^{n}}{2\Delta I}) + O(\Delta I^2)
\]
\[
\frac{\partial^2 V}{\partial S^2}_{S_j} = \alpha(\frac{V_{j+1}^{n+1} - 2V_{j}^{n+1} + V_{j-1}^{n+1}}{\Delta S^2}) + (1 - \alpha)(\frac{V_{j-1}^{n} - 2V_{j}^{n} + V_{j+1}^{n}}{\Delta S^2}) + O(\Delta S^2)
\]

to approximate the IBVP (2.21) for the average strike call. We choose the Crank-Nicolson approximation in time with backward Euler approximation for the first two time levels and 0 ≤ n ≤ N_t, 0 ≤ i ≤ M, 0 ≤ j ≤ N.

We approximate the boundary conditions and the initial condition as
\[
v_{i,N}^n = F(i\Delta I, S_m, n\Delta \tau)
\]
\[
v_{M,j}^n = 0
\]
\[
v_{i,0}^n = \max (j\Delta S - \frac{i\Delta I}{T}, 0).
\]
With \( I_i = i\Delta I, S_j = j\Delta S \), writing \( v_{i,j}^n \) for the approximation to \( V_{i,j}^n = V(n\Delta \tau, i\Delta I, j\Delta S) \), we obtain the finite difference approximation for 0 ≤ i < M, 0 ≤ j < N:
\[
(I - \Delta \tau \alpha L_h) V_{i,j}^{n+1} = (I + \Delta \tau (1 - \alpha) L_h) V_{i,j}^n, \tag{2.32}
\]
where
\[
L_h V_{i,j} = \frac{\sigma^2 j^2}{2} (V_{i,j+1} - 2V_{i,j} + V_{i,j-1}) + \frac{r j}{2} (V_{i,j+1} - V_{i,j-1}) + \frac{j\Delta S}{2\Delta I} (V_{i+1,j} - V_{i-1,j}) - rV_{i,j} \tag{2.33}
\]
Since no BC is required along \( i = 0 \) or along \( j = 0 \), we use an upwind approximation:
\[
V_{-1,j} = 2V_{0,j} - V_{1,j}, \quad 0 < j < N, \tag{2.34}
\]
\[
V_{i,-1} = 2V_{i,0} - V_{i,1}, \quad 0 \leq i < M. \tag{2.35}
\]
We take a second order accurate upwind scheme (2.36) with respect to \( I \). According to Fig. 2.1, we revise (2.33) with the upwind scheme:
\[
\frac{\partial V}{\partial I} = \begin{cases} 
\frac{j\Delta S}{2\Delta I} (V_{i+1,j} - V_{i,j}), & i = M-1 \tag{2.37} \\
\frac{j\Delta S}{2\Delta I} (V_{i,j+1} - 2V_{i,j} + V_{i,j-1}) + \frac{r j}{2} (V_{i,j+1} - V_{i,j-1}) & i = 0, 1, 2, \ldots, M-2 \tag{2.36}
\end{cases}
\]
We solve (2.32) with Bi-CGSTAB [27].
2.4 Numerical Tests

To test the performance of well-posed boundary conditions for Asian option valuation problems, we solve an average strike call as an example and take similarity reduction solutions (see Appendix A) as the benchmark solutions in the test. Numerical tests show that the new boundary conditions \((BC3)\) work better with a smaller relative error than \((BC1)\) and \((BC2)\) in Table 2.1. From the numerical example, we can also confirm that the well-posed boundary conditions for the two space variable problem (1.4) are working correctly.

**Boundary Estimate.** We expect to show the performance of well-posed boundary conditions. At the same time, we can test the performance of the three boundary conditions listed in Table 2.1. To compare with the similarity reduction solutions, we will solve the problem (1.4) for \(\sigma = 0.4\) and \(r = 0.1\) at three months before expiry and there has already been three months averaging [33, 32]. In the numerical implementation, \(I_m = 100, S_m = 100, Nt = 50, N = 100\). The value of the option calculated by \(V = SH\) (\(H\) is the one dimensional solution) on \(I \in [0, 100]\) and \(S \in [0, 100]\) is taken as the “exact value”.

We test the code with \(V(I, S_m, \tau) = S_mH(\tau)\) along \(S = S_m\) which is the “exact boundary value” provided that \(V = SH\) is the “exact value”. In the numerical implementation, \(R_{\text{max}} = 1, Nt = 50, M = N = 100\). When calculating the “exact value” \(V = SH\), we took \(H(\|\frac{I}{S}\|, \tau) = H(R, \tau)\), where \(R\) is a mesh point of a fine mesh (its mesh number is \(N = 10000\)), and \(\|\frac{I}{S}\|\) is rounded to the nearest \(R\).

![Figure 2.2: Comparison \(V(I, S, \tau)S_mH(\tau)\) (solid line) and \(SH\) (dash line)](image)
We compare the solution contours of numerical solutions with "exact boundary value" \( V(I, S, \tau)_{S_m H(\tau)} \) and "exact value" \( V = SH \) in Fig. 2.2, where the "exact value" is shown as the dash line and the calculated value with the exact boundary condition is given as the solid line. Fig. 2.2 shows that when the exact boundary condition is used, the numerical solution \( V \) of the two dimensional IBVP (2.21) is graphically precise with the similarity reduction solution (SH). We can conclude that well-posed boundary conditions are working well for the two space variable model based on (1.4).

We now compare the results computed with (BC1), (BC2) and (BC3) in Table 2.1 with the benchmark solutions \( V = SH \) respectively, as seen in Fig. 2.3, 2.4 and Fig. 2.5.

On the one hand, Fig. 2.3 shows that (BC1) leads to a very thick boundary layer caused by the discontinuity at \((S_m, I_m)\). In Fig. 2.4, we can see (BC2) improves the results, but still leads to a thick boundary layer. The boundary conditions (BC1) and (BC2) would be much larger than the "exact" boundary value.

On the other hand, (BC3) matches the "exact boundary condition" \( S_m H(\tau) \) significantly better (see Fig. 2.5). In comparison with the similarity reduction solution, Fig. 2.5 shows that the numerical result is consistent with the similarity reduction solution. The relative error of numerical solutions with (BC3) is 0.0485, significantly smaller than the relative errors 0.8225 and 3.4858 as shown in Table 2.2.
Figure 2.4: Comparison $V_{BC2}$ (solid line) and similarity reduction solution $V = SH$ (dash line)

Figure 2.5: Comparison $V_{BC3}$ (solid line) and similarity reduction solution $V = SH$ (dash line)
Table 2.2: Relative errors of numerical solutions with three boundary condition choices

<table>
<thead>
<tr>
<th>BC choice</th>
<th>relative error of numerical solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(BC1)</td>
<td>3.4858</td>
</tr>
<tr>
<td>(BC2)</td>
<td>0.8225</td>
</tr>
<tr>
<td>(BC3)</td>
<td>0.0485</td>
</tr>
</tbody>
</table>

To conclude, (BC3) outperforms (BC1) and (BC2) as an estimate of boundary values. We could use (BC3) as a reasonable boundary condition compared with (BC1) and (BC2). The following tests are about the performance of the numerical solution with (BC3).

**Numerical Domain.** In this section, we will run test examples to choose a numerical domain in a numerical implementation.

First, we discuss how to decide the numerical truncation domain.

- How do we choose $S_m$?
  Since $S_m$ is a truncation value of $S \to \infty$, the larger $S_m$ the better. However, to price the option value at $S_0$, numerical experience shows that twice the spot stock price is reasonable, i.e. $S_m = 2S_0$.

- How do we choose $I_m$?
  We choose $I_m = TS_m$. According to the final payoff $\max(S - \frac{I}{T}, 0)$ and Fig. 2.1, we only have trivial solutions when $\frac{I}{T} > S$. Therefore, we only need to test the domain when $\frac{I}{T} \leq S$ or even a smaller area. To make things easy, we choose $I_m = TS_m$.

In test example, we solve for the value of the option at the stock price $S_0 = 100$ for the interest rate $r = 0.1$, three month maturity $T = 0.25$ and the volatility $\sigma = 0.4$. The time step is set to one day, i.e. $N_t = 90$ [1]. Results 2, 3, 4 in Table 2.3 show that the same value of solution is obtained as long as the grid spacing is the same. There is no need to set $S_m$ even larger. Results 1, 2 in Table 2.3 show that there is also no need to set $I_m$ larger than $S_mT$.

Table 2.3: Test of the effect of the truncation of the domain on the solution

<table>
<thead>
<tr>
<th>result no.</th>
<th>$S_m$</th>
<th>$I_m$</th>
<th>$N(dS)$</th>
<th>$M(dI)$</th>
<th>Option Price at $S_0 = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>200</td>
<td>200</td>
<td>200(1)</td>
<td>800(0.25)</td>
<td>5.2220</td>
</tr>
<tr>
<td>2</td>
<td>200</td>
<td>50</td>
<td>200(1)</td>
<td>200(0.25)</td>
<td>5.2220</td>
</tr>
<tr>
<td>3</td>
<td>300</td>
<td>75</td>
<td>300(1)</td>
<td>300(0.25)</td>
<td>5.2220</td>
</tr>
<tr>
<td>4</td>
<td>400</td>
<td>100</td>
<td>400(1)</td>
<td>400(0.25)</td>
<td>5.2220</td>
</tr>
</tbody>
</table>

**Grid Convergence.** We show grid convergence of the numerical method in Section 2.3 with (BC3) for a test example with typical parameter values $T = 0.25$, $r = 0.1$, $\sigma = 0.4$ and $S_0 = 100$. We fix the number of time steps $N_t = 90$, numerical domain $I \in [0,50]$ and $S \in [0,200]$, increase the number of grid points $M = N = 100$, $M = N = 120$, $M = N = 150$, $M = N = 200$, $M = N = 300$, and use the numerical solutions when $M = N = 600$ as benchmark solutions. Fig. 2.6 shows second order convergence of the numerical method.
Figure 2.6: Numerical solutions have a second order convergence. We fix the number of time steps $N_t = 90$, numerical domain $I \in [0, 50]$ and $S \in [0, 200]$, increase the number of grid points $M = N = 100$, $M = N = 120$, $M = N = 150$, $M = N = 200$, $M = N = 300$, and use the numerical solutions when $M = N = 600$ as benchmark solutions.

In summary, we develop boundary conditions that make the two space variable Asian option valuation model well-posed. The numerical example to price an average strike call shows that well-posed boundary conditions are working correctly, since when the exact boundary condition is used the numerical solution (V) of the two dimensional IBVP (2.21) is graphically precise with the similarity reduction solution (SH). Furthermore, the new well-posed boundary conditions for the average strike option we derive from the financial arguments and the put-call parity provide more accurate numerical results than boundary conditions provided in the existing literature.
Swing options are more complex and difficult to price than options in the financial markets \[5\]. A fair valuation requires contributions from both modelling and numerical methods. Of particular interest to us are PDE approaches with well-posed boundary conditions. The PDE problem of the swing option valuation we solve in this dissertation is on one of the most popular SDE models for electricity prices that captures seasonality and mean-reversion, first introduced by Lucia and Schwartz in \[19\]. Lucia and Schwartz assume an Uhlenbeck-Ornstein process for the logarithm of the electricity price, from which one can develop a one factor mean reversion model in the Black-Scholes type. Since exact analytic pricing formulas are not available, the valuation of swing options is highly dependent on numerical methods.

For the one factor model, PDE methods would be faster and more accurate than tree methods or Monte Carlo methods under the same circumstances. PDE approaches have been developed for the one factor model in \[15, 31\], but boundary conditions have not been appropriately applied. We use the energy method to find well-posed boundary conditions for the one factor model.

Then we derive formulas for both \( S_{\text{min}}/x_{\text{min}} \) and \( S_{\text{max}}/x_{\text{max}} \). In \[12\], Kangro and Nicolaides develop a formula for \( S_{\text{max}} \) and we extend their work to the one factor model and derive a priori error estimates for both Dirichlet boundary conditions and Neumann boundary conditions. We test the results numerically on a single European type swing option problem.

We use the finite element penalty methods to solve typical swing option problems numerically. To test the performance of well-posed boundary conditions and the finite element penalty method, we use the limiting options, full swing and one swing, as numerical examples. The two limiting options provide important benchmarks for more general options. Numerical examples on the two limiting options show that the well-posed boundary conditions are working correctly for the one factor model and the finite element penalty method is working correctly to resolve the early exercise feature.

With confidence in the PDE approach, we extend numerical examples to complex forms of swing options. One example is a multiple right swing option with different strike prices. We identify that the initial condition of each right is no longer its payoff function. The payoff function of a right to be exercised later has a jump discontinuity since its optimal exercise region shrinks. We develop an algorithm to identify the optimal exercise boundary
at each time level which also allows us to record the optimal exercise time. Numerical results are accurate to one cent comparing with the benchmark solution computed by a Binomial tree method, which confirms that the algorithm to track the optimal exercise time of each swing right is working correctly.

We further extend applications to multiple right swing options with a waiting period restriction. To price swing options with multiple rights, we need to track the optimal exercise option prices. A waiting period exists between two swing rights to be exercised successively, so we cannot exercise the latter right when we see an optimal exercise opportunity within the waiting period, but have to wait for the first optimal exercise opportunity after the waiting period. Under the extreme case by setting the waiting period equal to the size of the time step, the numerical results show that the value of swing option price increases to the value of \( M \) times an American option price, as expected.

### 3.1 Boundary Conditions

One can choose (1.19) with the primitive variable \( S \) or (3.22) with \( \ln S \) as the space variable to solve. We provide well-posed boundary conditions for both. We use the energy method to develop well-posed boundary conditions for (1.19) and (3.22) on a finite domain. We develop well-posed boundary conditions in the simple case when \( \lambda = 0 \), since the market price risk function \( \lambda \) does not make the PDE approach fundamentally different. If we look at the advection term \( K_1(\rho(\tau) - \ln S)S\frac{\partial V}{\partial S} \) in (1.19), we find that

\[
K_1(\rho(\tau) - \ln S) < 0, \quad S \to \infty
\]

\[
K_1(\rho(\tau) - \ln S) > 0, \quad S \to 0
\]

so there are outflows along the left and right boundaries as shown in Fig. 3.1 and Neumann boundary conditions would be reasonably applied.

A formal derivation follows the intuitive analysis. Assume \( \phi \) is an arbitrary test function in \( H^1_0 \), we multiply (1.19) by \( \phi \) to get the weak formulation

\[
(V_\tau, \phi) = \frac{1}{2} \sigma^2 \left( S^2 \frac{\partial^2 V}{\partial S^2}, \phi \right) + K_1 \left( (\rho(\tau) - \ln S)S\frac{\partial V}{\partial S}, \phi \right) - r(V, \phi). \tag{3.1}
\]

Eq. (1.19) is well-defined for \( S \in (0, \infty) \) but we solve it on a numerical domain \([S_{\text{min}}, S_{\text{max}}] \subset (0, \infty)\), so we look for the well-posed boundary conditions on \([S_{\text{min}}, S_{\text{max}}] \). To apply the energy method, we replace \( \phi \) by \( V \), so that (3.1) becomes

\[
\frac{d}{d\tau} \| V \|^2 = \sigma^2 \int S^2 \frac{\partial^2 V}{\partial S^2} V dx + 2K_1 \int (\rho(\tau) - \ln S)S\frac{\partial V}{\partial S} V dS - 2r\| V \|^2. \tag{3.2}
\]

To get the weak formulation, we apply integration by parts to integrals in (3.2)

\[
2K_1 \int SV(\rho(\tau) - \ln S)\frac{\partial V}{\partial S} dS
\]

\[
= K_1 (\rho(\tau) - \ln S)SV^2|_{S_{\text{min}}}^{S_{\text{max}}} + K_1 \| V \|^2 - K_1 \int V^2(\rho(\tau) - \ln S)dS, \tag{3.3}
\]

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Figure 3.1: Neumann boundary conditions applied at left and right boundaries

and

\[ \int S^2 \frac{\partial^2 V}{\partial S^2} VdS = S^2 \frac{\partial V}{\partial S} \bigg|_{S_{\text{min}}}^{S_{\text{max}}} - \left\| S \frac{\partial V}{\partial S} \right\|^2 - 2 \int S \frac{\partial V}{\partial S} VdS. \quad (3.4) \]

Since

\[ 2 \int S \frac{\partial V}{\partial S} VdS = SV^2 \bigg|_{S_{\text{min}}}^{S_{\text{max}}} - \| V \|^2, \quad (3.5) \]

\[ \int S^2 \frac{\partial^2 V}{\partial S^2} VdS = S^2 \left( \frac{\partial V}{\partial S} \right)_{S_{\text{min}}}^{S_{\text{max}}} - SV^2 \bigg|_{S_{\text{min}}}^{S_{\text{max}}} - \left\| S \frac{\partial V}{\partial S} \right\|^2 + \| V \|^2. \quad (3.6) \]

So we can simplify (3.2) to

\[ \frac{d}{d\tau} \| V \|^2 = \sigma^2 S^2 V \frac{\partial V}{\partial S} \bigg|_{S_{\text{min}}}^{S_{\text{max}}} - \sigma^2 SV^2 \bigg|_{S_{\text{min}}}^{S_{\text{max}}} + K_1 S(\rho(\tau) - \ln S)V^2 \bigg|_{S_{\text{min}}}^{S_{\text{max}}} - \sigma^2 \left\| S \frac{\partial V}{\partial S} \right\|^2 + K_1 \int V^2(\rho(\tau) - \ln S)dS \\
+ (\sigma^2 + K_1 - 2r) \| V \|^2 \leq \sigma^2 S^2 V \frac{\partial V}{\partial S} \bigg|_{S_{\text{min}}}^{S_{\text{max}}} - \sigma^2 SV^2 \bigg|_{S_{\text{min}}}^{S_{\text{max}}} + K_1 S(\rho(\tau) - \ln S)V^2 \bigg|_{S_{\text{min}}}^{S_{\text{max}}} \\
+ K_1 \int V^2(\rho(\tau) - \ln S)dS + (\sigma^2 + K_1 - 2r) \| V \|^2. \quad (3.7) \]
We notice that $K_1 S(\rho(\tau) - \ln S)V^2|_{S_{\text{min}}}^{S_{\text{max}}}$ can contribute negative energy when numerical domain is well-chosen. If we choose $S_{\text{max}}$ large enough so that $\rho(\tau) - \ln S_{\text{max}} < 0$ and choose $S_{\text{min}}$ small enough so that $\rho(\tau) - \ln S_{\text{min}} > 0$, then

$$K_1 S(\rho(\tau) - \ln S)V^2|_{S_{\text{min}}}^{S_{\text{max}}} = K_1 S_{\text{max}}(\rho(\tau) - \ln S_{\text{max}})V^2(S_{\text{max}}) - K_1 S_{\text{min}}(\rho(\tau) - \ln S_{\text{min}})V^2(S_{\text{min}}) < 0.$$  \hspace{1cm} (3.9)

We also notice that

$$\int V^2(\rho(\tau) - \ln S)dS = \int_{S_{\text{min}}}^{S_{\text{max}}} \rho V^2 dS + \int_{S_{\text{min}}}^{S_{\text{max}}} V^2 \ln(\frac{1}{S})dS,$$  \hspace{1cm} (3.10)

and we can show $\int V^2(\rho(\tau) - \ln S)dS$ has an upper bound in term of $\|V\|_{L^2}$ that does not depend on $V$.

Since $\rho(t) = \frac{1}{K_1}(\frac{\sigma^2}{2} + f'(t)) + f(t)$, and the seasonality function $f(t)$ is finite in practice, we can reasonably assume that $\rho \leq C_1 < \infty$. Then

$$\int_{S_{\text{min}}}^{S_{\text{max}}} \rho V^2 dS \leq C_1 \int_{S_{\text{min}}}^{S_{\text{max}}} V^2 dS = C_1 \|V\|_{L^2(S_{\text{min}}, S_{\text{max}})}^2,$$  \hspace{1cm} (3.11)

so the problematic term is $\int_{S_{\text{min}}}^{S_{\text{max}}} V^2 \ln(\frac{1}{S})dS$. To estimate an upper bound for $\int_{S_{\text{min}}}^{S_{\text{max}}} V^2 \ln(\frac{1}{S})dS$, we consider two cases:

- **$S_{\text{min}} \geq 1$**

  Since

  $$\ln(\frac{1}{S_{\text{max}}}) \leq \ln(\frac{1}{S}) \leq \ln(\frac{1}{S_{\text{min}}}) \leq 0,$$  \hspace{1cm} (3.12)

  $$\int_{S_{\text{min}}}^{S_{\text{max}}} V^2 \ln(\frac{1}{S})dS \leq \int_{S_{\text{min}}}^{S_{\text{max}}} V^2 \ln(\frac{1}{S_{\text{min}}})dS = \ln(\frac{1}{S_{\text{min}}}) \|V\|_{L^2(S_{\text{min}}, S_{\text{max}})}^2 \leq 0,$$  \hspace{1cm} (3.13)

  and does not contribute to the energy. In this case,

  $$\frac{d}{dt}\|V\|^2 \leq \sigma^2 S^2 V \frac{\partial V}{\partial S} |_{S_{\text{min}}}^{S_{\text{max}}} - \sigma^2 V^2 |_{S_{\text{min}}}^{S_{\text{max}}} + (\sigma^2 + K_1 - 2\tau + C_1 K_1)\|V\|^2.$$  \hspace{1cm} (3.14)

- **$S_{\text{min}} < 1$**

  When $S_{\text{min}} < 1$ is small, we can split $\int_{S_{\text{min}}}^{S_{\text{max}}} V^2 \ln(\frac{1}{S})dS$ into two parts $I_1$ and $I_2$.

  $$\int_{S_{\text{min}}}^{S_{\text{max}}} V^2 \ln(\frac{1}{S})dS = \int_{S_{\text{min}}}^{1} V^2 \ln(\frac{1}{S})dS + \int_{1}^{S_{\text{max}}} V^2 \ln(\frac{1}{S})dS = I_1 + I_2.$$  \hspace{1cm} (3.15)
Shown in (3.13), $I_2$ does not contribute to the energy. So we are left with $I_1$. For $I_1$, the kernel is

$$\ln\left(\frac{1}{S_{\min}}\right) \geq \ln\left(\frac{1}{S}\right) \geq 0,$$

so

$$I_1 = \int_{S_{\min}}^1 V^2 \ln\left(\frac{1}{S}\right) dS \leq \ln\left(\frac{1}{S_{\min}}\right) \int_{S_{\min}}^1 V^2 dS \leq C_2 \|V\|_{L^2(S_{\min},S_{\max})}^2,$$

where $C_2 = \ln\left(\frac{1}{S_{\min}}\right)$.

Therefore,

$$\frac{d}{d\tau} \|V\|^2 \leq \sigma^2 S^2 V \frac{\partial V}{\partial S}\bigg|_{S_{\min}}^{S_{\max}} - \sigma^2 S^2 V\bigg|_{S_{\min}}^{S_{\max}}$$

$$+ (\sigma^2 + K_1 - 2r + C_1 K_1 + C_2 K_1) \|V\|^2.$$

(3.18)

Furthermore,

$$\sigma^2 S^2 V \frac{\partial V}{\partial S}\bigg|_{S_{\min}}^{S_{\max}} - \sigma^2 S^2 V^2\bigg|_{S_{\min}}^{S_{\max}}$$

$$= \sigma^2 S_{\max} V(\tau, S_{\max}) \frac{\partial V}{\partial S}\bigg|_{S_{\max}}^{S_{\max}} - \sigma^2 S_{\min} V(\tau, S_{\min}) \frac{\partial V}{\partial S}\bigg|_{S_{\min}}^{S_{\min}}$$

$$- \sigma^2 S_{\max} V^2(\tau, S_{\max})\bigg|_{S_{\max}}^{S_{\max}} + \sigma^2 S_{\min} V^2(\tau, S_{\min})\bigg|_{S_{\min}}^{S_{\min}}$$

$$\leq \sigma^2 S_{\max} V \frac{\partial V}{\partial S}\bigg|_{S_{\min}}^{S_{\max}} + \sigma^2 S_{\min} V\bigg\{V - S_{\min} \frac{\partial V}{\partial S}\bigg\}\bigg|_{S_{\min}}^{S_{\min}}.$$(3.19)

In summary, with Dirichlet BC at the boundary $S_{\min}$ and Dirichlet or Neumann BC at the boundary $S_{\max}$,

$$\frac{d}{d\tau} \|V\|^2 \leq (\sigma^2 + K_1 - 2r + CK_1) \|V\|^2,$$

(3.21)

where $C$ depends on $\rho$, $S_{\min}$ and $S_{\max}$ but not $V$, so those BVPs are well-posed.

One can choose (1.19) or (3.22) to solve, but we prefer to use the logarithmic price $\ln S$ to the underlying asset price $S$ as a variable, since a uniform grid in $\ln S$, when converted in $S$, has a large density of nodes near $S = 0$ and we can solve a PDE with constant coefficients.

Now we use the energy method to find well-posed boundary conditions for (3.22). By changing variable $x = \ln S$, (1.19) becomes

$$\frac{\partial V}{\partial \tau} = \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} + K_1(\rho'(\tau) - x) \frac{\partial V}{\partial x} - rV, \, x \in (-\infty, \infty), \quad (3.22)$$

29
where
\[ \rho'(\tau) = \rho(\tau) - \frac{\sigma^2}{2K_1} = \frac{f'(\tau)}{K_1} + f(\tau). \]

To obtain the weak formulation of (3.22), we multiply (3.22) by an arbitrary test function \( \phi \) in \( H^1_0 \) and integrate it
\[ (V_\tau, \phi) = \frac{1}{2} \sigma^2 \left( \frac{\partial^2 V}{\partial x^2}, \phi \right) + K_1 \int (\rho'(\tau) - x) \frac{\partial V}{\partial x} \, dx - r(V, \phi). \quad (3.23) \]
Since we solve the problem on a numerical domain \([x_{\text{min}}, x_{\text{max}}]\), we will find the boundary conditions on \([x_{\text{min}}, x_{\text{max}}]\). Let \( \phi = V \), so that (3.23) becomes
\[ \frac{d}{d\tau} \|V\|^2 = \sigma^2 \int \frac{\partial^2 V}{\partial x^2} \, dx + 2K_1 \int (\rho'(\tau) - x) \frac{\partial V}{\partial x} \, dx - 2r\|V\|^2. \quad (3.24) \]
With
\[ \int \frac{\partial^2 V}{\partial x^2} \, dx = V \frac{\partial V}{\partial x}\big|_{x_{\text{max}}} - \frac{\partial V}{\partial x}\big|_{x_{\text{min}}}^2, \quad (3.25) \]
and
\[ 2 \int (\rho'(\tau) - x) \frac{\partial V}{\partial x} \, dx = (\rho'(\tau) - x)V^2\big|_{x_{\text{min}}}^{x_{\text{max}}} + \|V\|^2, \quad (3.26) \]
we simplify (3.24) to
\[ \frac{d}{d\tau} \|V\|^2 = \sigma^2 V \frac{\partial V}{\partial x}\big|_{x_{\text{min}}}^{x_{\text{max}}} - \sigma^2 \frac{\partial V}{\partial x}\|^2 + K_1 V^2(\rho'(\tau) - x)\big|_{x_{\text{min}}}^{x_{\text{max}}} + (K_1 - 2r)\|V\|^2. \quad (3.27) \]
We also notice that with properly chosen \( x_{\text{min}} \) and \( x_{\text{max}} \)
\[ \left\{ \begin{array}{l} \rho'(\tau) - x_{\text{max}} \leq 0 \\ \rho'(\tau) - x_{\text{min}} \geq 0, \end{array} \right. \quad (3.28) \]
so that \( K_1 V^2(\rho'(\tau) - x)\big|_{x_{\text{min}}}^{x_{\text{max}}} \) doesn’t contribute any energy and thus
\[ \frac{d}{d\tau} \|V\|^2 \leq \sigma^2 V \frac{\partial V}{\partial x}\big|_{x_{\text{min}}}^{x_{\text{max}}} + (K_1 - 2r)\|V\|^2. \quad (3.29) \]
When \( V = 0 \) or \( \frac{\partial V}{\partial x} = 0 \) is applied at \( x = x_{\text{max}} \) and \( x = x_{\text{min}} \),
\[ \frac{d}{d\tau} \|V\|^2 \leq (K_1 - 2r)\|V\|^2, \quad (3.30) \]
so those BVPs are well-posed. Therefore, Dirichlet boundary conditions or Neumann boundary conditions are well-posed boundary conditions.

Because of the early exercise feature, the optimal exercise price is not known in advance as a function of time, which makes the swing option valuation challenging. We use a penalty method to resolve this difficulty, which adds a penalty term to the right hand side of eq.
We use the swing put as an example to show that the new IBVP with a penalty term is well-posed if Dirichlet boundary conditions or Neumann boundary conditions are applied.

The valuation of a swing put with a strike price \( K \) can be written as a linear complementary problem (LCP) \[6\]

\[
\begin{aligned}
\frac{\partial V}{\partial \tau} - \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} + K_1 (\rho' (\tau) - x) \frac{\partial V}{\partial x} - r V &\geq 0, \\
(V - V^*) &\geq 0, \\
\left( \frac{\partial V}{\partial \tau} - \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} + K_1 (\rho' (\tau) - x) \frac{\partial V}{\partial x} - r V = 0 \right) \lor (V - V^* = 0), \\
V(0, x) &= V^*(x),
\end{aligned}
\] (3.31)

where \( V^* = \max(K - S, 0) = \max(K - e^x, 0) \) is the payoff function, which is the initial condition.

To solve the LCP (3.31), we add a power-\( p \) penalty term (3.32) (suggested by Kovalo, Linetsky and Marcozzi in [16]) to (3.22),

\[
\left( \frac{1}{\epsilon} \max(V^* - V, 0) \right)^p \equiv \left( \frac{1}{\epsilon} (V - V^*)^- \right)^p. \quad (3.32)
\]

For some small value, \( \epsilon > 0 \), we approximate (3.31) by a non-linear PDE of the form

\[
\frac{\partial V_\epsilon}{\partial \tau} = \frac{1}{2} \sigma^2 \frac{\partial^2 V_\epsilon}{\partial x^2} + K_1 g(\tau, x) \frac{\partial V_\epsilon}{\partial x} - r V_\epsilon + \left( \frac{1}{\epsilon} (V_\epsilon - V^*)^- \right)^p,
\] (3.33)

subject to the homogeneous initial condition

\[
V_\epsilon(0, x) = V_{0^*} = \max(K - S, 0) = \max(K - e^x, 0), \quad (3.34)
\]

where \( g(\tau, x) = \rho'(\tau) - x \).

We use the energy method to show that the new term \( \left( \frac{1}{\epsilon} (V - V^*)^- \right)^p \) in (3.33) does not contribute to mathematical energy and IBVP with a penalty term is well-posed if Dirichlet boundary conditions or Neumann boundary conditions are applied.

To get the weak form of (3.33), we multiply it by a test function \( \phi \) and integrate over the numerical domain:

\[
\int_{x_{\min}}^{x_{\max}} \frac{\partial V_\epsilon}{\partial \tau} \phi dx = \int_{x_{\min}}^{x_{\max}} \frac{1}{2} \sigma^2 \frac{\partial^2 V_\epsilon}{\partial x^2} \phi dx + \int_{x_{\min}}^{x_{\max}} K_1 g(\tau, x) \frac{\partial V_\epsilon}{\partial x} \phi dx \\
- \int_{x_{\min}}^{x_{\max}} r V_\epsilon \phi dx + \int_{x_{\min}}^{x_{\max}} \left( \frac{1}{\epsilon} (V_\epsilon - V^*)^- \right)^p \phi dx. \quad (3.35)
\]

We replace \( \phi \) by \( V_\epsilon \) to look for well-posed boundary conditions with the energy method for the new PDE (3.33). Comparing (3.35) with (3.22) and (3.29), we have

\[
\frac{d}{d\tau} \|V_\epsilon\|^2 \leq \sigma^2 V_\epsilon \frac{\partial V_\epsilon}{\partial x} \bigg|_{x_{\min}}^{x_{\max}} + (K_1 - 2r) \|V_\epsilon\|^2 + 2 \int_{x_{\min}}^{x_{\max}} \left( \frac{1}{\epsilon} (V_\epsilon - V^*)^- \right)^p V_\epsilon dx. \quad (3.36)
\]
We can show that \( \int_{x_{\min}}^{x_{\max}} \left( \frac{1}{\epsilon} (V_\epsilon - V^*) \right)^p V_\epsilon dx \) does not depend on \( V \). When \( S \geq K \) (i.e. \( x \geq \ln K \)), \( V^* = 0 \) and the penalty vanishes. Therefore,

\[
\int_{x_{\min}}^{x_{\max}} \left( \frac{1}{\epsilon} (V_\epsilon - V^*) \right)^p V_\epsilon dx = \int_{x_{\min}}^{\ln S_f(\tau)} \left( \frac{1}{\epsilon} (V_\epsilon - V^*) \right)^p V_\epsilon dx + \int_{\ln S_f(\tau)}^{\ln K} \left( \frac{1}{\epsilon} (V_\epsilon - V^*) \right)^p V_\epsilon dx. \tag{3.37}
\]

where \( S_f \) is the moving optimal exercise boundary.

Eq. (3.37) is bounded by a finite value. On the one hand, the penalty term vanishes or nearly vanishes everywhere in the domain where it is not optimal to exercise. Therefore, the contribution of the integral

\[
\int_{x_{\min}}^{\ln K} \left( \frac{1}{\epsilon} (V_\epsilon - V^*) \right)^p V_\epsilon dx \tag{3.38}
\]

will be trivial. On the other hand, the penalty term increases the value of the option towards the payoff price where it is optimal to exercise early. Therefore, \( 0 \leq V_\epsilon \leq K \) for any \( S \) and \( \tau \). In the optimal exercise region \([x_{\min}, \ln S_f(\tau)]\), \( 0 \leq V_\epsilon < V^* \leq K \),

\[
\int_{x_{\min}}^{\ln S_f(\tau)} \left( \frac{1}{\epsilon} (V_\epsilon - V^*) \right)^p V_\epsilon dx = \int_{x_{\min}}^{\ln S_f(\tau)} \left( \frac{1}{\epsilon} (V^* - V_\epsilon) \right)^p V_\epsilon dx \\
\leq \frac{1}{\epsilon^p} \int_{x_{\min}}^{\ln S_f(\tau)} (V^*)^p V^* dx \\
\leq \frac{1}{\epsilon^p} \int_{x_{\min}}^{\ln S_f(\tau)} (K)^{p+1} dx \\
= \frac{K^{p+1}}{\epsilon^p} (\ln S_f - x_{\min}) \\
\leq \frac{K^{p+1}}{\epsilon^p} \ln K. \tag{3.39}
\]

So the integral of the penalty term is bounded by a constant.

When \( V = 0 \) or \( \frac{\partial V}{\partial x} = 0 \) is applied at \( x = x_{\max} \) and \( x = x_{\min} \), (3.36) becomes

\[
\frac{d}{d\tau} ||V_\epsilon||^2 \leq (K_1 - 2r)||V||^2 + C_1, \tag{3.40}
\]

where \( C_1 = \frac{K^{p+1}}{\epsilon^p} \ln K \) is a constant. Therefore, the new IBVP solving (3.33) is well-posed.

### 3.2 Size of the Numerical Domain

Choosing the size of the numerical domain is a practical issue. We will develop a priori estimates for both DBC and NBC, from which we derive formulas to select the size of numerical domain. In [12], Kangro and Nicolaides estimate \( S_{\max} \) for the Black-Scholes model, which provides the size of numerical domain for options based on the Black-Scholes
model. The one factor model is in the Black-Scholes type but is different from the Black-Scholes model. There is no previous theoretical work to help us to determine the size of a numerical domain. We will extend Kangro and Nicolaides’ work to the one factor model and develop formulas for both $S_{\text{min}}/x_{\text{min}}$ and $S_{\text{max}}/x_{\text{max}}$.

The space-like variable of (3.22), $x$, lies in $(-\infty, \infty)$, so to solve (3.22) numerically, we need to truncate $(-\infty, \infty)$ to a numerical domain $[x_{\text{min}}, x_{\text{max}}]$ and apply well-posed boundary conditions. We first define $[S_{\text{min}}, S_{\text{max}}]$ for the IVP (3.41) and then define $[x_{\text{min}}, x_{\text{max}}]$ via $[S_{\text{min}}, S_{\text{max}}]$, i.e. $x_{\text{min}} = \ln S_{\text{min}}$ and $x_{\text{max}} = \ln S_{\text{max}}$

\[
\begin{align*}
    LV &= 0, \\
    V(T, S) &= p(S),
\end{align*}
\]  

(3.41)

where letter $L$ denotes the partial differential operator for each time $t$:

\[
L = \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + K_1(\rho(t) - \ln S) S \frac{\partial}{\partial S} - r.
\]  

(3.42)

The weak maximum principle [4] for parabolic equations allows us to obtain the comparison principle. With the comparison principle, we can estimate the piecewise error in the interior of the domain in terms of the maximum error on the boundary, and then use the error estimates to determine a suitable numerical domain for the boundary conditions in terms of a given error tolerance. We need to introduce some notations for the weak maximum principle. We assume $U$ to be an open, bounded subset of $(0, \infty)$, $\overline{U}$ a compact subset of $(0, \infty)$ and $\Gamma_T = \overline{U}_T - U_T$, set $U_T = [0, T) \times U$ for some fixed time $T > 0$.

Theorem 1. (Weak Maximum Principle [4])

Assume $u \in C^2(\overline{U}_T) \cap C(\overline{U}_T)$,

1. If

\[
Lu \geq 0 \quad \text{in} \quad U_T,
\]  

(3.43)

then

\[
\max_{\overline{U}_T} u \leq \max_{\Gamma_T} u^+, \quad u^+ = \max(u, 0).
\]  

(3.44)

2. If

\[
Lu \leq 0 \quad \text{in} \quad U_T,
\]  

(3.45)

then

\[
\min_{\overline{U}_T} u \geq -\max_{\Gamma_T} u^-, \quad u^- = \max(-u, 0).
\]  

(3.46)

Remark 1. In particular, if $Lu = 0$ within $U_T$, then

\[
\max_{\overline{U}_T} |u| = \max_{\Gamma_T} |u|.
\]
Lemma 1. (Comparison Principle)

Let \( V, W \in C^2(U_T) \cap \mathcal{C}(\overline{U}_T) \), such that

\[
LV \leq LW \quad \text{in} \; U_T \tag{3.47}
\]

and

\[
V \geq W \quad \text{on} \; \Gamma_T \tag{3.48}
\]

Then \( V \geq W \) in \( U_T \).

Proof. Let \( u = W - V \),

\[
\begin{cases}
Lu = LW - LV \geq 0 \quad \text{in} \; U_T, \\
u = W - V \leq 0 \quad \text{on} \; \Gamma_T.
\end{cases} \tag{3.49}
\]

By Thm. 1, the weak maximum principle,

\[
\max_{U_T} u \leq \max_{U_T} u \leq \max_{\Gamma_T} u^+ = 0, \quad u^+ = \max(u, 0). \tag{3.50}
\]

Therefore, \( u \leq 0 \) in \( U_T \), that is \( V \geq W \) in \( U_T \).

Remark 2. Furthermore, we could conclude that \( V \geq W \) in \( U_T \) from (3.48).

Now we show that the assumption “\( U \) to be an open, bounded subset of \( (0, \infty) \)” is satisfied. The variable \( x \) of (3.22) is finite in practice, so \( S \) lies in an open, bounded subset of \( (0, \infty) \). Furthermore, the advection coefficient in the one factor model is finite. The one factor model is a parabolic equation.

Lemma 2. The variable \( x \) of (3.22) is finite in practice.

Proof. With the SDE for \( x_t \)

\[
dx_t = K_1(\rho'(t) - x_t)dt + \sigma dZ_t, \tag{3.51}
\]

the closed form formulas for \( x_t \) is obtained as

\[
x_t = f(t) + (\ln S_0 - f(0))e^{-K_1t} + \sigma \int_0^t e^{-K_1(t-s)}dZ_s. \tag{3.52}
\]

Therefore, \( x_t \) follows a normal distribution

\[
x_t \sim \mathcal{N}
\left(f(t) + (\ln S_0 - f_0)e^{-K_1t}, \frac{\sigma^2}{2K_1}(1 - e^{-2K_1t})\right), \quad f_0 = f(0). \tag{3.53}
\]

The probability is almost zero when \( |x| \) goes to infinity, so the variable \( x \) is finite in practice.
3.2.1 Estimate $x_{max}$

Kangro and Nicolaides estimate $S_{max}$ for Black-Scholes model in [12] and we will estimate both $S_{max}$ and $x_{max}$ by extending their work to the one factor model.

The one factor model is a parabolic equation with a bounded advection coefficient. When $S \leq S_{max}$, we can define a general solution to $LV = 0$

$$Z(t, S) = \frac{1}{\sqrt{T + \epsilon - t}} e^{-\beta \frac{(\ln S - \theta S_{max})^2}{2(T + \epsilon - t)}},$$  \hspace{1cm} (3.54)

where $\epsilon \geq 0$ ($\epsilon > 0$ is a must for $\tau = T$), $\beta > 0$, $\theta \geq 1$. Then

$$
\begin{align*}
\frac{\partial Z}{\partial t} &= \frac{1}{(T + \epsilon - t)^2} \left( \frac{T + \epsilon - t}{2} - \beta (\ln \frac{S}{\theta S_{max}})^2 \right) Z(t, S) \\
\frac{\partial Z}{\partial S} &= -\frac{2\beta}{S(T + \epsilon - t)} (\ln \frac{S}{\theta S_{max}}) Z(\tau, S) \\
\frac{\partial^2 Z}{\partial S^2} &= \frac{2\beta}{S^2(T + \epsilon - t)^2} \left[ (\ln \frac{S}{\theta S_{max}} - 1)(T + \epsilon - t) + 2\beta (\ln \frac{S}{\theta S_{max}})^2 \right] Z(t, S).
\end{align*}
$$  \hspace{1cm} (3.55)

Suppose $\beta > 0$, $\epsilon \geq 0$ and $\theta \geq 1$ are chosen so that

$$
\frac{\partial Z}{\partial t} + \frac{1}{2} \sigma^2 S \frac{\partial^2 Z}{\partial S^2} + K_1 (\rho(t) - \ln S) S \frac{\partial Z}{\partial S} - rZ \leq 0,  \hspace{1cm} (t, S) \in [0, T] \times (0, S_{max}].
$$  \hspace{1cm} (3.56)

With (3.55), (3.56) becomes

$$
\beta(2\sigma^2 \beta - 1) \left( \frac{S}{\theta S_{max}} \right)^2 + \beta [\sigma^2 - 2K_1 (\rho(t) - \ln S)] (T + \epsilon - t) \ln \frac{S}{\theta S_{max}} - \frac{1}{2} (2\sigma^2 \beta - 1)(T + \epsilon - t) - r(T + \epsilon - t)^2 \leq 0.  \hspace{1cm} (3.57)
$$

To get a point-wise error estimate between the classical solution $V$ and the numerical solution $W$, in addition to an estimate of the boundary error, we need to choose values for $\beta$, $\epsilon$ and $\theta$ to satisfy the constraint (3.57). In practice, we are interested in swing option prices when $S$ lies in $[S, \overline{S}]$ and we call $[S, \overline{S}]$ a desired domain. We use Lemma 1 to derive the point-wise error $V - W$ in $(0, \overline{S}]$, given a maximum error along $S = S_{max}$.

Lemma 1 implies the following result:

**Lemma 3.** Assume the constraint (3.57) is satisfied. Let $V$ be a classical solution of $LV = 0$ on $(0, \infty)$ and $W$ be a function satisfying the problem $LW = 0$ on a truncated domain $(0, S_{max}]$, then

$$|V(t, S) - W(t, S)| \leq \sup_{t \in [t, T]} \left\{ \frac{|V(t', S_{max}) - W(t', S_{max})|}{Z(t', S_{max})} \right\} Z(t, S),  \hspace{1cm} (t, S) \in [0, T] \times (0, S_{max}] ,
$$  \hspace{1cm} (3.58)

where $Z(t, S)$ is given by (3.54).
Proof. Denote

$$f(t, S) = \sup_{t' \in [t, T)} \left\{ \left| \frac{V(t', S_{\text{max}}) - W(t', S_{\text{max}})}{Z(t', S_{\text{max}})} \right| \right\} Z(t, S), \quad (t, S) \in [0, T) \times (0, S_{\text{max}}].$$

Provided that constraint (3.57) is satisfied,

$$Lf(t, S) = \sup_{t' \in [t, T)} \left\{ \left| \frac{V(t', S_{\text{max}}) - W(t', S_{\text{max}})}{Z(t', S_{\text{max}})} \right| \right\} (LZ(t, S)) \leq 0.$$  \hfill (3.60)

The solution $V$ satisfies $LV = 0$ and $W$ satisfies $LW = 0$, so

$$L(V(t, S) - W(t, S)) = 0.$$  \hfill (3.61)

Therefore, (3.47) is satisfied. Furthermore, by the definition of $f(t, S),

$$f(t, S) \geq \frac{|V(t, S_{\text{max}}) - W_2(t, S_{\text{max}})|}{Z(t, S_{\text{max}})} Z(t, S_{\text{max}}).$$

That is, along the right boundary

$$f(t, S) \geq |V(t, S_{\text{max}}) - W(t, S_{\text{max}})|.$$  \hfill (3.63)

Then (3.48) is satisfied and (3.58) follows from Lemma 1.

Lemma 3 allows us to derive a point-wise error estimate formula (3.64).

**Theorem 2.** Let $V$ be a classical solution of $LV = 0$ on $(0, \infty)$ and $W$ be a function satisfying the problem $LW = 0$ on a truncated domain $(0, S_{\text{max}}]$. Then at every point $(t, S) \in [0, T) \times (0, \bar{S}], \bar{S} \leq S_{\text{max}},$

$$|V(t, S) - W(t, S)| \leq \|V - W\|_{L^\infty(S_{\text{max}})} \max \left\{ \frac{(\ln \frac{S_{\text{max}}}{S})^2}{2\sigma^2(T - t)} \right\}.$$  \hfill (3.64)

Proof. We define $\|V - W\|_{L^\infty(S_{\text{max}})} = \sup_{t \in [0, T]} \{|V(t, S_{\text{max}}) - W(t, S_{\text{max}})|\}$. For any fixed $(t^*, S^*) \in [0, T) \times (0, \bar{S}]$, Lemma 3 gives

$$|V(t^*, S^*) - W(t^*, S^*)| \leq \|V - W\|_{L^\infty(S_{\text{max}})} M(t^*, S^*), \quad t \in [t^*, T),$$  \hfill (3.65)

where $M(t^*, S^*) = \frac{Z(t^*, S^*)}{\min_{t \in [t^*, T]} Z(t, S_{\text{max}})}$.

If we can show that for any fixed $(t^*, S^*) \in [0, T) \times (0, \bar{S}] M(t^*, S^*)$ has an upper bound

$$\exp \left\{ \frac{(\ln \frac{S_{\text{max}}}{S})^2}{2\sigma^2(T - t^*)} \right\}.$$  \hfill (3.66)

then for every point $(t, S) \in [0, T) \times (0, \bar{S}], \bar{S} \leq S_{\text{max}},$

$$M(t, S) \leq \exp \left\{ \frac{(\ln \frac{S_{\text{max}}}{S})^2}{2\sigma^2(T - t)} \right\}.$$  \hfill (3.67)

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So (3.64) holds.

By looking carefully (3.54), the expression of $\frac{\partial Z}{\partial t}$ at $S = S_{max}$ is

$$\frac{\partial Z(t, S_{max})}{\partial t} = \frac{1}{(T + \epsilon - t)^2} \left( T + \epsilon - t - \beta (\ln \frac{1}{\theta}) \right) Z(t, S_{max}),$$  \hspace{1cm} (3.68)

do so $Z(\cdot, S_{max})$ is either monotonic or convex down.

The absolute minimum value $\inf_{t \in [t^*, T]} Z(t, S_{max})$ is achieved at $t^*$ or $T$. If $Z(\cdot, S_{max})$ is increasing, then $\inf_{t \in [t^*, T]} Z(t, S_{max}) = Z(t^*, S_{max})$. If $Z(\cdot, S_{max})$ is decreasing, then $\inf_{t \in [t^*, T]} Z(t, S_{max}) = Z(T, S_{max})$. Therefore,

$$M(t^*, S^*) = \max(M_1, M_2),$$ \hspace{1cm} (3.69)

where

$$\left\{ \begin{array}{l}
M_1 = \exp \left( -\frac{\beta}{T + \epsilon - t} \ln S_{max}^\epsilon \ln \frac{S_{max}^2}{S^*} \right), \\
M_2 = \sqrt{\frac{\epsilon}{T + \epsilon - t}} \exp \left( -\frac{\beta}{T + \epsilon - t} \ln \frac{S_{max}^2}{S^*} \right) + \frac{\beta}{\epsilon} \left( \ln \theta \right)^2.
\end{array} \right.$$ \hspace{1cm} (3.70)

Define

$$\xi = \sigma^2 - 2K_1(\rho(t) - \ln S).$$ \hspace{1cm} (3.71)

We need to choose values for $\beta$, $\epsilon$ and $\theta$ to satisfy constraint (3.57). We consider two cases:

**Case (1)** $\xi \geq 0$: In $S \geq \rho(t) - \frac{\sigma^2}{2K_1} = \rho'(t)$, i.e. $\rho'(t) - x \leq 0$, then constraint (3.57) is satisfied independent of $\epsilon \geq 0$ if we choose

$$\left\{ \begin{array}{l}
\beta = \frac{1}{2\sigma^2}, \\
\theta = \exp \left( \sigma^2 - 2K_1 \ln \frac{S_{max}}{\rho(t) - \ln S} \right) \geq 1,
\end{array} \right.$$ \hspace{1cm} (3.72)

where $\theta = 1$ and $\sigma^2 - 2K_1(\rho(t) - \ln S) = 0$. By choosing $\epsilon = \frac{(T - t^*) \ln \theta}{\ln S_{max}} \geq 0$ (since we predefined $t \in [t^*, T]$), there is no singularity problem when $\epsilon = 0$, we get

$$\left\{ \begin{array}{l}
M_1 = \exp \left( -\frac{\beta}{T - t^*} (\frac{\ln \frac{S_{max}^2}{S^*}}{\ln \frac{S_{max}}{S^*}})(\ln \frac{S_{max}^2}{S^*})^2 \right), \\
M_2 = \sqrt{\ln \frac{\theta}{\ln S_{max}}} \exp \left( -\frac{\beta}{T - t^*} (\ln \frac{S_{max}^2}{S^*})^2 \right).
\end{array} \right.$$ \hspace{1cm} (3.73)

Since $\theta \geq 1$, $0 \leq \ln \frac{\theta}{\ln S_{max}} < 1$ and $1 \leq \frac{\ln \frac{S_{max}^2}{S^*}}{\ln \frac{S_{max}}{S^*}} = 1 + \frac{\ln \theta}{\ln S_{max}} < 2$, we see that

$$\left\{ \begin{array}{l}
\exp \left( -\frac{\beta}{T - t^*} (\frac{\ln \frac{S_{max}^2}{S^*}}{\ln \frac{S_{max}}{S^*}})(\ln \frac{S_{max}^2}{S^*})^2 \right) \leq \exp \left( -\frac{\beta}{T - t^*} (\ln \frac{S_{max}^2}{S^*})^2 \right), \\
\sqrt{\ln \frac{\theta}{\ln S_{max}}} \exp \left( -\frac{\beta}{T - t^*} (\ln \frac{S_{max}^2}{S^*})^2 \right) \leq \exp \left( -\frac{\beta}{T - t^*} (\ln \frac{S_{max}^2}{S^*})^2 \right).
\end{array} \right.$$ \hspace{1cm} (3.74)
When we further choose $\beta = \frac{1}{2\sigma^2}$, we find that

$$M(t^*, S^*) \leq \exp \left(-\frac{1}{2\sigma^2(T-t^*)} \left(\ln \frac{S_{\max}}{S^*}\right)^2\right). \quad (3.75)$$

**Case (2)** $\xi < 0$: i.e. $\rho'(t) - x > 0$. If we choose $0 < \beta < 1$, $\epsilon = \frac{(T-t^*) \ln \theta}{\ln \frac{S_{\max}}{S^*}} \geq 0$ and large enough $\theta \geq 1$, constraint (3.57) is satisfied independent of $\epsilon \geq 0$.

$$\begin{align*}
M_1 &= \exp \left(-\frac{\beta}{T-t^*} \left(\ln \frac{\sigma^2 S_{\max}}{\ln \frac{S_{\max}}{S^*}}\right)(\ln \frac{S_{\max}}{S^*})^2\right), \\
M_2 &= \sqrt{\ln \frac{\theta}{\ln \frac{S_{\max}}{S^*}}} \exp \left(-\frac{\beta}{T-t^*} (\ln \frac{S_{\max}}{S^*})^2\right).
\end{align*} \quad (3.76)$$

Since $\theta \geq 1$, $0 \leq \frac{\ln \theta}{\ln \frac{S_{\max}}{S^*}} < 1$ and $1 \leq \frac{\ln \frac{S_{\max}}{S^*}}{\ln \frac{S_{\max}}{S^*}} = 1 + \frac{\ln \theta}{\ln \frac{S_{\max}}{S^*}} < 2$, we see that

$$\begin{align*}
\exp \left(-\frac{\beta}{T-t^*} \left(\ln \frac{\sigma^2 S_{\max}}{\ln \frac{S_{\max}}{S^*}}\right)(\ln \frac{S_{\max}}{S^*})^2\right) &\leq \exp \left(-\frac{\beta}{T-t^*} (\ln \frac{S_{\max}}{S^*})^2\right), \\
\sqrt{\ln \frac{\theta}{\ln \frac{S_{\max}}{S^*}}} \exp \left(-\frac{\beta}{T-t^*} (\ln \frac{S_{\max}}{S^*})^2\right) &\leq \exp \left(-\frac{\beta}{T-t^*} (\ln \frac{S_{\max}}{S^*})^2\right),
\end{align*} \quad (3.77)$$

and we obtain

$$M(t^*, S^*) \leq \exp \left(-\frac{\beta}{T-t^*} (\ln \frac{S_{\max}}{S^*})^2\right). \quad (3.78)$$

Since the estimate (3.78) is valid for all $0 < \beta < \frac{1}{2\sigma^2}$, we obtain

$$M(t^*, S^*) \leq \exp \left(-\frac{1}{2\sigma^2(T-t^*)} \left(\ln \frac{S_{\max}}{S^*}\right)^2\right). \quad (3.79)$$

**Remark 3.** Constraint (3.57) is satisfied.

$$\beta(2\sigma^2 \beta - 1)(\ln \frac{S}{\theta S_{\max}})^2 + \beta \xi (T-t) \ln \frac{S}{\theta S_{\max}} - \frac{1}{2} (2\sigma^2 \beta - 1)(T-t) - r(T-t)^2$$

$$= \beta(\ln \frac{\theta S_{\max}}{S}) [(2\sigma^2 \beta - 1)(\ln \frac{\theta S_{\max}}{S}) - \xi (T-t)] - \frac{1}{2} (2\sigma^2 \beta - 1)(T-t) - r(T-t)^2. \quad (3.80)$$

Since $-1 < 2\sigma^2 \beta - 1 < 0$, $\ln \frac{\theta S_{\max}}{S} > 0$, as long as $\theta$ is large enough, the negative sign will dominate, i.e.

$$\beta(2\sigma^2 \beta - 1)(\ln \frac{\theta S_{\max}}{S})^2 - \beta \xi (T-t) \ln \frac{\theta S_{\max}}{S} - \frac{1}{2} (2\sigma^2 \beta - 1)(T-t) - r(T-t)^2 \leq 0 \quad (3.81)$$

For $\theta = 1$, as long as $\ln \frac{S_{\max}}{S}$ is large enough, the negative sign will dominate, i.e. $S_{\max}$ is larger enough than $S$. The constraint (3.57) is also satisfied.
In summary,

\[ |V(t^*,S^*) - W(t^*,S^*)| \leq \|V - W\|_{L^\infty(S_{\text{max}})} \exp \left( -\frac{(\ln \frac{S_{\text{max}}}{S^*})^2}{2\sigma^2(T-t^*)} \right) , \]

\[ (t^*,S^*) \in [0, T) \times (0, \overline{S}) . \]  

(3.82)

So (3.64) holds.

**Corollary 1.** Particularly, for European option strikes at price \( K \),

\[ \|V - W\|_{L^\infty(S_{\text{max}})} \leq K \]  

(3.83)

and

\[ |V(t, S) - W(t, S)| \leq K \exp \left( \frac{(\ln \frac{S_{\text{max}}}{S})^2}{2\sigma^2(T-t)} \right) . \]  

(3.84)

**Corollary 2.** In practice, we are interested in the present value, so we set \( t = 0 \). Given a tolerance \( \text{TOL} \), we obtain the estimate in general

\[ x_{\text{max}} = \ln S_{\text{max}} = \ln(\overline{S}) + \sqrt{2\sigma^2T \ln \frac{\|V - W\|_{L^\infty(S_{\text{max}})}}{\text{TOL}}} . \]  

(3.85)

### 3.2.2 Estimate \( x_{\text{min}} \)

In [12], Kangro and Nicolaides don’t need to estimate \( S_{\text{min}} \) for the Black-Scholes model, but we also need to estimate \( x_{\text{min}} \) for the one factor model. With the same idea that we used to estimate \( x_{\text{max}} \), we develop a formula for \( x_{\text{min}} \).

When \( S \geq S_{\text{min}} \), we define a general solution to \( LV = 0 \)

\[ Z(t, S) = \frac{1}{\sqrt{T + \epsilon - t}} e^{-\beta \frac{(\ln \frac{S}{S_{\text{min}}})^2}{T + \epsilon - t}} , \]  

(3.86)

where \( S_{\text{min}} \leq S, \epsilon \geq 0, \beta > 0, 0 < \theta \leq 1 \).

Then

\[
\begin{align*}
\frac{\partial Z}{\partial t} &= \frac{1}{(T+\epsilon-t)^{\frac{3}{2}}} \left[ (T+\epsilon-t) - \beta (\ln \frac{S}{S_{\text{min}}})^2 \right] Z(t, S) \\
\frac{\partial Z}{\partial S} &= -\frac{2\beta}{S(T+\epsilon-t)} (\ln \frac{S}{S_{\text{min}}}) Z(t, S) \\
\frac{\partial^2 Z}{\partial S^2} &= \frac{2\beta}{S^2(T+\epsilon-t)^2} (\ln \frac{S}{S_{\text{min}}}) - 1)(T+\epsilon-t) + 2\beta (\ln \frac{S}{S_{\text{min}}})^2 Z(t, S).
\end{align*}
\]  

(3.87)

Suppose \( \epsilon \geq 0, \beta > 0 \) and \( 0 < \theta \leq 1 \) are chosen so that

\[ \frac{\partial Z}{\partial t} + 1/2 \sigma^2 S^2 \frac{\partial^2 Z}{\partial S^2} + K_1 (\rho(t) - \ln S) S \frac{\partial Z}{\partial S} - rZ \leq 0 , \]  

\[ (t, S) \in [0, T] \times [S_{\text{min}}, 0) . \]  

(3.88)
With (3.87), (3.88) becomes
\[
\beta(2\sigma^2\beta - 1) \left( \ln \frac{S}{\theta S_{min}} \right)^2 + \beta[\sigma^2 - 2K_1(\rho(t) - \ln S)](T + \epsilon - t) \ln \frac{S}{\theta S_{min}} \\
- \frac{1}{2}(2\sigma^2\beta - 1)(T + \epsilon - t) - r(T + \epsilon - t)^2 \leq 0. \tag{3.89}
\]

To get a point-wise error estimate between the classical solution \(V\) and the numerical solution \(W\), in addition to an estimate of the boundary error, we need to choose values for \(\beta\), \(\epsilon\) and \(\theta\) to satisfy the constraint (3.57). We use Lemma 1 to derive the point-wise error \(V - W\) on \([S, \infty)\) given a maximum error along \(S = S_{min}\).

Lemma 1 implies a similar result Lemma 4 to Lemma 3 for \(S \in [S_{min}, \infty)\).

**Lemma 4.** Assume the constraint (3.57) is satisfied. Let \(V\) be a classical solution of \(LV = 0\) on \(S \in (0, \infty)\) and \(W\) be a function satisfying the problem \(LW = 0\) on a truncated domain \([S_{min}, \infty)\), then
\[
|V(t, S) - W(t, S)| \leq \sup_{t' \in \{t, T\}} \left\{ \frac{|V(t', S_{min}) - W(t', S_{min})|}{Z(t', S_{min})} \right\} Z(t, S),
\]
\((t, S) \in [0, T) \times [S_{min}, \infty)\) \tag{3.90}

where \(Z(t, S)\) is given by (3.86).

**Theorem 3.** Let \(V\) be a classical solution of \(LV = 0\) on \(S \in (0, \infty)\) and \(W\) be a function satisfying the problem \(LW = 0\) on a truncated domain \([S_{min}, \infty)\). Then at every point \((t, S) \in [0, T) \times [S, \infty), S \geq S_{min}\)
\[
|V(t, S) - W(t, S)| \leq \|V - W\|_{L^\infty(S_{min})} \exp \left( -\frac{(\ln \frac{S_{min}}{S})^2}{2\sigma^2(T - t)} \right). \tag{3.91}
\]

**Proof.** We also define \(\|V - W\|_{L^\infty(S_{min})} = \sup_{t \in [0, T]} \{ |V(t, S_{min}) - W(t, S_{min})| \}\). For any fixed \((t^*, S^*) \in [0, T) \times [S, \infty)\), Lemma 4 gives us
\[
|V(t^*, S^*) - W(t^*, S^*)| \leq \|V - W\|_{L^\infty(S_{min})} M(t^*, S^*), \quad t \in [t^*, T), \tag{3.92}
\]
where \(M(t^*, S^*) = \max_{t \in [t^*, T]} \frac{Z(t^*, S^*)}{Z(t, S_{min})}\).

If we can show that for any fixed \((t^*, S^*) \in [0, T) \times [S, \infty), M(t^*, S^*)\) has an upper bound
\[
\exp \left( -\frac{(\ln \frac{S_{min}}{S})^2}{2\sigma^2(T - t^*)} \right), \tag{3.93}
\]
then for every point \((t, S) \in [0, T) \times [S, \infty), S \geq S_{min},
\[
M(t, S) \leq \exp \left( -\frac{(\ln \frac{S_{min}}{S})^2}{2\sigma^2(T - t)} \right). \tag{3.94}
\]
So (3.91) holds.

Similar to (3.69), we can argue that

\[ M(t^*, S^*) = \max(M_1, M_2), \]  

(3.95)

where

\[
\begin{align*}
M_1 &= \exp \left( -\frac{\beta}{T-t} \ln \frac{S_{\min}}{S} \ln(\theta^2 S_{\min}) \right), \\
M_2 &= \sqrt{\frac{\epsilon}{T-t}} \exp \left( -\frac{\beta}{T-t} \left( \ln \frac{\theta S_{\min}}{S} \right)^2 + \frac{\beta}{\epsilon} (\ln \theta)^2 \right).
\end{align*}
\]

(3.96)

We need to choose values for \( \beta, \epsilon \) and \( \theta \) satisfying constraint (3.89). We consider two cases:

**Case (1) \( \xi < 0 \):** i.e. \( \rho'(t) - x > 0 \), then constraint (3.89) is satisfied independent of \( \epsilon \geq 0 \), if we choose

\[
\begin{align*}
\beta &= \frac{1}{2\sigma^2} \\
0 < \theta &\leq 1 \\
\epsilon &= \frac{(T-t^*) \ln \theta}{\ln \frac{\theta S_{\min}}{S}} \geq 0.
\end{align*}
\]

(3.97)

Since we predefined \( t \in [t^*, T] \), there is no singularity problem when \( \epsilon = 0 \). From (3.97), we get

\[
\begin{align*}
M_1 &= \exp \left( -\frac{\beta}{T-t} \left( \ln \frac{\theta^2 S_{\min}}{S} \right)(\ln \frac{S_{\min}}{S})^2 \right), \\
M_2 &= \sqrt{\frac{\ln \theta}{\ln \frac{\theta S_{\min}}{S}}} \exp \left( -\frac{\beta}{T-t} (\ln \frac{S_{\min}}{S^*})^2 \right).
\end{align*}
\]

(3.98)

Since \( 0 < \theta \leq 1 \), \( 0 \leq \frac{\ln \theta}{\ln \frac{\theta S_{\min}}{S}} < 1 \) and \( 1 \leq \frac{\ln \frac{\theta^2 S_{\min}}{S}}{\ln \frac{\theta S_{\min}}{S}} = 1 + \frac{\ln \theta}{\ln \frac{\theta S_{\min}}{S}} < 2 \), we see that

\[
\begin{align*}
\exp \left( -\frac{\beta}{T-t} \left( \ln \frac{\theta^2 S_{\min}}{S} \right)(\ln \frac{S_{\min}}{S})^2 \right) &\leq \exp \left( -\frac{\beta}{T-t} (\ln \frac{S_{\min}}{S^*})^2 \right), \\
\sqrt{\frac{\ln \theta}{\ln \frac{\theta S_{\min}}{S}}} \exp \left( -\frac{\beta}{T-t} (\ln \frac{S_{\min}}{S^*})^2 \right) &\leq \exp \left( -\frac{\beta}{T-t} (\ln \frac{S_{\min}}{S^*})^2 \right).
\end{align*}
\]

(3.99)

When we further choose \( \beta = \frac{1}{2\sigma^2} \),

\[
M(t^*, S^*) \leq \exp \left( -\frac{1}{2\sigma^2(T-t)}(\ln \frac{S_{\min}}{S^*})^2 \right).
\]

(3.100)

**Remark 4.** The constraint (3.89) is satisfied.

\[
\beta(2\sigma^2 \beta - 1)(\ln \frac{S}{\theta S_{\min}})^2 + \beta \xi(T-t) \ln \frac{S}{\theta S_{\min}} - \frac{1}{2}(2\sigma^2 \beta - 1)(T-t)^2 - r(T-t)^2
\]

\[
= \beta \xi(T-t) \ln \frac{S}{\theta S_{\min}} - r(T-t)^2 \leq 0.
\]

(3.101)
Case (2) $\xi \geq 0$: i.e. $\rho'(t) - x \leq 0$ and $\ln \frac{S_{\min}}{S} < -\xi(T - t^*)$, if we choose $\epsilon = \frac{(T - t^*)\ln \theta}{\ln \frac{S_{\min}}{S}} \geq 0$, $0 < \beta < \frac{1}{2\sigma^2}$ and small enough $0 < \theta \leq 1$, constraint (3.89) is satisfied independent of $\epsilon \geq 0$.

\[ M_1 = \exp \left( -\beta \frac{\ln \frac{\theta^2 S_{\min}}{S}}{T - t^*} (\ln \frac{S_{\min}}{S})^2 \right), \]
\[ M_2 = \sqrt{\frac{\ln \theta}{\ln \frac{S_{\min}}{S}}} \exp \left( -\beta \frac{\ln \frac{S_{\min}}{S}}{T - t^*} (\ln \frac{S_{\min}}{S})^2 \right). \] (3.102)

Since $0 < \theta \leq 1$, $0 \leq \frac{\ln \theta}{\ln \frac{S_{\min}}{S}} < 1$ and $1 \leq \frac{\ln \theta^2 S_{\min}}{\ln \frac{S_{\min}}{S}} = 1 + \frac{\ln \theta}{\ln \frac{S_{\min}}{S}} < 2$, we see that

\[ \exp \left( -\beta \frac{\ln \frac{\theta^2 S_{\min}}{S}}{T - t^*} (\ln \frac{S_{\min}}{S})^2 \right) \leq \exp \left( -\beta \frac{\ln \frac{S_{\min}}{S}}{T - t^*} (\ln \frac{S_{\min}}{S})^2 \right), \] (3.103)

and we obtain

\[ M(t^*, S^*) \leq \exp \left( -\beta \frac{\ln \frac{S_{\min}}{S}}{T - t^*} (\ln \frac{S_{\min}}{S})^2 \right). \] (3.104)

Since this estimate is valid for all $0 < \beta < \frac{1}{2\sigma^2}$, we obtain

\[ M(t^*, S^*) \leq \exp \left( -\beta \frac{1}{2\sigma^2(T - t^*)} (\ln \frac{S_{\min}}{S^*})^2 \right). \] (3.105)

**Remark 5.** Now we check constraint (3.89):

\[ \beta(2\sigma^2 \beta - 1)(\ln \frac{S}{\theta S_{\min}})^2 + \beta \xi(T - t) \ln \frac{S}{\theta S_{\min}} - \frac{1}{2}(2\sigma^2 \beta - 1)(T - t) - r(T - t)^2, \]
\[ = \beta(\ln \frac{S}{\theta S_{\min}})[(2\sigma^2 \beta - 1)(\ln \frac{S}{\theta S_{\min}}) + \xi(T - t)] - \frac{1}{2}(2\sigma^2 \beta - 1)(T - t) \]
\[ - r(T - t)^2. \] (3.106)

Since $-1 < 2\sigma^2 \beta - 1 < 0$, $\ln \frac{S}{\theta S_{\min}} \geq 0$, as long as $0 < \theta \leq 1$ is small enough, the negative sign will dominate, i.e.

\[ \beta(2\sigma^2 \beta - 1)(\ln \frac{S}{\theta S_{\min}})^2 + \beta \xi(T - t) \ln \frac{S}{\theta S_{\min}} - \frac{1}{2}(2\sigma^2 \beta - 1)(T - t) \]
\[ - r(T - t)^2 \leq 0. \] (3.107)

For $\theta = 1$, as long as $\ln \frac{S_{\min}}{S}$ is small enough, the negative sign will dominate, i.e. $S_{\min}$ is smaller enough than $S$. The constraint (3.89) is also satisfied.
In summary,

\[ |V(t^*, S^*) - W(t^*, S^*)| \leq \| V - W \|_{L^\infty(S_{\min})} \exp \left( \frac{(\ln S_{\min})^2}{2\sigma^2(T - t^*)} \right), \]

\[ t \in [t^*, T] \times [S, \infty). \tag{3.108} \]

So (3.91) holds.

**Corollary 3.** In practice, we are interested in the present value, so we set \( t = 0 \). Given a tolerance \( TOL \), we obtain the estimate in general

\[ x_{\min} = \ln S_{\min} = \ln(S) - \sqrt{2\sigma^2 T \ln \frac{\| V - W \|_{L^\infty(S_{\min})}}{TOL}}. \tag{3.109} \]

### 3.2.3 A-Priori Error Estimates For Neumann Boundary Conditions

We have developed a priori error estimates for Dirichlet boundary conditions and will continue with a priori error estimates for Neumann boundary conditions. Theorems 2 and 3 provide the interior error estimates associated with the errors of Dirichlet boundary conditions and from that we can further define \( x_{\min} \) and \( x_{\max} \). As we already discussed, it is better to apply Neumann boundary conditions to a swing option problem, so we need to find interior point-wise errors caused by the errors of Neumann boundary conditions.

**Theorem 4.** Let \( V \) be a classical solution of \( LV = 0 \) on \( S \in (0, \infty) \) and \( W \) be a function satisfying the problem \( LW = 0 \) on a truncated domain \( (0, S_{\max}] \). Then at every point \( (t, S) \in [0, T) \times (0, S), S < S_{\max}, \)

\[ |V(t, S) - W(t, S)| \leq S_{\max} \exp \left\{ -\frac{(\ln S_{\max})^2}{2\sigma^2(T - t)} \right\} \sup_{t \in [0, T]} \left| \frac{\partial V}{\partial S}(t, S_{\max}) - \frac{\partial W}{\partial S}(t, S_{\max}) \right|. \tag{3.110} \]

**Proof.** We define a general solution \( Z(t, S) \),

\[ Z(t, S) = \frac{1}{\sqrt{T + \epsilon - t}} e^{-\frac{(\ln \frac{S}{S_{\max}})^2}{4(T + \epsilon - t)}}, \tag{3.111} \]

where \( \epsilon \geq 0, \beta > 0, \theta \geq 1 \) and

\[ \begin{cases} \frac{\partial Z}{\partial t} = \frac{1}{(T + \epsilon - t)^{\frac{3}{2}}} \left[ \frac{T + \epsilon - t}{2} - \beta (\ln \frac{S}{S_{\max}})^2 \right] Z(t, S) \\ \frac{\partial Z}{\partial S} = -\frac{2\beta}{S(T + \epsilon - t)} (\ln \frac{S}{S_{\max}}) Z(t, S). \end{cases} \tag{3.112} \]

The maximum of \( \sup_{t \in [0, T]} \left| \frac{\partial Z}{\partial S}(t, S_{\max}) \right| \) satisfies the inequality

\[ \sup_{t \in [0, T]} \left| \frac{\partial Z}{\partial S}(t, S_{\max}) \right| = \sup_{t \in [0, T]} \left| \frac{2\beta \ln \theta}{S_{\max}(T + \epsilon - t)} Z(t, S_{\max}) \right| \]

\[ = \frac{2\beta \ln \theta}{S_{\max}} \sup_{t \in [0, T]} \left\{ \frac{1}{(T + \epsilon - t)} Z(t, S_{\max}) \right\} \]

\[ \geq \frac{2\beta \ln \theta}{S_{\max} T} \sup_{t \in [0, T]} \{ Z(t, S_{\max}) \}. \tag{3.113} \]
And

$$\sup_{t \in [0, T]} |Z(t, S_{\text{max}})| \leq \frac{S_{\text{max}} T}{2 \beta \ln \theta} \sup_{t \in [0, T]} \left| \frac{\partial Z}{\partial S}(t, S_{\text{max}}) \right|. \quad (3.114)$$

Let $V$ be classical solution of $LV = 0$ on $S \in (0, \infty)$ and $W$ be a function satisfying the problem $LW = 0$ on a truncated domain $(0, S_{\text{max}}]$. Then at every point $(t, S) \in [0, T) \times (0, \bar{S}], \bar{S} < S_{\text{max}}, V-W$ satisfies the problem $L(V-W) = 0$ on $(0, S_{\text{max}}]$. Therefore,

$$\|V-W\|_{L^\infty(S_{\text{max}})} = \sup_{t \in [0, T]} |V(t, S_{\text{max}}) - W(t, S_{\text{max}})| \leq \frac{S_{\text{max}} T}{2 \beta \ln \theta} \sup_{t \in [0, T]} \left| \frac{\partial V}{\partial S}(t, S_{\text{max}}) - \frac{\partial W}{\partial S}(t, S_{\text{max}}) \right|. \quad (3.115)$$

Choose $\theta$ large enough to satisfy $\frac{T}{2 \beta \ln \theta} \leq 1,$

$$|V(t, S) - W(t, S)| \leq S_{\text{max}} \exp \left\{ - \frac{(\ln \frac{S_{\text{max}}}{S})^2}{2 \sigma^2 (T-t)} \right\} \sup_{t \in [0, T]} \left| \frac{\partial V}{\partial S}(t, S_{\text{max}}) - \frac{\partial W}{\partial S}(t, S_{\text{max}}) \right|. \quad (3.116)$$

\[ \square \]

**Corollary 4.** Since we only care about the present value of an option, i.e. when $t = 0,$

$$|V(0, S) - W(0, S)| \leq S_{\text{max}} \exp \left\{ - \frac{(\ln \frac{S_{\text{max}}}{S})^2}{2 \sigma^2 T} \right\} \sup_{t \in [0, T]} \left| \frac{\partial V}{\partial S}(t, S_{\text{max}}) - \frac{\partial W}{\partial S}(t, S_{\text{max}}) \right|. \quad (3.117)$$

Similarly, we can derive the interior error with respect to the error at the boundary $S = S_{\text{min}}.$

**Theorem 5.** Let $V$ be a classical solution of $LV = 0$ on $S \in (0, \infty)$ and $W$ be a function satisfying the problem $LW = 0$ on a truncated domain $[S_{\text{min}}, \infty)$. Then at every point $(t, S) \in [0, T) \times [\bar{S}, \infty), \bar{S} \geq S_{\text{min}},$

$$|V(t, S) - W(t, S)| \leq S_{\text{min}} \exp \left\{ - \frac{(\ln \frac{S_{\text{min}}}{S})^2}{2 \sigma^2 (T-t)} \right\} \sup_{t \in [0, T]} \left| \frac{\partial V}{\partial S}(t, S_{\text{min}}) - \frac{\partial W}{\partial S}(t, S_{\text{min}}) \right|. \quad (3.118)$$

**Proof.** We define a general solution $Z(t, S),$

$$Z(t, S) = \frac{1}{\sqrt{T + \epsilon - t}} e^{-\beta \left( \frac{\ln \frac{S}{S_{\text{min}}}}{\sqrt{T + \epsilon - t}} \right)^2}, \quad (3.119)$$

where $\epsilon \geq 0, \beta > 0, 0 < \theta \leq 1.$ Its derivatives satisfy

\[
\begin{align*}
\frac{\partial Z}{\partial S} &= \frac{1}{(T + \epsilon - t)^{3/2}} \left[ \frac{T + \epsilon - t}{2} - \beta \left( \frac{\ln \frac{S}{S_{\text{min}}}}{\sqrt{T + \epsilon - t}} \right)^2 \right] Z(t, S) \\
\frac{\partial Z}{\partial S} &= -\frac{2 \beta}{\sqrt{T + \epsilon - t}} \left( \frac{\ln \frac{S}{S_{\text{min}}}}{\sqrt{T + \epsilon - t}} \right) Z(t, S).
\end{align*}
\]
The maximum of $\sup_{t \in [0,T]} |\frac{\partial Z}{\partial S}(t, S_{\min})|$ satisfies the inequality

$$
\sup_{t \in [0,T]} |\frac{\partial Z}{\partial S}(t, S_{\min})| = \sup_{t \in [0,T]} \left| \frac{2\beta \ln \theta}{S_{\min}(T + \epsilon - t)} Z(t, S_{\min}) \right|
$$
$$
= - \frac{2\beta \ln \theta}{S_{\min}T} \sup_{t \in [0,T]} \left\{ \frac{1}{(T + \epsilon - t)} Z(t, S_{\min}) \right\}
$$
$$
\geq - \frac{2\beta \ln \theta}{S_{\min}T} \sup_{t \in [0,T]} \{ Z(t, S_{\min}) \}.
$$

(3.121)

And

$$
\sup_{t \in [0,T]} |Z(t, S_{\min})| \leq - \frac{S_{\min}T}{2\beta \ln \theta} \sup_{t \in [0,T]} |\frac{\partial Z}{\partial S}(t, S_{\min})|.
$$

(3.122)

Let $V$ be a classical solution of $LV = 0$ on $S \in (0, \infty)$ and $W$ be a function satisfying the problem $LW = 0$ on a truncated domain $[S_{\min}, \infty)$, $V - W$ satisfies the problem $L(V - W) = 0$ on $[S_{\min}, \infty)$. Therefore,

$$
\|V - W\|_{L^\infty(S_{\min})} = \sup_{t \in [0,T]} |V(t, S_{\min}) - W(t, S_{\min})|
$$
$$
\leq - \frac{S_{\min}T}{2\beta \ln \theta} \sup_{t \in [0,T]} \left| \frac{\partial V}{\partial S}(t, S_{\min}) - \frac{\partial W}{\partial S}(t, S_{\min}) \right|
$$

(3.123)

Choose $\theta$ small enough, s.t. $- \frac{T}{2\beta \ln \theta} \leq 1$,

$$
|V(t, S) - W(t, S)| \leq S_{\min} \exp \left\{ - \frac{(\ln \frac{S_{\min}}{S})^2}{2\sigma^2(T - t)} \right\} \sup_{t \in [0,T]} \left| \frac{\partial V}{\partial S}(t, S_{\min}) - \frac{\partial W}{\partial S}(t, S_{\min}) \right|.
$$

(3.124)

**Corollary 5.** Since we only care about the present value of an option, i.e. when $t = 0$,

$$
|V(0, S) - W(0, S)| \leq S_{\min} \exp \left\{ - \frac{(\ln \frac{S_{\min}}{S})^2}{2\sigma^2T} \right\} \sup_{t \in [0,T]} \left| \frac{\partial V}{\partial S}(t, S_{\min}) - \frac{\partial W}{\partial S}(t, S_{\min}) \right|.
$$

(3.125)

**Remark 6.** From the payoff function of a call, i.e. $\max (S - K, 0)$,

$$
\frac{\partial V}{\partial S}(t, S_{\max}) \sim 1
$$

(3.126)

and

$$
\frac{\partial V}{\partial S}(t, S_{\min}) \sim 0,
$$

(3.127)
and from the payoff function of a put, i.e. \( \max(K - S, 0) \),

\[
\frac{\partial V}{\partial S}(t, S_{max}) \sim 0 \tag{3.128}
\]

and

\[
\frac{\partial V}{\partial S}(t, S_{min}) \sim 1, \tag{3.129}
\]

we can assume

\[
\sup_{t \in [0, T]} \left| \frac{\partial V}{\partial S}(t, S_{max}) - \frac{\partial W}{\partial S}(t, S_{max}) \right| \leq 1 \tag{3.130}
\]

and

\[
\sup_{t \in [0, T]} \left| \frac{\partial V}{\partial S}(t, S_{min}) - \frac{\partial W}{\partial S}(t, S_{min}) \right| \leq 1. \tag{3.131}
\]

So (3.117) and (3.125) can be written as:

\[
|V(0, S) - W(0, S)| \leq S_{max} \exp \left( -\frac{(\ln S_{max})^2}{2\sigma^2 T} \right) \tag{3.132}
\]

and

\[
|V(0, S) - W(0, S)| \leq S_{min} \exp \left( -\frac{(\ln S_{min})^2}{2\sigma^2 T} \right). \tag{3.133}
\]

Theorems 4 and 5 provide the interior error estimates caused by the maximum error of Neumann boundary conditions.

To solve a swing option problem, we can use the point-wise error estimate (3.132) and (3.133) to find a suitable numerical domain \([x_{min}, x_{max}]\) for Neumann boundary conditions provided that an error tolerance on a desired domain \([\underline{S}, \overline{S}]\) is given. To test the performance, we take a single European type swing call with strike price \(K\) as an example, assuming that the desired domain is \([\underline{S}, \overline{S}] = [1, 30]\). Parameters \(K, T, r, \sigma, K_1, \rho'(t)\) are calibrated in [30] and shown in Table 3.1:

| Table 3.1: Parameters in numerical tests |
|---|---|---|---|---|---|---|
| \(K\) | \(T\) | \(r\) | \(\sigma\) | \(K_1\) | \(\rho'(t)\) |
| 15 | 1 | 0.05 | 0.392 | 0.525 | 3.032 |

We interpret the stock prices as dollar values and require that artificial boundaries and Neumann boundary conditions are such as to create a maximum error of \(K/100\) dollars in the option value of interest. Therefore, we replace \(S\) with \(\overline{S}\) in (3.132) and with \(\underline{S}\) in (3.133):

\[
\begin{align*}
\frac{K}{100} &= S_{max} \exp \left( -\frac{(\ln S_{max})^2}{2\sigma^2 T} \right), \\
\frac{K}{100} &= S_{min} \exp \left( -\frac{(\ln S_{min})^2}{2\sigma^2 T} \right). \tag{3.134}
\end{align*}
\]
\[ \begin{cases} S_{\text{max}} \exp \left( -\frac{(\ln \frac{S_{\text{max}}}{40})^2}{2 \cdot 0.392^2} \right) = 0.15 \\ S_{\text{min}} \exp \left( -\frac{(\ln \frac{S_{\text{min}}}{10})^2}{2 \cdot 0.392^2} \right) = 0.15 \end{cases} \] (3.135)

From (3.135), we find that \( S_{\text{max}} = 126.4863 \) and \( S_{\text{min}} = 0.5351 \) by solving (3.135) and thus \( x_{\text{max}} = 4.8401, x_{\text{min}} = -0.6253 \).

Numerical tests show that numerical solutions of a swing option problem on a domain \([-0.6253, 4.8401]\) have a maximum error less than 0.15. We compare numerical solutions computed on \([x_{\text{min}}, x_{\text{max}}]\) with benchmark solutions to show point-wise errors. Benchmark solutions refer to numerical solutions computed on \([\ln(K/10), \ln(20K)] = [\ln(0.15), \ln(300)]\) (large enough to cover \([x_{\text{min}}, x_{\text{max}}]\)). We fix \( \Delta x = 0.0171 \) and \( \Delta t = 0.0096 \) in numerical tests. Fig. 3.2 shows that the numerical domain \([-0.6253, 4.8401]\) makes the point-wise errors less that \( K/100 = 0.15 \) dollars as expected on desired domain \([0, \ln 30]\). The maximum absolute error on the desired domain calculated on \([x_{\text{min}}, x_{\text{max}}]\) is \( 2.5849e - 006 \).

![Figure 3.2: Point-wise error of numerical solutions computed on \([x_{\text{min}}, x_{\text{max}}]\) and benchmark solutions refer to numerical solutions computed on \([\ln(K/10), \ln(20K)]\).](image)

Fig. 3.3 shows the maximum error on the desired domain \([1, 30]\) caused by different choices of \( x_{\text{min}} \) or \( x_{\text{max}} \). The maximum error is an increasing function of \( S_{\text{min}} \) and a decreasing function of \( S_{\text{max}} \). If we want to have five significant figures, \( S_{\text{min}} \) has to be set to \( \frac{K}{100} \) and \( S_{\text{max}} \) has to be \( 20K \) as shown in Fig. 3.4.
Figure 3.3: The maximum error of option prices on the desired domain is a decreasing function of $S_{\text{min}}$ and an increasing function of $S_{\text{max}}$.

Figure 3.4: Common log of the maximum error of option prices on the desired domain $[1, 30]$ in terms of $S_{\text{min}}$ or $S_{\text{max}}$. 
We can compute a smaller numerical domain from (3.137) than the numerical domain computed from (3.135). If we have better knowledge about the boundary, like

\[
\begin{align*}
\sup_{t \in [0, T]} |\frac{\partial V}{\partial S}(t, S_{\max}) - \frac{\partial W}{\partial S}(t, S_{\max})| < \frac{1}{N}, N > 1, \\
\sup_{t \in [0, T]} |\frac{\partial V}{\partial S}(t, S_{\min}) - \frac{\partial W}{\partial S}(t, S_{\min})| < \frac{1}{N}, N > 1.
\end{align*}
\]  

(3.136)

(3.135) becomes

\[
\begin{align*}
S_{\max} \exp \left(-\frac{(\ln S_{\max})^2}{2 \cdot 0.392^2}\right) &= N \frac{K}{100}, \\
S_{\min} \exp \left(-\frac{(\ln S_{\min})^2}{2 \cdot 0.392^2}\right) &= N \frac{K}{100}.
\end{align*}
\]  

(3.137)

3.3 Numerical Examples

In this section, we use finite element methods to solve typical swing option problems numerically. As introduced in Sec. 1.1, swing options have two special cases. In the first, the number of swings is equal to the number of delivery times within the delivery periods ("full-swing"). In the second, a single swing right is purchased ("one-swing"). The two special cases are useful tools that allow us to test the performance of well-posed boundary conditions and the finite element penalty method.

We extend the finite element penalty method to price more general swing options, e.g. swing options with multiple identical swing rights. Numerical results show that the finite element penalty method is thousands times faster than the Binomial tree method with an error at the same level.

Then we extend the typical case, namely a swing option with identical swing rights, to a multiple right swing option with different strike prices. We identify that the initial condition of a right to be exercised later is no longer its payoff function. Instead it has a jump discontinuity, since its optimal exercise region shrinks. We develop an algorithm to identify the optimal exercise boundary at each time level, which also allows us to record the optimal exercise time. Numerical results are accurate to one cent compared with the benchmark solutions computed by a Binomial tree method.

We further extend applications to multiple right swing options with a waiting period restriction. For a swing option with multiple rights, a waiting period exists between two swing rights to be exercised successively, so we cannot exercise the latter right when we see an optimal exercise opportunity within the waiting period, but have to wait for the first optimal exercise opportunity after the waiting period. Therefore, we keep track of the optimal exercise time when pricing each swing right. When the value of a waiting period goes to zero, we expect the option price of each right approached the value of the American option from below. With a waiting period equal to the size of the time step in the numerical scheme, the numerical results show that as the size of the time step decreases the value of swing option price increases to the value of $M$ times an American option price as expected.

3.3.1 Full Swing

We will start with the limiting options to test the numerical performance of well-posed boundary conditions and the code. Since full swing is equivalent to several individual
European options with different expiry dates, the price of a full swing is the sum of the present value of each individual European type right. So we do not need to price a full swing but price a European put as an example to show that the code is working properly with the well-posed boundary conditions.

We solve (3.22) to price European options. We use a finite element discretization to derive the matrix system (3.144). A brief introduction of the finite element discretization follows. Applying integration by parts to (3.23),

\[
(V_{\tau}, \phi) = -\frac{1}{2} \sigma^2 \left( \frac{\partial V}{\partial x}, \frac{\partial \phi}{\partial x} \right) + K_1 \left( (\rho'(\tau) - x) \frac{\partial V}{\partial x}, \phi \right) - r(V, \phi)
\]

\[
+ \frac{1}{2} \sigma^2 \left( \frac{\partial V}{\partial x} \bigg|_{x_{\text{max}}} - \frac{\partial V}{\partial x} \bigg|_{x_{\text{min}}} \right).
\]

(3.138)

Let \( V_h = \text{span}\{ \varphi_1, \varphi_2, \ldots, \varphi_M \} \), where \( \varphi_1, \varphi_2, \ldots, \varphi_M \) are linearly independent, \( u = \sum_{i=1}^{M} u_i \varphi_i(x) \) be the approximation of \( \phi \) in \( V_h \), \( \psi = \sum_{j=1}^{M} \psi_j \varphi_j(x) \) be the approximation of \( V \) in \( V_h \). For any \( \psi \in V_h \),

\[
(u_{\tau}, \psi) = -\frac{1}{2} \sigma^2 \left( \frac{\partial u}{\partial x}, \frac{\partial \psi}{\partial x} \right) + K_1 \left( (\rho'(t) - x) \frac{\partial u}{\partial x}, \psi \right) - r(u, \psi)
\]

\[
+ \frac{1}{2} \sigma^2 \left( \frac{\partial u}{\partial x} \psi \bigg|_{x_{\text{max}}} - \frac{\partial u}{\partial x} \psi \bigg|_{x_{\text{min}}} \right).
\]

(3.139)

When we substitute for \( u \) and \( \psi \),

\[
\sum_{i=1}^{M} \sum_{j=1}^{M} u_i \psi_j(\varphi_i, \varphi_j) = -\frac{\sigma^2}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} u_i \varphi_j(\varphi_i', \varphi_j') + \sum_{i=1}^{M} \sum_{j=1}^{M} u_i \varphi_j(K_1(\rho'(t) - x)\varphi_i', \varphi_j)
\]

\[
- r \sum_{i=1}^{M} \sum_{j=1}^{M} u_i \varphi_j(\varphi_i, \varphi_j) + \frac{\sigma^2}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \frac{\partial u_i}{\partial x} \varphi_i \varphi_j.
\]

(3.140)

For every \( \varphi_j, j = 1, 2, \ldots, M \),

\[
\sum_{i=1}^{M} u_i \varphi_j(\varphi_i, \varphi_j) = -\frac{\sigma^2}{2} \sum_{i=1}^{M} u_i \varphi_j(\varphi_i', \varphi_j') + \sum_{i=1}^{M} u_i \varphi_j(K_1(\rho'(t) - x)\varphi_i', \varphi_j)
\]

\[
- r \sum_{i=1}^{M} u_i \varphi_j(\varphi_i, \varphi_j) + \frac{\sigma^2}{2} \sum_{i=1}^{M} \frac{\partial u_i}{\partial x} \varphi_i \varphi_j.
\]

(3.141)

For notational convenience, we write (3.141) in matrix form

\[
BU = AU + A^{BC},
\]

(3.142)
where \( A = -\frac{\sigma^2}{2} A^{DIF} + K_1 A^{CON} - r A^F, \)

\[
A^{DIF} = \begin{pmatrix}
\frac{1}{\Delta x_1} & \frac{-1}{\Delta x_1} & \frac{-1}{\Delta x_2} & \cdots & \frac{-1}{\Delta x_{M-1}} \\
\frac{-1}{\Delta x_1} & \frac{1}{\Delta x_1} + \frac{1}{\Delta x_2} & \cdots & \frac{1}{\Delta x_{M-1}} \\
\frac{-1}{\Delta x_2} & \frac{-1}{\Delta x_2} & \ddots & \cdots & \frac{1}{\Delta x_{M-1}} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{-1}{\Delta x_{M-1}} & \frac{1}{\Delta x_{M-1}} & \cdots & \cdots & \frac{1}{\Delta x_{M-1}}
\end{pmatrix},
\]

\[
A^{CON} = \begin{pmatrix}
-\frac{\rho'(t) - x_1}{2} + \frac{\Delta x_1}{\Delta x_1} & \frac{-\rho'(t) - x_1}{2} - \frac{\Delta x_1}{\Delta x_1} & \frac{-\rho'(t) - x_2}{2} + \frac{\Delta x_2}{\Delta x_2} & \cdots & \cdots \\
\frac{\Delta x_1}{\Delta x_1} & \frac{-\rho'(t) - x_1}{2} - \frac{\Delta x_1}{\Delta x_1} & \frac{-\rho'(t) - x_2}{2} + \frac{\Delta x_2}{\Delta x_2} & \cdots & \cdots \\
\frac{-\rho'(t) - x_2}{2} + \frac{\Delta x_2}{\Delta x_2} & \frac{\Delta x_2}{\Delta x_2} & \ddots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \ddots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \frac{\Delta x_{M-1}}{\Delta x_{M-1}}
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
\frac{\Delta x_1}{6} & \frac{\Delta x_1}{6} & \frac{\Delta x_1}{6} & \cdots & \frac{\Delta x_1}{6} \\
\frac{\Delta x_2}{6} & \frac{\Delta x_2}{6} & \frac{\Delta x_2}{6} & \cdots & \frac{\Delta x_2}{6} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{\Delta x_{M-1}}{6} & \frac{\Delta x_{M-1}}{6} & \cdots & \cdots & \frac{\Delta x_{M-1}}{6}
\end{pmatrix},
\]

\[
A^{BC} = \frac{\sigma^2}{2} \begin{pmatrix}
\frac{-\partial V}{\partial x}\bigg|_{x_{\min}} \\
0 \\
\vdots \\
0 \\
\frac{-\partial V}{\partial x}\bigg|_{x_{\max}}
\end{pmatrix}, \quad U = \begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_{M-1} \\
u_M
\end{pmatrix}.
\]

We use the Crank-Nicolson (CN) approximation in time with the backward Euler (BE) approximation for the first two time levels to integrate the system (3.142).

\[
\begin{align*}
B.E. \quad B \frac{U^n - U^{n-1}}{\Delta t} &= A^n U^n + A^{BC}, \\
C.N. \quad B \frac{U^n - U^{n-1}}{\Delta t} &= A^n U^n + A^{BC} + A^{n-1} U^{n-1} + A^{BC}.
\end{align*}
\]

If we rearrange (3.143) and take \( \Delta t = \frac{T}{N}, \) we obtain

\[
\begin{align*}
B.E. \quad (B - \Delta t A^n) U^n &= BU^{n-1} + \Delta t A^{BC}, \quad n = 1, 2, \\
C.N. \quad (2B - \Delta t A^n) U^n &= (2B + \Delta t A^{n-1}) U^{n-1} + \Delta t A^{BC}, \quad n = 3, 4, \ldots, N_t.
\end{align*}
\]

Typical parameters for this European type swing put are the exercise price \( K = 15, \) \( r = 0.05, \) \( \sigma = 0.392, \) \( K_1 = 0.525, \rho'(t) = 3.032, \) time to maturity 1 year. We compute the option value at \( S = 15 \) on the truncated domain \([-0.6253, 4.8401]\) and obtain the second order convergence as the expected in Fig. 3.5 on the desired domain \([0, \ln(30)]\). Benchmark solutions refer to the numerical solutions computed with 416 time steps on a 561 by 561 point mesh. We denote the number of time steps as \( Nt \) and the number of grid points as \( M + 1 \) by \( N + 1. \)
Numerical solutions of a single European type swing put show second order convergence. Both the number of time steps and the number of grid points, are increased at the same time, i.e. $N_t = 52, M = N = 70$, $N_t = 104, M = N = 140$, $N_t = 208, M = N = 280$. Benchmark solutions refer to the numerical solutions computed with 416 time steps on a mesh with 561 by 561 points.

Numerical solutions of a single European type swing put show that the code is working properly with the boundary conditions we derived and we obtain second order convergence as expected.

### 3.3.2 Single Swing

We price a single swing right to verify that the code is working correctly using a finite element penalty method. Since a real single swing right has an early exercise feature, i.e. one-swing, the optimal exercise price is unknown in advance, which makes the swing option valuation challenging. We use a finite element penalty method to resolve this difficulty.

We solve (3.33) to price single swing options. We use a finite element discretization to derive the matrix system (3.151). Here is an introduction to the discretization we use. Applying integration by parts to (3.35),

\[
(V_\tau, \phi) = -\frac{1}{2}\sigma^2 \left( \frac{\partial V}{\partial x}, \frac{\partial \phi}{\partial x} \right) + K_1 \left( (\rho'(\tau) - x) \frac{\partial V}{\partial x}, \phi \right) - r(V, \phi) \\
+ \left( \left( \frac{1}{\epsilon} (V - V^*) \right)^p, \phi \right) + \frac{1}{2}\sigma^2 \left( \frac{\partial V}{\partial x} \phi \bigg|_{x_{\text{max}}} - \frac{\partial V}{\partial x} \phi \bigg|_{x_{\text{min}}} \right). \tag{3.145}
\]
Comparing (3.145) with (3.138) and (3.142), we write the following matrix form:

\[ B\dot{U} = AU + A^{BC} + B\Pi_{\epsilon} \]  

(3.146)

where \( \Pi_{\epsilon} |_{i} = (\frac{1}{\epsilon}(U_{i} - U_{i}^{*})^{-})^{p} \) is non-linear. We use one-step of Newton’s method to linearize the penalty term \( \Pi_{\epsilon} \):

\[ \Pi_{\epsilon}^{n} = \Pi_{\epsilon}^{n-1} + J^{n-1}(U^{n} - U^{n-1}), \]  

(3.147)

where \( J^{n-1} \) is the diagonal Jacobian matrix at time \( n - 1 \) with

\[ J_{ii} = - \frac{1}{\epsilon} \left( \frac{\partial (\frac{1}{\epsilon}(U - U^{*})^{-})^{p-1}}{\partial U} \right) |_{ii} = - \frac{p}{\epsilon} \left( \frac{1}{\epsilon}(U_{i} - U_{i}^{*})^{-} \right)^{p-1}, \quad p = 2, 3, 4, \ldots \]  

(3.148)

and when \( p = 1 \),

\[ J_{ii} = \begin{cases} - \frac{1}{\epsilon} & \text{if } U_{i} < U_{i}^{*} \\ 0 & \text{otherwise} \end{cases} \]  

(3.149)

Next, we use the Crank-Nicolson (CN) approximation in time with the backward Euler (BE) approximation for the first two time levels to integrate the system (3.146).

\[
\begin{cases}
B.E. \quad B^{U^{n} - U^{n-1}}_{\Delta t} = A^{n}U^{n} + A^{BC} + B\Pi_{\epsilon}^{n}, \\
C.N. \quad B^{U^{n} - U^{n-1}}_{\Delta t} = (A^{n}U^{n} + A^{BC} + B\Pi_{\epsilon}^{n}) + (A^{n-1}U^{n-1} + A^{BC} + B\Pi_{\epsilon}^{n-1})/2.
\end{cases}
\]

(3.150)

If we rearrange (3.150) and take \( \Delta t = \frac{T}{N_t} \), we obtain

\[
\begin{cases}
B.E. \quad (B - \Delta tA^{n} - \Delta tBJ^{n-1})U^{n} = (B - \Delta tBJ^{n-1})U^{n-1} + \Delta t(A^{BC} + B\Pi_{\epsilon}^{n-1}), \\
C.N. \quad (2B - \Delta tA^{n} - \Delta tBJ^{n-1})U^{n} = (2B + \Delta tA^{n-1} - \Delta tBJ^{n-1})U^{n-1} + \Delta t(A^{BC} + 2B\Pi_{\epsilon}^{n-1}), \quad n = 3, 4, \ldots, N_t.
\end{cases}
\]

(3.151)

We price an American put on B-S model for a set of typical parameter values, \( r = 0.25, \sigma = 0.05, K = 100 \), the time to maturity of 3 months and use numerical solutions computed by the Binomial tree method with 500,000 time steps on \( [0, 4K] \) as benchmark solutions. Fig. 3.6 and Table 3.2 show that second order convergence with both penalty parameters \( p = 1 \) and \( p = 2 \).

We will choose \( p = 2 \) when we price a single swing option. The finite element penalty method with \( p = 2 \) requires one tenth the number of time steps for the same error compared to \( p = 1 \), as shown in Fig. 3.6 and Table 3.2. Furthermore, we will not choose \( p \geq 3 \) because it leads to ill-conditioned matrices. We solve three examples with three sets of parameters shown in Table 3.2 to calculate the condition number of the linear system (3.151). Example 1: \( r = 0.25, \sigma = 0.05, K = 100 \), the time to maturity of 3 months. Example 2: \( r = 0.01, \sigma = 0.8, K = 100 \), the time to maturity of 3 months. Example 3: \( r = 0.05, \sigma = 0.15, K = 100 \), the time to maturity of 3 months.
Table 3.2: The maximum error computed by the finite element penalty method

<table>
<thead>
<tr>
<th>Nt</th>
<th>1000</th>
<th>2000</th>
<th>4000</th>
<th>8000</th>
<th>16000</th>
</tr>
</thead>
<tbody>
<tr>
<td>p = 1</td>
<td>0.024517</td>
<td>0.0039743</td>
<td>0.001047</td>
<td>2.67327e-004</td>
<td>6.626e-005</td>
</tr>
<tr>
<td>p = 2</td>
<td>0.003976</td>
<td>0.0010477</td>
<td>2.6763e-004</td>
<td>6.6187e-005</td>
<td>1.58e-005</td>
</tr>
</tbody>
</table>

Table 3.3: Condition number of the matrix in the linear system (3.151).

<table>
<thead>
<tr>
<th></th>
<th>p = 1</th>
<th>p = 2</th>
<th>p = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1</td>
<td>8.9E + 05</td>
<td>2.3E + 11</td>
<td>5.9E + 16</td>
</tr>
<tr>
<td>Example 2</td>
<td>3.9E + 05</td>
<td>5.7E + 09</td>
<td>1.00E + 14</td>
</tr>
<tr>
<td>Example 3</td>
<td>7.3E + 03</td>
<td>5.8E + 09</td>
<td>5.8E + 15</td>
</tr>
</tbody>
</table>

Figure 3.6: Numerical solutions when $p = 1$ and $p = 2$ have second order convergence. We increase the number of grid points $M = N = 800$, $M = N = 1600$, $M = N = 3200$, $M = N = 6400$ and increase the number of time steps at the same time as shown in the legend. We use the numerical solutions computed by the Binomial tree method with 500,000 time steps on $[0, 4K]$ as benchmark solutions.
Numerical experiments show that the code is working correctly using a finite element penalty method and why we choose two as the value of the penalty parameter.

### 3.3.3 Identical Multiple Swing Rights

To test the numerical performance on a multiple right case, we solve a swing put problem with up to five exercise rights. Since the solution to the Black-Scholes model is a good estimate and a useful tool to test the performance of numerical schemes, we will use the Black-Scholes model with a set of typical parameter values $T = 1$, $K = 100$, $\sigma = 0.3$, $r = 0.05$, $S_0 = 100$. Numerical results show second order grid convergence in Fig. 3.7.

![Figure 3.7: Numerical results show second order grid convergence. Numerical solutions are computed by the finite element penalty method with $N_t = 1000, 2000, 4000, 8000$. We plot the common log of relative errors of the numerical solutions. Exact solutions refer to numerical solutions computed by the finite element penalty method with $N_t = 16000$ time steps.](image)

To show the finite element penalty method we use in this dissertation is much faster than the Binomial tree method, we compute the prices of $M$ right swing options at $S_0 = 100$, $M = 1, 2, 3, 4, 5$ using both the finite element penalty method and the Binomial tree method. We fix 4001 grid points and increase the number of time steps as $N_t = 1000, 2000, 4000, 8000$ in the finite element penalty method, and increase the number of time steps $N_t = 400, 800, 1600, 3200, 6400$ in the Binomial tree method. We compare the relative error of the swing option price at $S_0 = 100$ and the CPU time of running both. Benchmark solutions refer to numerical solutions computed by the Binomial tree method with time steps $N_t = 25, 600$. 

55
Relative errors obtained by the finite element penalty method are in the left-lower corner in Fig. 3.8. Relative errors obtained by the Binomial tree method are in the right-upper corner.

Fig. 3.8 shows that to obtain the same number of significant figures in an option price, the finite element penalty method is thousands times faster than the Binomial tree method. Furthermore, one can obtain additional option values and optimal exercise opportunities at different current stock prices simultaneously using the finite element penalty method, but only obtain one option value each time using the Binomial tree method.

![Figure 3.8: Comparison of CPU time for a single stock price. We fix the number of grid points to be 4001 and increase the number of time steps $Nt = 1000$, 2000, 4000, 8000 when using the finite element penalty method. We increase the number of time steps $Nt = 400$, 800, 1600, 3200, 6400 when using the Binomial tree method. Relative errors obtained by the finite element penalty method are in the left-lower corner. Relative errors obtained by the Binomial tree method are in the right-upper corner. Benchmark solutions refer to numerical solutions computed by the Binomial tree method with time steps $Nt = 25600$.](image)

### 3.3.4 Multiple Swing Rights with Different Strikes

We extend numerical examples to a more general case, a multiple swing right problem with different strikes. For swing options with identical rights, one doesn’t have to worry about the order of the right to be exercised. However, for swing options with different rights, one has to find the optimal exercise order of all the rights. As an example, we price a swing option with five put rights whose exercise strikes are 80, 90, 100, 110 and 120
respectively using the Black-Scholes model for a set of typical parameter values \( T = 1, K = 100, \sigma = 0.3, r = 0.05, S_0 = 100. \)

At a time \( t \leq T \), the larger value of the strike price the higher the value of that swing right given a current stock price. Therefore, the swing right with the largest strike price value has the largest optimal exercise region and one would always exercise the highest strike price right of the existing rights to attain the maximum value of the swing option.

![Figure 3.9: Optimal exercise time boundary for American options with different strikes](image)

Fig. 3.9 shows that the swing right with a larger strike price value has a larger optimal exercise region. We use backward time in Fig. 3.9, i.e. \( t = 0 \) is the expiry. Each line represents for the optimal exercise boundary \( S_f(K) \) for the swing right with strike price \( K \) and the left-lower region to \( S_f(K) \) is the optimal exercise region. If \( S \leq S_f(K) \) at a specific time level, then the swing right with strike price \( K \) will be exercised.

We develop an algorithm to identify the optimal exercise boundary at each time level which also allows us to record the optimal exercise time. We solve a multiple swing right problem with five strikes 80, 90, 100, 110 and 120 and numerical results are accurate to one cent. We use solutions from the Binomial Tree method as benchmark solutions and 4001 grid points in the finite element penalty method. The value of this five right swing option at \( S = 100 \) calculated from the finite element penalty method has five significant figures comparing with the benchmark solutions, as shown in Table 3.4.
Table 3.4: Comparison of swing option values of five rights with strike price 80, 90, 100, 110, 120 computed by the finite element penalty and Binomial tree methods.

<table>
<thead>
<tr>
<th>Nt</th>
<th>Finite element penalty method</th>
<th>Binomial tree method</th>
</tr>
</thead>
<tbody>
<tr>
<td>4000</td>
<td>56.3770349646</td>
<td>56.377585803</td>
</tr>
<tr>
<td>8000</td>
<td>56.3771628463</td>
<td>56.377887342</td>
</tr>
<tr>
<td>16000</td>
<td>56.3771993806</td>
<td>56.3775490103</td>
</tr>
</tbody>
</table>

3.3.5 Multiple Swing Rights with a Waiting Period

In practice, the delivery of power is limited by capacity constraints, so there is usually a waiting time period for a contract with multiple rights. That is to say, the $j^{th}$ right has to be held a time $\delta$ after the $(j-1)^{st}$ right being exercised even if there is an optimal exercise opportunity. We extend the application to a more general swing option problem, i.e multiple swing rights with a waiting period $\delta$. We solve a swing put problem with $M$ exercise rights $M = 1, 2, 3, 4, 5$ as an example for a set of typical parameter values $T = 1$, $K = 100$, $\sigma = 0.3$, $r = 0.05$, $S_0 = 100$, $\delta = 0.1$ using the Black-Scholes model.

![Figure 3.10: Optimal exercise time boundary for five swing rights with waiting time $\delta = 0.1$. Each line represents the optimal exercise time boundary of the $j^{th}$ swing right to be exercised. The region below that line is the optimal exercise region.](image)

With a waiting time $\delta = 0.1$, the optimal exercise region of the second right to be exercised shrinks compared with that of the first right to be exercised, the optimal exercise
region of the third right to be exercised shrinks compared to that of the second right to be exercised, and so on, as shown in Fig. 3.10. We use backward time in Fig. 3.10, i.e. $t = 0$ is the expiry. Each line represents for the optimal exercise boundary $S^j$ for $j^{th}$ swing right and the left-lower region to $S^j$ is the optimal exercise region. If $S \leq S^j$ at a specific time level, then the $j^{th}$ swing right will be exercised.

At expiry, there is a region between $S^2$ and $S^1$, where a single right will be exercised but the second right cannot be. When the current stock price is greater than $S^2$, the second right is worthless and we need to adjust the payoff function of the second right to be (3.152)

$$Payoff^{j=2} = \begin{cases} 
K - S, & \text{when } S \leq S^{j=2}, \\
0, & \text{when } S > S^{j=2}.
\end{cases}$$

(3.152)

A plot of (3.152) is shown in Fig. 3.11.

Table 3.5 lists the optimal exercise boundary $S^j$ at the expiry for $j^{th}$ swing right and we can adjust the payoff function of $j^{th}$ right accordingly using (3.152).

Table 3.5: The optimal exercise boundary $S^j$ at the expiry for $j^{th}$ swing right

<table>
<thead>
<tr>
<th>$S^1$</th>
<th>$S^2$</th>
<th>$S^3$</th>
<th>$S^4$</th>
<th>$S^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>84.3</td>
<td>80.3</td>
<td>77.6</td>
<td>75.7</td>
</tr>
</tbody>
</table>

Figure 3.11: Payoff function of the second right to be exercised
We keep track of the optimal exercise time when pricing each swing right by identifying the optimal exercise boundary at each time level, since we cannot exercise a swing right when we see an optimal exercise opportunity within the waiting period. When the waiting period equals the size of the time step, the swing option becomes a multiple identical right swing option.

Table 3.6 shows that the value of swing option price increases to the value of $M$ times an American option price as expected. When the value of a waiting period goes to zero, we expect the option price of each right approaches the value of an American option. We price a swing option with $M$ identical rights and set a waiting period equal to the size of the time step. We compute numerical solutions at $S_0 = 100$ using the Binomial tree method and $\delta = \frac{T}{N_t}$. As we increase the number of time steps $N_t$ to infinity $\delta$ goes to zero. We use the numerical solution when $N_t = 51,200$ as the American option price and increase $N_t = 12800, 25600$ and 51200 to compute the swing option price with $M$ rights at $S_0 = 100$, i.e. $V_M$. As we increase $N_t$, $\delta$ decreases.

Table 3.6: The swing option price with $M$ identical rights goes to the value of $M$ times a single American option price as $\delta$ goes to zero.

<table>
<thead>
<tr>
<th>$N_t$</th>
<th>12800</th>
<th>25600</th>
<th>51200</th>
<th>$M \times 9.87004$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{M=1}$</td>
<td>9.86996</td>
<td>9.87001</td>
<td>9.87004</td>
<td>9.87004</td>
</tr>
<tr>
<td>$V_{M=2}$</td>
<td>19.7398</td>
<td>19.74</td>
<td>19.7401</td>
<td>19.7401</td>
</tr>
<tr>
<td>$V_{M=3}$</td>
<td>29.609</td>
<td>29.6096</td>
<td>29.6099</td>
<td>29.6101</td>
</tr>
<tr>
<td>$V_{M=4}$</td>
<td>39.4782</td>
<td>39.4792</td>
<td>39.4797</td>
<td>39.4801</td>
</tr>
<tr>
<td>$V_{M=5}$</td>
<td>48.3466</td>
<td>49.3485</td>
<td>49.3494</td>
<td>49.3502</td>
</tr>
</tbody>
</table>

When $\delta = 0.1$, we compute the $M$ right swing option price at $S_0 = 100$ using the finite element penalty method as shown in Table 3.7.

Table 3.7: The swing option price at $S_0 = 100$

<table>
<thead>
<tr>
<th>$M$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_M$</td>
<td>9.86948</td>
<td>18.9513</td>
<td>27.4936</td>
<td>35.5665</td>
<td>43.2166</td>
</tr>
<tr>
<td>CPU time</td>
<td>1.75</td>
<td>3.359</td>
<td>4.906</td>
<td>6.406</td>
<td>7.859</td>
</tr>
</tbody>
</table>
CHAPTER 4

CONCLUSIONS

In this dissertation, we develop well-posed boundary conditions to price average strike options and swing options in the energy market.

We use the energy method to find boundary conditions that make a two space variable model of Asian options well-posed on a finite domain. To test the performance of well-posed boundary conditions, we price an average strike call as an example and use similarity solutions as benchmark solutions. We also develop new boundary conditions for the average strike option from financial arguments and the put-call parity. Numerical results show that well-posed boundary conditions are working appropriately. Furthermore, solutions with new boundary conditions match the similarity solution significantly better than the boundary conditions found in the existing literature.

In the swing option valuation, we derive formulas to estimate the size of the numerical domain. Kangro and Nicolaides estimate $S_{\text{max}}$ for the Black-Scholes model in [12]. We extend their work to the one factor model and estimate both $S_{\text{max}}/x_{\text{max}}$ and $S_{\text{min}}/x_{\text{min}}$. We also develop a priori error estimates for both Dirichlet boundary conditions and Neumann boundary conditions. Numerical experiments show that formulas to determine the size of numerical domain are working correctly. We also plot the point-wise error as a function of $S_{\text{max}}/S_{\text{min}}$, one can decide the size of a numerical domain for a given precision.

Each swing right has an early exercise feature, so the optimal exercise price is not known in advance as a function of time and we use a finite element penalty method to resolve the difficulty. Numerical results on a full swing and a one swing show that the well-posed boundary conditions are appropriately applied and the finite element penalty method is working correctly.

Then we apply the finite element penalty method to a swing option with multiple identical rights and the numerical results show that the finite element penalty method is thousands times faster than the Binomial tree method at the same level of accuracy. Furthermore, the finite element penalty method allow us to obtain additional option values and optimal exercise opportunities at different current stock prices simultaneously. The Binomial tree method can provide one option value each time.

Furthermore, we price a multiple right swing option with different strike prices. We identify that the initial condition of each right can be no longer its payoff function. Any two swing rights cannot be exercised at the same time, so exercising one swing right can force the optimal region of the other swing right to shrink. As a result, the initial condition has a
jump discontinuity. We develop an algorithm to identify the optimal exercise boundary at each time level which also allows us to record the optimal exercise time. Numerical results are accurate to one cent comparing with the benchmark solutions computed by a Binomial tree method. This also implies that the algorithm is working correctly.

We also extend the application to a swing option with a waiting period restriction. When a waiting period exists between two swing rights to be exercised successively, we cannot exercise the latter right when we see an optimal exercise opportunity within the waiting period. Therefore, we keep track of the optimal exercise time by identifying the optimal exercise boundary at each time level. We also verify an extreme case numerically. When the waiting time decreases, the value of swing option price increases to the value of $M$ times an American option price as expected.
BENCHMARK SOLUTIONS TO THE AVERAGE STRIKE OPTION USING A SIMILARITY REDUCTION METHOD

We will use the similarity reduction problem solutions as benchmark solutions to compare boundary conditions in Table 2.1. The similarity reduction is to substitute $V = SH$ with $R = \frac{t}{S}$ into (1.4) and reduce (1.4) to the one space variable PDE problem:

$$
\begin{align*}
&\frac{\partial H}{\partial \tau} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + \frac{\partial H}{\partial R} - rR \frac{\partial H}{\partial R} = 0 \\
&H(\infty, t) = 0 \\
&\frac{\partial H}{\partial \tau} + \frac{\partial H}{\partial R} = 0 \quad \text{at} \quad R = 0 \\
&H(R, T) = \max (1 - \frac{R}{T}, 0). \\
\end{align*}
$$

(A.1)

Next let $\tau = T - t$, so that we have the initial boundary value problem

$$
\begin{align*}
&\frac{\partial H}{\partial \tau} = \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + \frac{\partial H}{\partial R} - rR \frac{\partial H}{\partial R} \\
&H(\infty, \tau) = 0 \\
&-\frac{\partial H}{\partial \tau} + \frac{\partial H}{\partial R} = 0 \quad \text{at} \quad R = 0 \\
&H(R, 0) = \max (1 - \frac{R}{T}, 0).
\end{align*}
$$

(A.2)

To show that (A.2) is well-posed, we will use the weak form of the equation. For any test function $\phi \in H^1$,

$$
\int_R \frac{\partial H}{\partial \tau} \phi dR = \int_R \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} \phi dR + \int_R \frac{\partial H}{\partial R} \phi dR - \int_R rR \frac{\partial H}{\partial R} \phi dR,
$$

(A.3)

choose $\phi = H$,

$$
\int_R \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} HdR = \frac{1}{2}\sigma^2 \int_R H \frac{\partial H}{\partial R} dR - \frac{1}{2} \sigma^2 \int R \frac{\partial H}{\partial R} dR - \frac{1}{2} \sigma^2 \int R \frac{\partial^2 H}{\partial R^2} dR,
$$

(A.4)
then
\[
\int_{\mathbb{R}} \frac{\partial H}{\partial \tau} H dR = -\frac{1}{2} \sigma^2 \left[ \int_{\mathbb{R}} \left( R \frac{\partial H}{\partial R} \right)^2 dR + \frac{2}{\sigma^2} \int_{\mathbb{R}} (r + \sigma^2) H \left( R \frac{\partial H}{\partial R} \right) dR \right] \\
+ \frac{1}{2} \int_{\mathbb{R}} dH^2 + \frac{1}{2} \sigma^2 R^2 H \frac{\partial H}{\partial R} \bigg|_{0}^{\infty}
\]

(A.5) can be written as
\[
\frac{1}{2} \frac{d}{d\tau} \|H\|^2_2 = -\frac{1}{2} \sigma^2 \left[ \left\| R \frac{\partial H}{\partial R} \right\|^2_2 + 2(1 + \frac{r}{\sigma^2}) \left( R \frac{\partial H}{\partial R} , H \right) \right] \\
+ \frac{1}{2} H^2 \bigg|_{0}^{R_{\max}} + \frac{1}{2} \sigma^2 R^2 H \frac{\partial H}{\partial R} \bigg|_{0}^{R_{\max}} \\
= -\frac{1}{2} \sigma^2 \left[ \left\| R \frac{\partial H}{\partial R} + (1 + \frac{r}{\sigma^2})H \right\|^2_2 - (1 + \frac{r}{\sigma^2})^2 \|H\|^2_2 \right] \\
+ \frac{1}{2} H^2 \bigg|_{R_{\max}} + \frac{1}{2} \sigma^2 R^2 H \frac{\partial H}{\partial R} \bigg|_{R_{\max}} \\
\leq \frac{1}{2} \left( \sigma + \frac{r}{\sigma} \right)^2 \|H\|^2_2 + \frac{1}{2} H^2 \bigg|_{R_{\max}} + \frac{1}{2} \sigma^2 R^2 H \frac{\partial H}{\partial R} \bigg|_{R_{\max}} (A.6)
\]

When \( H = 0 \) is applied at \( R = R_{\max} \), the boundary integrals have no contribution to the growth of energy, so we have \( \frac{d}{d\tau} \|H\|^2_2 \leq \left( \sigma + \frac{r}{\sigma} \right)^2 \|H\|^2_2 \) and thus \( \|H\|^2_2 \leq \|H_{t=0}\|^2_2 e^{\left( \sigma + \frac{r}{\sigma} \right)^2 \tau} \).
According to the Def. 2, the problem is well-posed. Therefore, a Dirichlet boundary condition along \( R = R_{\max} \) is sufficient to guarantee well-posedness.

To approximate (A.4), we use a finite difference approximations
\[
\left( \frac{\partial H}{\partial \tau} \right)_{t_n} = \frac{H^{n+1}_j - H^n_j}{\Delta \tau} + O(\Delta \tau) \\
\left( \frac{\partial H}{\partial R} \right)_{R_j} = \alpha \left( \frac{H^{n+1}_{j+1} - H^{n+1}_j}{2\Delta R} \right) + (1 - \alpha) \left( \frac{H^n_{j+1} - H^n_{j-1}}{2\Delta R} \right) + O(\Delta R^2) \quad (A.7) \\
\left( \frac{\partial^2 H}{\partial R^2} \right)_{R_j} = \alpha \left( \frac{H^{n+1}_{j+1} - 2H^n_{j+1} + H^n_{j+1}}{\Delta R^2} \right) + (1 - \alpha) \left( \frac{H^n_{j-1} - 2H^n_{j} + H^n_{j+1}}{\Delta R^2} \right) + O(\Delta R^2)
\]
to approximate
\[
\begin{align*}
\frac{\partial H}{\partial \tau} &= \frac{1}{2} \sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + \frac{\partial H}{\partial R} - rR \frac{\partial H}{\partial R} \\
H(R_{\max}, \tau) &= 0 \\
-\frac{\partial H}{\partial R} + \frac{\partial H}{\partial R} &= 0 \quad \text{at} \; R = 0 \\
H(R, 0) &= \max (1 - \frac{R}{T}, 0), \quad (A.8)
\end{align*}
\]
where $\alpha = \frac{1}{2}$ gives a Crank-Nicolson (C-N) approximation and $\alpha = 1$ gives a backward-Euler (B-E) approximation.

At $R_j = j \Delta R$, we approximate

$$\frac{\partial H}{\partial \tau} = \frac{1}{2} \sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR) \frac{\partial H}{\partial R}$$

(A.9)

by

$$\frac{H_j^{n+1} - H_j^n}{\Delta \tau} = \frac{\sigma^2 j^2}{2} \left( \alpha (H_{j+1}^{n+1} - 2H_j^{n+1} + H_{j-1}^{n+1}) + (1 - \alpha)(H_{j+1}^n - 2H_j^n + H_{j-1}^n) \right)$$

$$+ \frac{1 - rj \Delta R}{2 \Delta R} \left( \alpha (H_{j+1}^{n+1} - H_{j-1}^{n+1}) + (1 - \alpha)(H_{j+1}^n - H_{j-1}^n) \right)$$

(A.10)

Rearranging and taking $h_j^n$ as the finite difference approximation of $H_j^n = H(n \Delta \tau, j \Delta R)$, we obtain the finite difference scheme

$$A_j h_{j-1}^{n+1} + B_j h_j^{n+1} + C_j h_{j+1}^{n+1} = a_j h_{j-1}^n + b_j h_j^n + c_j h_{j+1}^n$$

(A.11)

where

$$A_j = \left( -\frac{\sigma^2 j^2}{2} + \frac{1}{2 \Delta R} (1 - rj \Delta R) \right) \alpha \Delta \tau,$$

$$B_j = 1 + \frac{\sigma^2 j^2 \alpha}{2} \Delta \tau,$$

$$C_j = \left( -\frac{\sigma^2 j^2}{2} - \frac{1}{2 \Delta R} (1 - rj \Delta R) \right) \alpha \Delta \tau,$$

and

$$a_j = \left( \frac{\sigma^2 j^2}{2} - \frac{1}{2 \Delta R} (1 - rj \Delta R) \right) (1 - \alpha) \Delta \tau,$$

$$b_j = 1 - \frac{\sigma^2 j^2 (1 - \alpha)}{2} \Delta \tau,$$

$$c_j = \left( \frac{\sigma^2 j^2}{2} + \frac{1}{2 \Delta R} (1 - rj \Delta R) \right) (1 - \alpha) \Delta \tau.$$

At $j = 0$, i.e. $R_j = 0$,

$$\frac{H_0^{n+1} - H_0^n}{\Delta \tau} = \frac{\alpha (H_1^{n+1} - H_0^{n+1}) + (1 - \alpha)(H_1^n - H_0^n)}{\Delta R},$$

(A.14)

which we write as

$$B_0 h_0^{n+1} + C_0 h_1^{n+1} = b_0 h_0^n + c_0 h_1^n,$$

(A.15)

where

$$B_0 = 1 + \alpha \frac{\Delta \tau}{\Delta R},$$

$$C_0 = -\alpha \frac{\Delta \tau}{\Delta R},$$

$$b_0 = 1 - (1 - \alpha) \frac{\Delta \tau}{\Delta R},$$

$$c_0 = (1 - \alpha) \frac{\Delta \tau}{\Delta R}.$$
The other boundary condition at \( R = R_{\text{max}} \) can be approximated by the condition that

\[
h^n_J = 0 \tag{A.17}
\]

We may therefore approximate the problem (A.2) by the system

\[
Mh^{n+1} = y^n \tag{A.18}
\]

where

\[
M = \begin{bmatrix}
B_0 & C_0 & 0 & 0 & \cdots & 0 \\
A_1 & B_1 & C_1 & 0 & \cdots & 0 \\
0 & A_2 & B_2 & C_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & A_{J-2} & B_{J-2} & C_{J-2} \\
0 & \cdots & 0 & A_{J-1} & B_{J-1} & 0 \\
\end{bmatrix},
\]

\[
y^n = \begin{bmatrix}
y^n_0 \\
y^n_1 \\
y^n_2 \\
\vdots \\
y^n_{J-2} \\
y^n_{J-1} \\
\end{bmatrix},
\]

\[
h^{n+1} = \begin{bmatrix}
h^{n+1}_0 \\
h^{n+1}_1 \\
h^{n+1}_2 \\
\vdots \\
h^{n+1}_{J-2} \\
h^{n+1}_{J-1} \\
\end{bmatrix},
\]

\[
h^0_j = \max(1 - j\Delta R/T, 0) \tag{A.20}
\]

Eq. (A.18) can be solved by the Thomas algorithm [24] at each time step.

We solve the problem (A.2) for \( \sigma = 0.4 \) and \( \tau = 0.1 \) at three months before maturity \( T = 0.5 \) and there has already been three months averaging [33, 32]. In this case, we calculate the price at \( t = 0.25 \) instead of the price at \( t = 0 \), i.e. we are calculating over the time span \( 0 \leq \tau \leq 0.5 - 0.25 = 0.25 \). To smooth out the singularity of the payoff function, we take two steps with backward Euler (\( \alpha = 1 \)) first, then switch to Crank-Nicolson (\( \alpha = \frac{1}{2} \)). We show the solution in Fig. A.1, which is the same as the result shown in [33, 32].
The European average strike call option

Figure A.1: Option value $H$ versus $R$, $\sigma = 0.4, r = 0.1$ at three months before maturity $T = 0.5$ and there has already been three months averaging [33, 32]


[34] Z. Zhu and N. Stokes, A Finite Element Platform for Pricing Path Dependent Exotic Options, CSIRO working paper, 1999.


BIOGRAPHICAL SKETCH

Jinhua Yan

Jinhua Yan enrolled in the Financial Mathematics program at FSU in the fall of 2006 and did research under the advisement of Professor David A. Kopriva. Jinhua also worked as a Teaching Assistant and was awarded the title of Distinguished Teaching Assistant by the Mathematics department.