2007

An Investigation of College Students' Covariational Reasonings

Onder Koklu
THE FLORIDA STATE UNIVERSITY

COLLEGE OF EDUCATION

AN INVESTIGATION OF COLLEGE STUDENTS’ COVARIATIONAL REASONINGS

By

ONDER KOKLU

A Dissertation submitted to the
Department of Middle and Secondary Education
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Degree Awarded:
Summer Semester, 2007
The members of the Committee approve the dissertation of Onder Koklu defended on 01/03/2007.

____________________________
Elizabeth Jakubowski
Professor Directing Dissertation

____________________________
Sande Milton
Outside Committee Member

____________________________
Leslie Aspinwall
Committee Member

____________________________
Kenneth Shaw
Committee Member

Approved:

____________________________
Pamela Carroll, Chair, Department of Middle and Secondary Education

The Office of Graduate Studies has verified and approved the above named committee members.
For
My father Erol Koklu

ACKNOWLEDGMENTS
I would like to thank my father, Erol Koklu, for encouraging me to pursue PhD degree in the US. He sowed me the way and helped me to open my mind to new endeavors throughout my life. Without his wisdom, vision, and encouragement I couldn’t accomplish this goal. I am also thankful to my mother. My wife has always been by my side and shared my burden in every minute throughout my study. I would like to apologize, if I have broken her heart in my stressful times. She never complained about anything and supported me all the time. I love you Ebru and I always will.

I would like to express my sincere appreciation to my major professor, Dr. Elizabeth Jakubowski, for her guidance, support, inspiration and encouragement. For eight years, she has been more than a major professor to me. Dr. Jakubowski, you are one of the most beautiful people in my life. I will always be grateful for your support and guidance throughout my graduate study. I would like to thank to my committee members, Dr. Leslie Aspinwall, Dr. Kenneth Shaw and Dr. Sande Milton for their valuable feedbacks throughout my study.

I also would like to express my gratitude to Turkish Ministry of National Education for providing an incredible opportunity to me and many other people to study in the US. Without its financial support, this dream couldn’t come true.

TABLE OF CONTENTS

List of Tables                        vii
List of Figures                       viii
<table>
<thead>
<tr>
<th>Section</th>
<th>Start Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>xi</td>
</tr>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>Covariational Reasoning and Graphical Context</td>
<td>3</td>
</tr>
<tr>
<td>Theoretical Framework</td>
<td>5</td>
</tr>
<tr>
<td>Problem Statement and Purpose of The Study</td>
<td>6</td>
</tr>
<tr>
<td>Significance of The Study</td>
<td>8</td>
</tr>
<tr>
<td>II. LITERATURE REVIEW</td>
<td>9</td>
</tr>
<tr>
<td>Overview</td>
<td>9</td>
</tr>
<tr>
<td>Rate of Change</td>
<td>10</td>
</tr>
<tr>
<td>Three Perspectives to Rate of Change</td>
<td>12</td>
</tr>
<tr>
<td>Images of Rate and Images of Covariation</td>
<td>13</td>
</tr>
<tr>
<td>Static Vs. Dynamic Conceptions of Functions</td>
<td>18</td>
</tr>
<tr>
<td>Changing Nature of Functions</td>
<td>19</td>
</tr>
<tr>
<td>Reasoning about Change</td>
<td>26</td>
</tr>
<tr>
<td>Three Types of Reasoning</td>
<td>26</td>
</tr>
<tr>
<td>Covariational Reasoning</td>
<td>26</td>
</tr>
<tr>
<td>III. MATERIALS AND METHOD</td>
<td>30</td>
</tr>
<tr>
<td>Overview</td>
<td>30</td>
</tr>
<tr>
<td>Methodology</td>
<td>30</td>
</tr>
<tr>
<td>Research Procedures</td>
<td>32</td>
</tr>
<tr>
<td>Data Collection</td>
<td>33</td>
</tr>
<tr>
<td>Data Analysis</td>
<td>36</td>
</tr>
<tr>
<td>Quality Criteria</td>
<td>37</td>
</tr>
<tr>
<td>IV. DATA ANALYSES AND FINDINGS</td>
<td>39</td>
</tr>
<tr>
<td>Overview</td>
<td>39</td>
</tr>
<tr>
<td>The Case of Jay</td>
<td>40</td>
</tr>
<tr>
<td>Introduction</td>
<td>40</td>
</tr>
<tr>
<td>Analysis of the Case</td>
<td>42</td>
</tr>
<tr>
<td>Summary of the Case</td>
<td>76</td>
</tr>
<tr>
<td>The Case of Karl</td>
<td>79</td>
</tr>
<tr>
<td>Introduction</td>
<td>79</td>
</tr>
<tr>
<td>Analysis of the Case</td>
<td>81</td>
</tr>
<tr>
<td>Summary of the Case</td>
<td>114</td>
</tr>
<tr>
<td>Cross-case Analyses</td>
<td>119</td>
</tr>
<tr>
<td>V. CONCLUSIONS</td>
<td>123</td>
</tr>
<tr>
<td>Assertions</td>
<td>125</td>
</tr>
<tr>
<td>Implications</td>
<td>128</td>
</tr>
<tr>
<td>Limitations</td>
<td>129</td>
</tr>
<tr>
<td>Issues for Future Research</td>
<td>130</td>
</tr>
</tbody>
</table>
APPENDICES 131
REFERENCES 141
BIOGRAPHICAL SKETCH 146

LIST OF TABLES

Table 1.1 Mental actions and corresponding behaviors 6
Table 3.1 Mental actions 36
<table>
<thead>
<tr>
<th>Table 4.1</th>
<th>Summary of two cases based upon five mental actions</th>
<th>123</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 5.1</td>
<td>Common patterns of interpretation and representation of simultaneous changes</td>
<td>125</td>
</tr>
</tbody>
</table>

**LIST OF FIGURES**
Figure 2.1  Speed as a rate  14
Figure 2.2  A snapshot of the cars position  15
Figure 2.3  A graph of completed motion of the car  16
Figure 2.4  Bottle filling with water  28
Figure 4.1  Task-1  42
Figure 4.2  Jay’s graph of distance vs. speed  43
Figure 4.3  Jay’s graph of distance vs. time  44
Figure 4.4  Jay’s second graph of distance vs. time  45
Figure 4.5  Task-2  47
Figure 4.6  Jay’s graph of distance vs. time  47
Figure 4.7  Jay’s second graph of distance vs. time  48
Figure 4.8  Task-3  49
Figure 4.9  Jay’s graph of distance vs. time  49
Figure 4.10  Task-4  51
Figure 4.11  Jay’s graph of amount of chemical X vs. time  51
Figure 4.12  Jay’s second graph of amount of chemical X vs. time  53
Figure 4.13  Task-5  54
Figure 4.14  Jay’s two graphs of amount of chemical X and Y vs. time  54
Figure 4.15  Task-6  55
Figure 4.16  Jay’s two graphs: Area of a circle vs. radius and Area of a circle vs. time  56
Figure 4.17  Task-7  57
Figure 4.18  Jay’s presentation of position of the ladder  58
Figure 4.19  Task-8  58
Figure 4.20  Jay’s presentation of change in the angle  59
Figure 4.21  Jay’s presentation of horizontal disposition  59
Figure 4.22  Jay’s graph of vertical vs. horizontal position of the ladder  60
Figure 4.23  Task-10  61
Figure 4.24  Jay’s first graph of height vs. amount of water  62
Figure 4.25  Jay’s second graph of height vs. amount of water  63
Figure 4.26  Jay’s work on the figure.  63
Figure 4.27  Task-11  65
ABSTRACT

Present study investigates college students’ covariational reasoning in light of five mental actions described in covariation framework introduced by Marilyn P. Carlson. More specifically,
this study focuses on college students’ understanding and reasoning about simultaneous changes of two variables when they interpret a functional situation and use their interpretations to demonstrate simultaneous changes of two variables in graphical representations.

Two high performing college students’ reasoning was investigated in a multiple case study design. Data was obtained from a detailed examination of students’ thinking and reasoning processes through the task based in-depth clinical interviews. Data obtained from students’ verbal expressions and graphical representations were analyzed in light of the theoretical lens. Carlson et al.’s (2002) covariation framework provided a skeletal structure for the description and interpretation of findings in each case. Specifically, five mental actions defined in the framework were used to describe each student’s covariational reasoning.

Analysis of data disclosed that functional situations are conceived as static rather than dynamic. This static approach prevents students from evaluating the whole process as it is happening at once. In other words, students have difficulties to represent continues changes of two variables in a functional situation and coordinate the simultaneous changes of two variables on entire domain. In addition, students’ difficulties in graphical representations produce inconsistencies between interpretations and representations of simultaneous changes of two variables. It is also revealed that Students’ strong procedural tendency hinders reasoning and meaningful interpretations about change in functional situations.

CHAPTER I
INTRODUCTION

“We live in a world characterized by change. Motion and growth are prevalent part of our everyday lives and no time is motion more obviously a characteristic of our daily lives than today”

Garnet Hauger

The idea of change- both how things change and at what rate things change with respect to each other- is fundamental to a study of calculus, which is a critical course for students majoring in mathematics, sciences (physics, chemistry, biology etc.), engineering, business and several other majors (Carlson, Larsen, & Jakobs, 2001; Saldhana & Thompson, 1998; Noble, Nomirovsky, Wright, & Tierney, 2001). Calculus is the study of properties of continuous functions and their derivatives and integrals. It is also recognized as a study of motion in which changing values are studied. Therefore, understanding patterns of change and reasoning about simultaneous changes of variables in a functional relationship has been shown to be essential and foundational for understanding the major concepts in calculus (Carlson & Oehrtman, 2005; Kaput, 1992; Thompson, 1994a). These reasoning abilities are also said to be important to represent and interpret a wide array of dynamic events in which the variables continuously change in tandem.

What is a formal definition of this type of reasoning? According to Carlson, Jacobs, Larsen, and Hsu (2002), it is “cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (p.354) and they call it covariational reasoning. Another definition comes from Saldanha & Thompson (1998). According to these researchers, it is “…holding in mind a sustained image of two quantities’ values (magnitudes) simultaneously” (p.298). It has been shown that these reasoning abilities are necessary for analyzing, interpreting and representing the patterns of change in continuously changing dynamic events (Carlson et al, 2001; Monk, 1992).

Because covariational reasoning involves mental activities of coordinating the simultaneous changes of two variables, it has been shown to be important for understanding main concepts in calculus, such as limits, derivatives, and definite integrals (Carlson et al, 2001; Thompson, 1994b; Monk & Nemirovsky, 1994). According to Thompson (1994b) these
reasoning abilities are foundational for even understanding the Fundamental Theorem of Calculus. He also stated that difficulties in learning the limit concept have been linked to weak covariational reasoning abilities.

According to Carlson & Oehrtman (2005) “as students begin to explore dynamic function relationships, such as how speed varies with time or how the height of water in a bottle varies with volume, they will need to begin considering how one variable changes while imagining the changes in the other” (p.11). They also stated that being able to imagine changes of variables simultaneously in a dynamic functional relationship is difficult. Because it requires one to be able to picture the whole process at once and to be able to imagine running through the several input and output pairs simultaneously instead of performing specific computations for every input value. Thompson (1994b) expressed this ability as imagining the expression evaluate it self very rapidly over a continuum of possible domain. This ability of imagining the whole process as it is happening at once has been linked to students’ underlying function conceptions.

Studies (Carlson & Oehrtman, 2005; Monk, 1992; Monk & Nemirovsky, 1994; Thompson, 1994b) have revealed that students who posses strong procedural skills such as symbol manipulations and weak conceptual structures are unable to construct images of simultaneous covariation of two quantities in a functional relation. Monk (1992), for example, classified students function conceptions as “point-wise” and “across-time”. Monk observed that students who had point-wise view were not able to imagine the whole functional situation in once when they asked to graph given dynamic functional situation. Rather students attempted to collect x and y values by making precise measurements and graph the situation. College students’ understanding of function concept and their underlying conceptions have been widely documented by many researchers (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992; Monk, 1992; Thompson, 1994b). According to investigations college students are entering the universities with little or no function conception. Even high performing undergraduates have weak function understanding (Breidenbach et al. 1992; Carlson, 1998). Although college students function conceptions include dependence, correspondence and different representations of functions, it has been shown they are still limited to procedural computations and skills (Kaput, 1992; Monk, 1992; Carlson; 1998). Thus they have difficulties in imagining and coordinating the changes in two variables of functions simultaneously.
Considering college students’ difficulties about understanding changing nature of functional relationship and imagining and coordinating the changes in variables concurrently, researchers (Kaput, 1994; Orton, 1983; Thompson, 1994b; Confrey & Smith, 1995) suggest that we need to start introducing the concept of function as covariation and addressing the problems of calculus such as coordination of changes of variables in tandem, and giving the students more opportunity to work with concept of rate of change in earlier grades. According to Confrey and Smith (1994), correspondence approach to learning functions in the current curriculum is formula dependent. And they criticized this approach because it focuses on rules and formulas to describe how to obtain the output value from a given input value. They stated that covariation approach is essential for coordinating movement (changes) between input and output values. Thus the authors claimed that the covariation approach makes the concept of rate of change more visible to students.

In connection with the findings and suggestions of these research studies, NCTM (1989) emphasized the inclusion of function related activities in algebra curriculum beginning with fourth grade will provide students a good basis for developing reasoning abilities to be able to see and analyze patterns of change in different contexts in early ages. In 2000, NCTM released Principles and Standards for School Mathematics. Beside the statements that stressed the importance of the inclusion of function related activities in algebra curriculum, it is also recommended that students should be able to analyze change in various real-world contexts.

Students should be able to interpret statements such as “the rate of inflation is decreasing”. The study in change in grades 9-12 is intended to give students a deeper understanding of the ways in which changes in quantities can be represented mathematically and of the concept of rate of change. (NCTM, 2000 p.305)

**Covariational Reasoning and Graphical Context**

Functions represent relationships between varying quantities. These relationships between variables can be represented in different systems as equations, graphs, tables of values. Because students’ reasoning abilities in this study is referred as ability to coordinate changes of variables in tandem, graphical representations are playing important role for representing and interpreting simultaneous changes of variables in dynamic events.
Graphs are widely used way of representing functions and as Selden & Selden (1992) stated “…they can provide an immediately accessible pictorial image useful in explaining increasing, decreasing, maxima, minima, and inflection points” (p.3). By constructing and interpreting the graph of a dynamic functional relationship, students may reveal their thoughts about patterns of change such as relative changes of input and output, and direction of these changes. Students’ reasoning abilities in coordinating simultaneous changes have been shown to be important to demonstrate ones ability to represent dynamic functional events in a graphical context and interpret important features in the shape of a graph of dynamic functional relationship (Carlson et al, 2002; Carlson & Oehrtman, 2005).

Representing the dynamic events in graphical context involves constructing the graph of given situation to demonstrate the relationship of variables that characterize the situation. Constructing the graph of functional relations is said to be one of the most difficult process observed from early grades to advance levels (Leinhardth, Zavlavsky, & Stein, 1990). Construction of graphs requires one to make transition from functional situation to graphical representation of that situation. This process of transition is referred as modeling (O’Callahan, 1998) or as mathematization of the functional relationships (Yerushalmy, 1997) given in a context. Yerushalmy (1997) clearly stated that modeling is powerful activity for examining the meaningful reasoning. It is also shown that covariational reasoning abilities are playing important role in modeling dynamic functional events (Carlson, 2002; Kaput, 1994; Monk, 1992).

Interpretation of a graph is said to be another important action as well as construction of a graph (Leinhardth, Zavlavsky, & Stein, 1990). Interpretation is referred as making sense out of a partial or overall shape of a graph by extracting information from the graph. As Leinhardth, Zavlavsky, and Stein (1990) stated that interpretation can be global and general or local and specific. Students’ reasoning abilities in coordinating the simultaneous changes have been shown to be important to interpret graphical representation of dynamic functional situations (Carlson et al, 2002). Students use their reasoning when they try to decide, for example, how x and y changes with respect to each other or what is happening when a curve is concave up or concave down or what is the meaning of a point at which graph is changing its shape.

It has been reported that college students even those who have taken calculus have difficulties in representing and interpreting the dynamic functional situations (Carlson et al 2002;
Hauger, 1998; Monk, 1992; Thompson, 1994a). Carlson et. al. (2002), for example, stated that even the most talented undergraduate students have difficulties in modeling functional relationships of situations involving rate of change of one variable as it continuously changes in a dependent relationship with another variable. Research studies (Monk, 1992; Monk & Nemirovsky, 1994; Thompson 1994b) have also revealed that calculus students have difficulties representing and interpreting the graphs of dynamic function relationships.

Research reports mostly concentrated around: a) students difficulties in representing the dynamic functions in graphical context, in other words, difficulties in constructing the graph of given dynamic situations; b) students difficulties in interpreting the dynamic functions given in a graphical context. Students’ difficulties in both representing and interpreting the dynamic function situations have been linked to students’ weak reasoning abilities in coordinating the simultaneous changes of variables. More specifically research findings reveal that when representing and interpreting the dynamic functional events, students have weak or limited conceptions of coordinating the direction of change, coordinating the amount of change simultaneously, identifying and expressing the concavity and inflection points.

**Theoretical Framework**

The theoretical framework for this study provided a guide in explaining students’ covariational reasoning abilities about dynamic functional events. This framework also provided a basis for data analysis and interpretation. Eisenhart (1991) described the conceptual framework as a skeletal structure of justification for adopting certain ideas that establish the research perspective and provide a guide for data analysis and interpretation. The work of Marilyn P. Carlson will help to analyze and explain students’ covariational reasoning abilities in dynamic functional events.

Covariation framework was introduced by Carlson (1998) as a result of several investigations in which behaviors of undergraduate students have been identified as they were responding to tasks that involve interpreting and representing dynamic function situation. Carlson et al. (2002) defined covariational reasoning as

…cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other.(p. 354)
Mental actions and corresponding behaviors have been classified and defined by Carlson et al (2002). Table below represents the descriptions of five mental actions of covariational reasoning and behaviors classified considering these mental actions. In order to determine an individual’s covariational reasoning ability, examination of mental actions and behaviors observed while working on the task is necessary (Carlson et al., 2002).

**Table 1.1 Mental actions and corresponding behaviors**

<table>
<thead>
<tr>
<th>Mental Action</th>
<th>Description of Mental Action</th>
<th>Behaviors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mental Action 1</td>
<td>An image of two variables changing simultaneously. (Coordinating the value of one variable with changes in the other.)</td>
<td>• Labeling the axes with verbal indications of coordinating the two variables(e.g., y changes with changes in x)</td>
</tr>
<tr>
<td>(MA1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mental Action 2</td>
<td>A loosely coordinated image of how the variables are changing with respect to each other. (Coordinating the direction of change of one variable with changes in the other)</td>
<td>• Constructing an increasing straight line</td>
</tr>
<tr>
<td>(MA2)</td>
<td></td>
<td>• Verbalizing an awareness of the direction of change of the output while considering changes in the input.</td>
</tr>
<tr>
<td>Mental Action 3</td>
<td>An image of amount of change of the output variable while considering changes in fixed amounts of the function’s domain. (Coordinating the amount of change of one variable with changes in the other)</td>
<td>• Plotting points / constructing secant lines</td>
</tr>
<tr>
<td>(MA3)</td>
<td></td>
<td>• Verbalizing an awareness of the amount of change of the output while considering changes in the input.</td>
</tr>
<tr>
<td>Mental Action 4</td>
<td>An image of rate/slope for contiguous intervals of the function’s domain. (Coordinating the average rate of change of function with uniform increments of change in the input variable.)</td>
<td>• Constructing contiguous secant lines for the domain.</td>
</tr>
<tr>
<td>(MA4)</td>
<td></td>
<td>• Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input.</td>
</tr>
<tr>
<td>Mental Action 5</td>
<td>An image of the continuously changing rate over the entire domain. (Coordinating the instantaneous rate of change of function with continues changes in the independent variable for the entire domain of the function.)</td>
<td>• Constructing a smooth curve with clear indications of concavity changes</td>
</tr>
<tr>
<td>(MA5)</td>
<td></td>
<td>• Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)</td>
</tr>
</tbody>
</table>

**Problem Statement and Purpose of the Study**

Concept of rate of change has been shown by research studies (Carlson, Larsen, & Jakobs, 2001; Saldhana & Thompson, 1998; Noble, Nomirovsky, Wright, & Tierney, 2001; Orton, 1983; Hauger, 1995; Hauger 1997; White & Mitchelmore, 1996) to be important and central concept to the study of calculus. It is also documented in research literature (Thompson, 1994a; Carlson et al., 2001; Monk & Nemirovsky, 1994) that lack of ability to reason about
change in continuously changing functional relationships cause difficulties in learning basic calculus concepts such as limit, derivative.

Research reveals that students are entering the colleges with limited or no conceptions of functions (Breidenbach et al. 1992; Carlson, 1998). Thus they have difficulties in coordinating the simultaneous changes of two variables in continuously changing dynamic functions (Kaput, 1992; Monk, 1992; Carlson; 1998). Even high performing calculus students have difficulties the imagining the simultaneous changes of one variable with changes of the other (Carlson; 1998).

Research literature also reveals that college students have difficulties constructing and interpreting the graph of dynamic functional situations (Carlson et al, 2002; Monk, 1992). Even though they perform well in routine tasks, they demonstrate poor understanding when engaged in non-routine problems. It has been shown that even high performing college students have difficulties in modeling non-routine dynamic events and in interpreting the models of these dynamic events.

Despite variety of research studies (Thompson, 1994a; Orthon, 1983; White & Mitchelmore, 1996) that emphasized the effects of students’ understanding of rate of change on students’ understanding in calculus concepts such as limit, derivative, integrals, there is little information about how college students’ reason about continuously changing functional relationships. Thus, aim of this study is to explore, describe and analyze the college students’ covariational reasoning abilities. More specifically, this study investigates how college students use their understanding and reasoning to interpret a functional situation and use their interpretations to demonstrate covariation of two variables in graphical representation, and how they use their understanding and reasoning to interpret important features in the shape of a graph of functional events. The specific research questions are:

What thinking and reasoning processes do college students’ use as they attempt to coordinate simultaneous changes of two variables in continuously changing functional situations?

- How do college students interpret simultaneous changes of two variables in functional situations?
- How do college students use their interpretations to represent covariation of two variables in graphical context?
- How do college students interpret covariation of two variables in given graphs of functional events?
Significance of the Study

Believing that it is important to find out how well students imagine and coordinate changes of two variables simultaneously in continuously changing dynamic functional events, this study will contribute to understanding college students’ covariational reasoning abilities.

Although students’ knowledge about rate of change have been studied largely (Hauger, 1998), it is not clear how students represent and interpret continuously changing dynamic functional relationships. In other words, there is little information about students’ covariational reasoning abilities. Therefore it is important to understand how students coordinate changes of two variables simultaneously in continuously changing dynamic functional events.

Literature reveals that graphical representations from early grades to advance levels are used from quantitative perspective rather than qualitative view (Leinhardth, Zavlavsky, & Stein, 1990). In other words, students are either asked to construct graphs of given symbolic representations (equations etc.) or asked to construct a graph by extracting numerical values from a given situation. This approach is poor in the way of assessing the students’ reasoning and conceptual understanding. This study will contribute to literature by examining the students’ covariational reasoning in graphical context from qualitative perspective.

Moreover, understanding students’ covariational reasoning abilities can provide sufficient information to develop more effective instruction. Considering the difficulties students encounter in traditional calculus classrooms, understanding students’ reasoning patterns about change and continuous change can suggest some alternative ways in instruction. Further, identification of students’ covariational reasoning abilities would give educators and curriculum designers an opportunity to consider inclusion of some aspects of function concept in earlier grades.
CHAPTER II
LITERATURE REVIEW

Overview

This chapter demonstrates the review of literature. Although there is plenty of literature pertaining to students’ understanding of rate of change, literature related to students’ reasoning about coordinating the simultaneous changes is limited. Chapter is designed to provide examples of previous research studies related to students’ thinking and understandings of change from different perspectives.

First, research studies about students’ understanding of rate of change are provided. While studies of Orthon (1983) and Thompson (1994a) pointed out the role of students’ conceptions of rate of change in their understanding of calculus concepts, Hauger investigated the students’ understanding of rate of change in a broad view in terms of three perspectives—global, interval, and point.

Second, literature about development of images of rate and images of covariation are provided. In this part, studies of Thompson are main sources in explaining how students develop mature images of rate and images of covariation.

Third, chapter reviews the studies about students’ understanding of relationship between function and its derivative. This review provides reader useful information about how students use variation and covariation of variables to make connections between functions and their derivatives.

Forth, literature related to dynamic conceptualization of functions is provided. An overview of nature of change of functions is provided by Sierpinska. Carlson and Oehrtman have pointed out the necessities of having process view of functions in order to conceptualize functions dynamically rather than having action view of function which is more static conception. Study of Dubinsky and Harel, have given the detailed explanation of process and action view of functions. Likewise, Study of Monk characterized the students’ conceptions of functions as point-wise (static conception) and across-time (dynamic conception). He investigated students’ conceptions of functions having to do with their understanding of related rates. Study of Confrey and Smith has classified students’ function conceptions as correspondence (static view) and covariation (dynamic view) by examining their approaches to solve rate of change problems in the context of linear and exponential functions.
Last but not least, literature about reasoning types and covariational reasoning are discussed. Hauger proposed three different types of reasoning about rate of change. And Carlson with her colleagues have conducted several researches about students’ covariational reasoning abilities. Basically they investigated how students coordinate simultaneous changes of two variables in dynamic events.

**Rate of Change**

Concept of rate of change has been informed by many research studies (Carlson, Larsen, & Jacobs, 2001; Saldhana & Thompson, 1998; Noble, Nemirovsky, Wright, & Tierney, 2001; Hauger, 1995; Hauger, 1997) as one of the fundamental concepts and a gate keeper to access advance mathematical concepts such as limit, derivative, and definite integrals. Noh (2004) stated that conceptual understanding of the concept of rate of change is especially essential for the study of calculus and physics.

Orton (1983) emphasized the importance of the conceptual understanding of rate of change in the study of calculus. He has investigated the students’ understanding of integration in calculus. In this investigation he observed that students’ difficulties in understanding the calculus concept like concept of integration stem from their difficulties in understanding the earlier concepts such as concept of rate of change. According to Orton, students’ weak conception of rate of change stem from their poor understanding of ratio, proportion, interpreting graphical information involving curves. Then he suggested that these concepts should be considered in earlier grades long before introducing the calculus concepts. He states:

> We must certainly take every opportunity to lay foundations of ideas of rate of change throughout a pupil’s school life and, as with limits, not leave the study of this important idea until it is required in order to make sense of differentiation. . . . no opportunity should be lost by teachers to develop these ideas, and that it is wrong to attempt to introduce calculus without a long and persistent study of graphs and rate of change. The same applies to ideas of limit. (page 243)

According to Thompson (1994a), concept of rate of change is fundamental for college students to better understand the *Fundamental Theorem of Calculus* which is “the realization that the accumulation of a quantity and rate of change of its accumulation are tightly related” (Thompson 1994a, p. 8). In the same study Thompson also refers the Swokowski’s (1991) explanation of Fundamental Theorem of Calculus as:
Suppose \( f \) is continuous on a closed interval \([a,b]\).

**Part I.** If the function \( G \) is defined by
\[
G(x) = \int_a^x f(t)\,dt
\]
for every \( x \) in \([a,b]\) then \( G \) is an anti-derivative of \( f \) on \([a,b]\).

**Part II.** If \( F \) is any anti-derivative of \( f \) on \([a,b]\), then
\[
\int_a^b f(x)\,dx = F(b) - F(a)
\]

Then Thompson (1994a) gave couple examples in this study to clarify how concept of rate of change is important to understand Fundamental Theorem of Calculus. As a first example he stated:

In a changing, multiplicative quantity, the total accumulation changes at the rate of accruals of the constitutive quantities. For example, suppose you have driven a car for \( x \) miles, and that in the next 0.0001 seconds you average 93 km/hr. During that 0.0001 seconds, your total driving distance is changing at the rate of 93 km/hr- regardless of how far you have driven. If we imagine that during each infinitesimal period of time you drove at some average speed and if we could know each of those average speeds, we could reconstruct your total driving distance at each infinitesimal moment of time. Thus, if we were to have an analytic expression which gave us your speed during each infinitesimal period of time, we could, in principle, recover your distance function. The problem is now one of technique- construct an analytic function whose rate of change differs at most infinitesimal from the rates of change we know you had. This method is not unique to speed and distance, but will apply to any quantity constructed multiplicatively from a rate and another quantity. (p.10)

In the same study, Thompson investigated undergraduate and graduate students’ understanding of rate. He addressed the students’ difficulties in interpreting the average rate of change functions. First, he gave a function \( X(t) = v(t + .1) - v(t) / .1 \) where \( v(t) \) is the volume in cubic meters of a cooling object \( t \) hours after removing a heat source. And he asked students to state what information \( x(t) \) gives about this object. Students’ responses indicated that only 4 students out of 19 have referred to an average rate of change of volume. Even most of the students could not give an appropriate unit for \( x(t) \). Only 7 students out of 19 stated the unit as “cubic meters per hour”. Similarly Thompson has observed that students have difficulties when interpreting the function \( r(x) = d(x + .1) - d(x) / .1 \), where \( d(x) = 16x^2 \) and representing the
distance an object falls \( t \) seconds after being released. Most students interpreted \( r(x) \) as amount of the speed as object fell during some tenth of a second. According to Thompson these difficulties are related to their images of functions and they have weak schemes in concept of rate of change and average rate of change, in particular.

**Three Perspectives to Rate of Change**

Garnet Hauger has conducted several research studies (1994, 1995, and 1999) investigating students understanding of concept of rate of change. In these studies he examined students understanding of the concept of rate of change in three categories: a) students’ interpretations of the overall shape of a graph, b) students’ understanding of average rate of change and c) students’ understanding of instantaneous rate of change. He was also interested in how students construct knowledge of instantaneous rate of change by using numerical and graphical approaches involving average rate of change.

To understand students’ interpretation of overall shape of a graph, he presented a graph of the number of yeast cells in a culture changing with time. Students were asked to interpret situation shown in that graph. He observed that while some students tried to divide graph into three approximately linear parts, other students divided the graph into two parts at the point of inflection. Students who divided the graph into three parts did not seem to have an understanding of concavity or inflection point. However students who divided the graph into two parts at the inflection point gave more sophisticated answers by mentioning the change of the growth rate. Hauger also noticed that some students correctly used the terms “inflection point,” “concave up,” and “concave down” to describe features of the curve.

To understand the students’ conceptions of average rate of change, he then asked students to identify the periods in which number of yeast cells was changing rapidly and periods in which number of yeast cells was changing slowly. After examining the students’ responses, he stated that students have given different explanations about slow and rapid growth. Some students divided the graph into equal consecutive time periods and compared the changes in these periods. Other students used the steepness of slope of the graph to explain slow or rapid change. For example, some stated that if the graph had steep slope, population was changing rapidly and if the graph had a shallower slope population was changing slowly.
To understand students’ understanding of instantaneous rate of change, students were asked to determine the rate of change of the yeast population at a specific point of time. He noticed that most students used the average change over an interval to estimate the rate of change at the point. Hauger stated that although students did not correctly find the rate of change at that point, the process they used is useful for understanding the idea of finding the limit of the slopes of secant lines. By this way students can understand the concept of derivative. Some students, on the other hand, attempted to draw a tangent line to the curve at the point and used the slope of that line to determine the rate of growth.

Hauger (1999) was also interested in developing students understanding of concept of instantaneous rate of change by using numerical approach involving the concept of average rate of change. In the same study mentioned above, he gave students a table of values corresponding to points on the graph and asked them to determine the rate of change in a specific point. He observed that students who used average change over an interval in graphical representation have given more accurate estimates by using the values in table and applying the same method. However, students who drew a tangent line to find the instantaneous rate of change in graphical representation were confused about what to do because they couldn’t draw tangent lines on the table, according to Hauger (1999).

**Images of Rate and Images of Covariation**

Thompson (1994a) mentioned the relationships between accumulation of quantities and accruals which are constructed by accumulation of quantities. According to Thompson, development of mature images of rate involves schematic coordination of these relationships mentioned above. To clarify what is meant by schematic coordination of the relationship between accumulation of quantities and accruals, Thompson gave a constant speed example shown in the figure below. He stated:

The total distance traveled in relation to the duration of the trip can be imagined as each having accumulated through accruals of distance and accruals of time so that at any moment during the trip the total distance traveled at that moment in relation to the total time of the trip is the same as the accrual of distance in relation to the accrual of time.
Figure 2.1 Speed as a rate.

Thompson (1994a) then explained the stages of the development of images of rate as follows:

- Image of change in some quantity (e.g., displacement of position, increase in volume),
- A loosely coordinated images of two quantities (e.g., displacement of position and duration of displacement),
- An image of the covariation of two quantities so that their measures remain in constant ratio. (p.5)

According to Thompson, constructing an image of the covariation of two quantities is the last stage of the development of the mature image of rate. Moreover, Saldanha and Thompson (1998) stated that images of covariation are also developmental and this development is defined in following stages:

- Coordination of two quantities’ values – think of one, then the other, then the first, then the second and so on.
- Construction of an operative image of covariation in which a person imagines both quantities having been tracked for some duration. (p.2)

Saldanha and Thompson (1998) investigated the images of covariation of an 8th grade student by engaging him in a teaching experiment. In this study researchers employed a simulation of a car movement activity by utilizing Geometer’s Sketchpad. By dragging a point with a computer mouse, an 8th grade student displayed the cars distances from two cities represented by points A and B. He had also an option to display distances as bars perpendicular to each other as shown in the figure 2.2.
He was asked to describe each distances behavior in relation to the car’s position along the road. He first observed the behavior of car’s distance from A. Researchers noted that when he moves the C (car) along the road he just focused on the change in the height of vertical bar which represents the distance between C (car) and City A. After researchers asked him whether the distance between two points changed faster in some places other than others, he focused on the changes in the AC with respect to the changes in the position of C. Then he observed that AC changes differently in different places with the same changes in C’s position. Although he was able to coordinate simultaneous changes of bar’s height and C’s position, it is reported that this is not coordinating changes in AC with changes in C’s distance from its starting point (Saldanha & Thompson, 1998). Then he observed the behavior of BC, which represents the distance from C (car) and the city B, likewise. Finally when he was asked to describe behavior of AC and BC together, he stated that AC and BC are both decreasing when the car getting closer to the cities and at one point bar representing the BC pauses although AC is still decreasing. He understood that it is the closest point to city B and same thing happened to AC after some time.

In the second phase of this study, Saldanha and Thompson (1998) presented different graphs, such as a graph shown in figure 2.3, plotting AC versus BC. Then he was asked to explore and predict possible locations of the two cities relative to the road so that the car’s movement produces that graph.
For the graph shown above an 8th grade student Shawn stated that extremes of the graph shows the closest and farthest points to A and the closest point to B must be in the middle of these points. After analyzing these responses, researchers reported that he could intricately coordinate images of two quantities varying individually and simultaneously.

Saldanha and Thompson (1998) also referred the unpublished study of Coulombe and Brenson describing the development of the construction of covariation. Saldanha and Thompson (1998) stated that Coulombe and Brenson suggested following properties to be involved in the concept of covariation.

- The identification of two data sets
- The coordination of two data patterns to form associations between increasing, decreasing, and constant patterns
- The linking of two patterns to establish specific connections between data values
- The generalization of the link to predict unknown data values.

### Relationship Between Function and Its Derivative

Researchers in Technical Education Research Center (TERC) have conducted several studies (Nemirowski & Rubin, 1991, 1992; Rubin & Nemirowsky, 1991; Monk & Nemirowsky, 1992; Nemirowsky, 1994) concerned with the high school students’ knowledge of rate of change. In these studies students’ concepts of the relationship between a function and its derivative were investigated in graphical context. Students worked with a device which simulates a car moving on a track and also generates velocity and position graphs. Starting point of the
track was assigned a position value zero. When the car was moving away from the starting point, position value was increasing until other end point and when the car was moving from the end point towards the starting point position value was decreasing. But in all cases position value was always positive. On the other hand, velocity of the car was considered positive when the car was moving away from starting point and velocity was negative when the car was moving towards the starting point.

Students were presented a velocity graph and asked to predict position graph. Students’ responses were revealed that they have difficulties in making connections between a function and its derivative in graphical context. For example they mostly assumed that graphs of a function and its derivative must closely resemble. In other words they assumed that two graphs had same shape for example, if the velocity graph was line then their predicted position graphs were also line or if they have presented a concave up graph for velocity they predicted that position graph was also concave up. They expressed that if the graph of velocity increased then graph of position increased too.

They based their work on three assumptions:

1. “Every normal human being, from early stages in life, has some intuitive knowledge about the relationship between function and derivative. . . . [W]e constructs complex bodies of knowledge that enable us to make sense of situations involving change.”

2. “The relationship between function and derivative is one of those notions that always remain open to further elaboration, with new and unresolved issues involving the fundamental nature of space, time, and number.”

3. “Students’ performance in solving problems involving the function/derivative relationship is strongly affected by contextual parameters.” (P. 3-4)

Considering students’ difficulties in relating the functions and their derivatives, Nemirowsky and Rubin claimed that covariational approach to functions needed to be addressed to overcome the difficulties stem from resemblance approach. According to authors, resemblance approach assumes that the graph of a function and graph of its derivative resemble each other and this approach makes students think that function and its derivative have same shape and properties, in other words they assume that they are alike. On the other hand variation approach
deals with, how changes in the one variable (position) over an interval of time results in change in other variable (velocity) in the same interval of time.

According to Nemirowsky and Rubin (1992), one of the important reasons that students believed that velocity and position graphs are alike is that students every day experiences with velocity and position or speed and distance. Similarly other research studies have also revealed that students’ intuitive knowledge and past experiences have effect on their understanding the concept of rate of change.

Confrey and Smith (1995), for example, claimed that they have witnessed students’ intuitions about rate of change and they stated that this primitive understanding of rate of change stem from their experiences with physical world. Beziudenhout (1998) on the other hand, discussed negative effects of these primitive understanding or intuitions on understanding the concept of rate of change. She investigated undergraduate students’ understanding of the concept of rate of change and found that many students demonstrated an inadequate intuition about the concept of rate of change. She claimed that students generally described the notion of ‘average rate of change’ as same with arithmetic mean. She also noticed that students in her study had difficulties in understanding the differences between instantaneous rate of change and average rate of change.

According to Hauger (1998), students usually have some experiences with rate of change before taking calculus. However their experiences in middle and high school are limited to constant or uniform rate of change. In addition these constant and uniform rate of change largely represented in symbolic way such as \(d = r \times t\). According to Hauger, studying only uniform or constant rate may lead students to believe that all rates are uniform and they will not have a concept of variable rate.

**Static vs. Dynamic Conceptions of Functions**

Functions represent relationships between varying quantities and as Tall (1997) stated “one purpose of function is to represent how things change” (p.1). Studies have revealed that students’ underlying function conceptions are playing important role in imagining the simultaneous covariation of variables to be able to reason dynamically. It has been consistently presented by many research studies that students who posses strong procedural skills such as symbol manipulations and weak conceptual structures are unable to construct images of
simultaneous covariation of two quantities in a functional relation (Carlson & Oehrtman, 2005; Monk & Nemirovsky, 1994).

Research studies (Carlson & Oehrtman, 2005; Monk, 1992; Confrey & Smith, 1995; Thompson, 1994b) revealed that dynamic conceptualizing of functions is essential to be able to coordinate changes in two variables simultaneously. Monk (1992), for example, showed that students who had point-wise conceptions of functions, which is a static conception, had difficulties in describing patterns of change in the value of a function that results from a pattern of change in the values of the input variables. Students who had point-wise conceptions had tendency to calculate value of a function for every input values by making precise measurement in a given situation. On the other hand, students who had across-time conception, which is a dynamic conception, were able to describe the simultaneous changes of dependent and independent variables.

Moreover, Carlson and Oehrtman (2005) pointed out that a student with a process view of function can imagine the whole situation as it is happening once rather than calculating every single pairs of input and output values. Thompson (1994b) has also supported that idea by stating that process view of function allows one to imagine an expression evaluating itself rapidly.

According to Confrey and Smith (1994), correspondence approach is concentrated around the application of certain rules and formulas to describe how to obtain the output value from a given input value. They criticized this approach because it is static. They stated that covariation approach is needed for coordinating movement (changes) between input and output values.

**Changing nature of functions.** Sierpinska (1992) emphasizes the changing nature of function concept by defining the functions as *world of relationships* or *world of processes* and she describes the variables X and Y as *world of changes* or *changing objects*. According to her, “these relationships or processes have to be well defined and this refers to the world of rules, patterns, laws” (p.31).

After defining a function as relationship between changing magnitude of x and changing magnitude of y, she stated that first condition of understanding function concept is to be aware of changes and relationships between them. Therefore she proposed following two acts of understanding functions as first and most fundamental acts.
• Identification of changes observed in the surrounding world as a practical problem to solve.
• Identification of regularities in relationships between changes as a way to deal with the changes.

Sierpinska (1992) suggested that students should be given opportunities to explore and identify the phenomena in their everyday life by using knowledge about functions. She pointed out the importance of modeling situations in real life or science. Functions, according to her, may appear as models of certain relationships that they observe in their social and economic lives. She also emphasized the importance of understanding the concept of variable by stating that realization and understanding the subject of change in a functional relationship is important. According to her, students have difficulties in identifying what is changing or changing objects.

**Action vs. Process view of functions.** According to Carlson & Oehrtman (2005) possessing a strong process view of function is crucial and students must move from action view of functions to process view of functions. They stated that a process view of function does not necessarily mean a success in all functional reasoning, however, it is essential in order to coordinate changes in dynamic functional relationship where input and output values change in tandem. More specifically they explained why having process view of function is necessary in developing covariational reasoning abilities and why having action view of function is not sufficient in this matter as follows

…reasoning dynamically is difficult because it requires one to be able to disregard specific computations and to be able to imagine running through several input-output pairs simultaneously. This ability is not possible with an action view in which each individual computation must be explicitly or mentally performed. …input and output are not conceived except as a result of values considered one at a time, so the student cannot reason about a function acting on entire intervals. A student with a process view can conceive of the entire process as happening at once, and is able to conceptually run through a continuum of input values while attending to the resulting impact on output values. This is precisely the ability required for covariational reasoning. (p.8)

And it is has been also shown that process view of functions is important for understanding the basic calculus concepts. Furthermore, Asiala, Cottrill, Dubinsky and Schwingendorf (2001) stated that calculus instructors assume that students coming to study this
subject will bring with them strong process conception of function. As an example of how a process view foster the understanding of limit, Carlson & Oehrtman (2005) stated following:

In order to understand the definition of a limit, a student must coordinate an entire interval of output values, imagine reversing the function process and determine the corresponding region of input values. The action of a function on these values must be considered simultaneously since another process (one of reducing the size of the neighborhood in the range) must be applied while coordinating the results. (p.9)

According to Dubinsky and Harel (1992), action view of function involves the ability to plug numbers into algebraic expressions and obtain corresponding output for these numbers. This view of function is static and it does not allow one to imagine whole evaluation process as it happens once. In other words, students who have action view of function may not be able to think several input and output pairs simultaneously. Rather, they think about the process one step at a time. On the other hand, they stated that process view of function is dynamic and involves dynamic transformations of quantities. Students who have process view are able to think about the whole process as it happens at ones and are able to imagine several input and output pairs simultaneously.

Thompson (1994b) defined the process view of function as building an image of “self evaluating” expressions. Then he claimed that process view of function enable students to imagine whole process without evaluating the expression for exactly every input in order to think of the results. Then he stated “A process conception of function opens the door to a wealth of imagery. Student can begin to imagine running through a continuum of numbers and letting an expression evaluate itself (very rapidly) at each number” (p.27).

Breidenbach et al (1992), Dubinsky (1991) and Dubinsky & Harel (1992) have emphasized the necessity of development of students’ process conception of functions. They claimed that existence and developments of process conception are indications of conceptual understanding beyond mere manipulations of variables or procedural skills. They also pointed out the necessity for students to move from action view of function concept which is a static conception involving ability to plug numbers into algebraic expressions and calculate, to process view of function concept which is a dynamic conception involving ability to transform object by some actions to obtain new object, to build the conceptual understanding.
Considering the studies of Breidenbach et al (1992), Dubinsky (1991) and Dubinsky & Harel (1992), Carlson and Oehrtman (2005) also stated that process views of function is important and necessary to represent and interpret dynamic functional situations. According to authors, this reasoning about dynamic functional situations is difficult because:

…it requires one to be able to disregard specific computations and to be able to imagine running through several input-output pairs simultaneously. This ability is not possible with an action view in which each individual computation must be explicitly or mentally performed. (p.7)

Carlson and Oehrtman (2005) expressed the general characteristics of action and process views of functions as follows:

- In action view, a function is tied to a specific rule, formula, or computation and requires the completion of specific computations and/or steps. In process view, a function is a generalized input-output process that defines a mapping of a set of input values to a set of output values.
- In action view, one must perform or imagine each action. In process view, one can imagine the entire process without having to perform each action.
- In action view, answer is dependent on the formula. In process view, process is independent of the formula.
- In action view, one can only imagine a single value at a time as input or output. In process view, one can imagine all input and output pairs at once or run through a continuum of inputs. A function is transformation of entire spaces.
- In action view, composition is substituting a formula expression for x. In process view, composition is a coordination of two input-output processes.
- In action view, inverse is switching x and y and solve for y or reflecting the graph with respect to the origin (y=x). In process view, inverse is the reversal of a process that defines a mapping from a set of output values to a set of input values.
- In action view, domain and range are conceived at most as an algebra problem (denominator cannot be zero and radicand cannot be negative). In process view, domain and range are produced by reflecting on a set of all possible inputs and outputs.
- In action view, functions are conceived as static. In process view, functions are conceived dynamic.
In action view, a function’s graph is a geometric figure. In process view, a function’s graph defines a specific mapping of a set of input values to a set of output values.

**Correspondence vs. Covariation approach to functions.** Confrey and Smith (1994) claimed that covariation approach to functions is necessary for better conceptualizing the notion of change. While focusing on understanding students’ concept of rate of change in their study, they stated that covariation approach is essential for coordinating *movement* (changes) between input and output values. According to authors, correspondence approach in the current curriculum is formula dependent. Confrey and Smith criticized this approach because it focuses on rules and formulas to describe how to obtain the output value from a given input value. Thus the authors claim that the covariation approach makes the concept of rate of change more visible to students. Then, they gave an example to describe a student using the covariation approach as follows:

Students working in a problem situation first fill down a table column with x-values, typically by adding 1, fill down a y column through an operation they construct within the problem context. Such an approach has the benefit of emphasizing rate of change.

Confrey and Smith (1994, 1995) investigated high school students’ ideas about rate of change in the context of linear and exponential functions. Students were given a table of values expressing the exponential growth of a given situation. More specifically, situation was the growth of the cell population over nine hours. Table values demonstrated the number of cell population for every hour from t = 0 to t = 9. Then students were asked to predict the number of cells at t = 10. They discussed the two ways of thinking about functions that shape the students’ thinking about rate of change.

In this prediction task, some students tried to relate y values by calculating the differences between consecutive values of y (Y1, Y2-Y1) for every hour (same increments in x values). Confrey and Smith called this additive rate of change. In linear function situations, students were able to predict the number of cells in t = 10 by realizing that the differences between consecutive y values were same. For exponential functions, students realized that the difference between consecutive Y values did not yield the same constant. After working on the table of values, some students discovered that the ratio of consecutive Y values (Y2 / Y1, Y3 / Y2) was same. Confrey and Smith called this multiplicative rate of change.
Authors stated that these studies showed evidence of students thinking of functions. According to Confrey and Smith, most students find it reasonable to study functions in terms of change in the dependent variable for fixed changes in the independent variable rather than as a relationship between the dependent variable and independent variable. They claimed that this is the evidence that students think about functions as covariation rather than correspondence.

Importance of covariation approach in learning functions has also been emphasized by Thopmson (1994b). He claimed that concept of function must be introduced as covariation first in K-14 mathematics curriculum. He suggested that a reflection of historical development of function concept as a model to teaching functions can provide students greater chance to construct strong conceptual background.

**Point-wise vs. Across-time approach to functions.** Monk (1992a, 1992b) has conducted two research studies to investigate students’ conceptions of functions having to do with their understanding of related rates. Monk categorized students’ conceptions as pointwise and across-time considering their responses to the problems given by physical models. According to Monk, as an alternative representational system, hands-on physical models are useful to represent particular functions if these models are carefully chosen.

In the first study, students were asked to relate changes in the length of the person’s shadow to changes in his distance from the lamp, when a six foot man is walking away from 16 foot street lamp. Monk then reported that some of the students used their intuitions for responding the question. For example, one of the students stated that since the tip of the shadow is farther away from the lamp than the person, tip of the shadow moves faster. Monk’s observations revealed that some students believe that greater distance means greater speed or vice versa. Some students stated that since the tip of the shadow getting longer, tip of the shadow is moving at an increasing rate.

Monk observed that students had difficulties in using the comparison methods to relate changes in the length of the person’s shadow to changes in his distance from the lamp. For example, Monk stated that some students confused about “within comparisons” and “between comparisons”. According to Monk, comparisons of increments in same quantity are “within comparison” and comparisons of increments of one quantity to increments of another quantity are “between comparisons”. He stated that some students use wrong comparison method to make conclusions. For example, some students measured the increments of the length of the shadow
for one foot increments in the man’s distance from the street lamp. One of the student found that
increments in the length of the shadow equal to each other but they are not same with the
increments in the man’s distance from the street lamp. Then this student concluded that since
these increments were not same, shadow could not move at a constant rate. Monk stated that this
student compared the increments of one quantity to increments of another quantity to make
decision about rate of change in one quantity. According to Monk this is a between comparison
and she should have used within comparisons to be able to determine the rate of change in the
length of the shadow. On the other hand, some students just measured the total length of the
shadow and total length of the man’s distance from lamp and concluded that since the length of
the shadow is greater than the man’s distance from lamp, tip of the shadow is moving faster than
man. Monk stated that this reasoning is not suitable for making this kind of decision. According
to him, they should have make comparisons of increments of each of the quantities.

In the second study, Monk posed a three-part problem involving a ladder in almost
vertical position against the wall. In the first two part of the question, bottom of the ladder have
been moved along the horizontal line by same amounts of increments and student were asked to
decide if the top of the ladder drops down by more or less or same amounts of increments. In the
third part, bottom of the ladder have been moved along the horizontal line with a constant speed
and students were asked if the top of the ladder drops down at an increasing or decreasing or
constant rate. And students were also asked to graph relating the changes in the disposition of the
bottom of the ladder to changes in the disposition of the top of the ladder.

Monk observed that students attempted to collect x and y values and by making precise
measurements of the increments of both bottom and top of the ladder. Then most students tried
to answer first two part of the question by looking at the data. According to Monk this approach
shows the “Pointwise” view of a function. This view of a function resembles with what Confrey
and Smith (1995) called correspondence view of a function.

As mentioned before, focus of this view is on relating the values of independent
variables to corresponding values of dependent variables. He also stated that evidence from the
student use of the model indicated that many students have difficulties in dealing with what he
called “Across-time” questions because of their “Pointwise” view of function. In part three, some
students stated that since there is a single object (ladder) in the model, if the bottom moves with
a constant speed, top should drop down with a constant speed too because they believed that the
system was fixed. Some students stated that sum of the two lengths (position of the bottom and top of the ladder) is always same so if one moves at a constant rate other should move at a constant rate too. According to Monk these students used what he called “student-generated principles, global rules” (p.186).

**Reasoning About Change**

**Three types of reasoning.** Hauger (1998) mentioned about three types of reasoning that students use when engaged in different tasks. First is the qualitative type of reasoning which involves making judgments and constructing graphs of given situation and interpreting the important futures (increasing, decreasing, concavity, inflection points) in the shape of graphs to make conclusions about the patterns of change. Second type of reasoning is numerical reasoning in which students deal with numerical values by making precise measurements and calculations in order to make decisions about change. For example in Monk’s study students made precise measurements of increments in position of the top of the ladder as it drops down over unit changes in the position of bottom end of the ladder. Third reasoning type according to Hauger (1998) is algebraic reasoning in which students make judgments about patterns of change by using formulas or equations.

**Covariational reasoning.** Covariational reasoning is defined by Carlson, Jacobs, Larsen, and Hsu (2002), as “cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (p.354). Another definition comes from Saldanha & Thompson (1998). According to these researchers, it is “…holding in mind a sustained image of two quantities’ values (magnitudes) simultaneously” (p.298). It has been shown that these reasoning abilities are necessary for analyzing, interpreting and representing the patterns of change in continuously changing dynamic events. (Carlson et al, 2001; Monk, 1992).

Carlson (1998) conducted a cross-sectional research study to investigate the college students function conceptions from different perspectives. In this study Carlson (1998) investigates the students’ abilities to:

- Characterize real world functional relationships using function notations;
- Operate with a particular type of function representation, such as formula, a table, or a graph;
• Interpret and describe local and global function properties: slope, continuity, and differentiability;
• Construct functions using formulas and other functions;
• Recognize functions, non-functions, and function types;
• Conceptualize a function both as a process and as an object;
• Interpret and understand the language of functions;
• Characterize the relationship between a function and an equation.

Besides investigation of students’ understanding of various function aspects, she also attempted to clarify students’ understanding of covariant aspects in this study. She tried to better understand students’ abilities to:

• Represent and interpret covariant aspect of the function situation (i.e. recognize and characterize how change in variable affects change in another);
• Interpret static and dynamic functional information (i.e. interpret graphs representing position and rate of change).

This investigation was one of the fundamental studies for examining and describing college students’ covariational reasoning abilities. Carlson designed 18-item written exam to measure the major aspects of the function concept including students’ abilities to represent and interpret covariant aspect of functional situation. Then she conducted follow-up interviews with 15 students to better understand the students’ conceptions. Findings of the study reveal that even high performing college algebra students have difficulties representing and interpreting the covariant aspect of function situation. She stated that although high performing second semester calculus students have broader views on functions than college algebra students, they were also mostly unable to represent and interpret the covariant aspect of a real world functional situations. Beginning graduate students, on the other hand, showed much greater tendency to represent and interpret the covariant aspects of functional situations, according to Carlson (1998).

As an example she summarized and demonstrated the result of students’ responses to the “bottle filling with water” problem. In this problem, students were asked to sketch a graph of the height as a function of the amount of water that is in the bottle shown in figure 2.4, while imagining the bottle filling with water.
Written exam results demonstrated that more than half of the college algebra students drew a straight line to show the relationship. Carlson later explained that these students have a loosely coordinated image of how the variables changing with respect to each other. Although these students were able to coordinate the direction of change they did not have mature image of coordination of changes in variables. On the other hand almost 25% of the college algebra students drew a strictly concave up graph. This could be evidence that these students have constructed more accurate images than those drew only straight line but they still could not recognize the inflection point where graph is turning from concave up to concave down.

In the contrary, most of the beginning graduate students drew absolutely correct graph for the situation. In the follow up interview sessions with students, one who drew the correct graph was asked to explain how he constructed the graph. Student stated that bottom and top part are different and in the bottom part change could not be explained by a straight line because this part is circular but top part has straight walls. And he suggested that thinking about putting the same amount of water every time and imagining the changes in the height helped him to draw correct graph. And he also mentioned that in the last part of the graph, slope of the straight line is about the slope of the last point of the curve.

The most striking finding in this study is that only around 15% of the second semester calculus students drew the correct graph of the situation. And Carlson stated that development of this ability to represent and interpret covariant aspect of a dynamic function is very slow and even students in mathematics major have fully develop this ability in the end of the undergraduate program or in the beginning of their graduate program.

Based on the findings of this study, a framework that describes the students’ mental actions and corresponding behaviors has been produced by the author. This framework consists of following five mental actions:

- An image of two variables changing simultaneously. (Coordinating the value of one variable with changes in the other.)
• A loosely coordinated image of how the variables are changing with respect to each other. (Coordinating the direction of change of one variable with changes in the other)
• An image of amount of change of the output variable while considering changes in fixed amounts of the function’s domain. (Coordinating the amount of change of one variable with changes in the other)
• An image of rate/slope for contiguous intervals of the function’s domain. (Coordinating the average rate of change of function with uniform increments of change in the input variable.)
• An image of the continuously changing rate over the entire domain. (Coordinating the instantaneous rate of change of function with continuous changes in the independent variable for the entire domain of the function.)

Using this framework Carlson et al (2002) conducted another study with 20 2nd semester high performing calculus students to understand how these students represent and interpret continuously changing dynamic functional relationship. She observed that students have little difficulty in constructing images of functions’ dependent variable changing in tandem with the imagined change in independent variable. But she also observed that even high performing calculus students had difficulties in constructing images of continuously changing rate. Therefore they have difficulties in interpreting images of increasing and decreasing rate and representing the inflection points. She concluded that calculus students:

• Were able to coordinate the value of one variable with changes in the other.
• Were able to coordinate the direction of change of one variable with changes in the other variable.
• Were able to coordinate the amount of change of one variable with changes in the other variable
• Were unable to consistently coordinate changes in the average rate of change with fixed changes in the independent variable for a function’s domain.
• Were not consistently able to coordinate the instantaneous rate of change with continuous changes in the independent variable.
• Had difficulties in explaining why a curve is smooth and what is conveyed by an inflection point on a graph.
CHAPTER III
MATERIALS AND METHOD

Overview

Based on the research questions, an aim of this study is to explore, describe and analyze college students’ covariational reasoning abilities. This chapter presents the methodology that was used to conduct this study considering proposed research questions in the light of the theoretical lens. Specific research questions for this study are:

- What thinking and reasoning processes do college students’ use as they attempt to coordinate simultaneous changes of variables in continuously changing functional situations?
- How do college students interpret simultaneous changes of two variables in functional situations?
- How do college students use their interpretations to represent covariation of two variables in graphical context?
- How do college students interpret covariation of two variables in given graphs of functional events?

This study was conducted with college students who were enrolled in a third semester Calculus course offered at Florida State University. Primarily, case study design and techniques were used in this study to provide a thick description about students’ thinking and reasoning processes. Data were collected through task-based clinical interviews to elicit in-depth information about 1) how calculus students interpret the information in dynamic functional events; 2) how they use this information to represent simultaneous changes of variables in graphical context; and 3) how they interpret graphical representations of dynamic functional relationships. Cases for this study were two third semester calculus students. Since the research involves two cases, Creswell’s (1998) format for the analysis of data for multiple cases was utilized. Creswell (1998) suggested within-case analysis and cross-case analyses for multiple case studies.

Methodology

Constructivist theory of learning has increasingly influenced mathematics education research. According to Paul Ernest (1998), widespread acceptance of constructivism brought a shift toward qualitative methods especially in the field of mathematics education. Since
constructivist methodology is concerned with a deeper understanding of an individual’s reasoning, perspectives and purposes, case study investigation was used to examine students’ insights more carefully and to provide detailed information and thick description of the phenomenon for the reader. Stake (1995), emphasized the importance of constructivist perspective in case study design:

Case study research shares the burden of clarifying descriptions and sophisticating interpretations. Constructivist view encourages providing readers with good raw material for their own generalizing. The emphasis is on description of things that readers ordinarily pay attention to, particularly places, events, and people, not only commonplace description, but ‘thick description’, the interpretation of the people most knowledgeable about the case. Constructivism helps a case study researcher justify lots of narrative description in the final report. (p.102)

Case study is defined by Stake (1995) as “the study of particularity and complexity of a single case, coming to understand its activity within important circumstances” (p. xi). Merriam (1989), on the other hand says “A qualitative case study is an intensive, holistic description and analysis of a single instance, phenomenon or a social unit” (p. 27). According to Yin (1994) “A case study is an empirical inquiry that investigates a contemporary phenomenon within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident.” (p.13). He then claimed that case studies are useful if the variables are so embedded in phenomenon being studied and it is hard to identify or separate phenomenon’s variables from their context. Merriam (1998) stated that one of the main reasons to employ case studies is because of a researcher’s interest in insight, discovery, and interpretation rather than testing hypothesis. According to Merriam (1998), qualitative case studies can be characterized as being particularistic, descriptive and heuristic. She describes these features as:

Particularistic means that case studies focus on particular situation, event, or phenomenon. …Descriptive means that the end product of case study is rich, thick description of the phenomenon under study. Thick description is a term from anthropology and means the complete, literal description of the incident or entity being investigated. … Heuristic means that case studies illuminate the reader’s understanding of the phenomenon under study. They can bring about the discovery of meaning, extend the reader’s experience or confirm what is known. (p. 29)
Uniqueness of case study research has been identified by several researchers in terms of knowledge learned from the study. For example, Merriam (1998) stated that uniqueness of case study research lies in the relationship between questions and end product. Stake (cited in Merriam, 1998) claims that case study knowledge differs from other research knowledge in the following four ways.

- More concrete: Case study knowledge resonates with our own experience because it is more vivid, concrete and sensory than abstract.
- More contextual: Our experiences are rooted in context as is knowledge in case studies. This knowledge is distinguishable from the abstract, formal knowledge derived from other research designs.
- More developed by reader interpretation: Readers bring to a case study their own experience and understanding, which lead to generalization when new data for the case are added to old data.
- Based more on reference populations determined by the reader: In generalizing as described above, readers have some population in mind. Thus unlike traditional research the reader participates in extending generalization to reference populations. (p.31)

Research Procedures

This research was conducted with college students who were enrolled in Calculus with Analytic Geometry – III course. With much of the content of this course providing a basis for further studies in mathematics, sciences (e.g., physics, chemistry) and engineering, it is required for undergraduate students in these different programs.

Calculus W/ Analytic Geometry III: Course content consists of functions with multiple variables and their graphical representations, vectors, partial derivatives, optimization, multiple integration, coordinate systems (polar, spherical, and cylindrical), curves, line integrals, flux integrals divergence theorem. Students must have taken the second semester calculus course (Calculus W/ Analytic Geometry II). Approximately 200 undergraduate students were enrolled in this course in fall term of 2005. There were 6 sections with different sizes. The instructors of all 6 sections were contacted and permission was asked to conduct the study in her/his class. From those who volunteered one class was selected.
**Participants.** Subjects participating in this study were drawn from high performing volunteer students. 7 high performing students were identified considering students grades on past quizzes and tests. Students whose average grades on tests and quizzes were “A” were considered high performing in the selection process. The course instructor provided me a list representing students’ average grades in our meeting that took place 2 days after contacting with the instructor. From these high performing students four were chosen to participate in this study based on their majors (one mathematics major, one physics major, one chemistry major and one engineering major). My intention was to select a cross-section of students who were studying in different academic disciplines and had varying experience with calculus concepts. Among these four students two students withdrew in the middle of data collection process. After identifying the subjects, students were contacted and asked to join clinical interviews. The consent letters were presented to students with brief description of the study. Each consent letter was obtained with subject’s agreement signature on it at the very first interview with each subject.

**Data Collection**

**Thinking and reasoning interviews.** To obtain a detailed understanding of college students’ covariational reasoning abilities and their strategies, and thinking skills data were collected through in-depth clinical interviews with the selected students while they were working in a problem situation. Thinking and reasoning interviews focused on exploring college students’ thinking and reasoning processes they use as they attempt to represent and interpret covariant aspects of the real world dynamic function situation. Since clinical interviews were major sources in case studies, thinking and reasoning interviews provided most of the raw data in this study.

These interviews concentrated on the participants’ thinking and reasoning processes rather than right or wrong answers that they produced. Interviews were semi-structured and task based. In order to improve the quality of the data representing participants’ thinking and reasoning participants were always urged to use a think aloud process. Students were encouraged to verbalize everything while working on the tasks. Probing questions were used to encourage participants for sufficient verbalization of their thoughts.

Interviews in this study were semi-structured task based. Mainly, tasks and problems that participants engaged in were preplanned tasks and problems, as Goldin (2000) stated below, and
some heuristic questions were prepared by researcher to handle contingencies. The researcher strived to not pose questions or give hints that might lead the participant to the solution of the problem or task. From this perspective, the format of the interviews in this study is neither fully structured nor unstructured. In other words, interviews in this study have a semi-structured format that has taken into consideration Goldin’s description of structured interviews and his explanation about distinction between structured and unstructured interview. According to Goldin (2000):

Structured task based interviews for the study of mathematical behavior involve minimally a subject (problem solver) and an interviewer (a clinician), interacting in relation to one or more tasks (questions, problems, activities) introduced to the subject by clinician in a preplanned way. The latter component justifies the term task based, so that the subjects’ interactions are not merely with the interviewers, but with the task environments. (p.519)

And he also discussed the explicit provisions and the way structured interviews are different from unstructured interviews. He stated:

Normally provision is made for observing and recording for later analysis what takes place during the interview… Explicit provision is made too for contingencies that may occur as the interview proceeds, possibly by means of branching sequences of heuristic questions, hints, and related problems in sequence, retrospective questions, or other interventions by the clinician. It is this explicit provision for contingencies, together with the attention to the sequence and structures of the tasks that distinguishes the “structured” interviews from “unstructured” interviews. (p. 519)

All interviews were audio taped and transcribed to record students’ answers to the given questions. Students were informed that there was no right or wrong answer for the questions and their names were not mentioned anywhere in this study.

**Interview Content**

**Background questions**: Background questions were employed in the very first interview for gathering information about students’ prior experiences with mathematics such as high school and college level mathematics courses that they have taken before enrolling in the Calculus course, reasons for taking calculus, course grades, and their future educational goals
such as intended major and college level mathematics courses that they were planning to take. Some questions were also posed to find out about the students’ non-academic activities for the purpose of providing rich and informed descriptions of students.

**Interview tasks:** Interview tasks (Appendix A) consisted of a series of problems designed to explore students thinking and reasoning about simultaneous changes of two variables when they attempt to represent and interpret dynamic functional situations. Most of the tasks were adapted from the literature and others designed by the researcher. There were two types of tasks: a) representation tasks and b) interpretation tasks. Each task was presented in separate piece of paper. Each task was first verbally presented to subjects then subjects was given time to read the task by themselves. On each task subjects were urged to use a think aloud process.

In representation tasks students were given a dynamic functional situation and asked to represent this situation in a graphical context. In these tasks all situations were presented verbally and some of these with accompanying figures and shapes. Representing the given situation in graphical context meant constructing a graph of that situation. Rather than plotting individual points in a given scaled coordinate system, students were asked to produce qualitative graphs of concrete situations and asked to view them globally instead of point-wise which is explained by Monk (1992). In other words, participants were asked to attend to the entire graph as an expression of the relationship between two simultaneously changing variables and express that relationship in words rather than numbers. My rationale for probing the constructing of a qualitative graph of a given situation was to understand the participants’ thought process such as how they construct an image and make transformations on that image when they attempt to make a transition from given situation to graphical representation of that situation by coordinating two variables changing in tandem. I am not interested in each participant’s manipulative skills such as plotting points and connecting the plotted points.

In the interpretation task students were given a graphical representation of a functional event and asked to describe the overall behavior of the graph by interpreting important features in the shape of the graph. Participants were asked to interpret global features of a given graph such as general shape of the graph, pattern change (interpreting direction of change or changes from concave up to concave down or vise versa), intervals of increase or decrease and intervals of extreme increase and decrease (e.g. determining the part in which dependent variable increasing or decreasing most). My intention was to understand how participants of this study
have constructed meaning about the relationship between the two variables and their pattern of covariation by looking at the entire graph.

**Data Analysis**

This section presents the analysis of data which were collected through clinical interviews. Considering the proposed research questions, qualitative data analysis techniques were employed. Students’ words and their responses supporting their reasoning about a covariant aspect of real world dynamic function situations were examined. Coding procedures were used by the researcher after collecting the interview data. Coding task is explained by Merriam (1998) as comparing one unit of information with the next in looking for recurring regularities. According to Bogdan and Biklen (1998), collected data involve these regularities in the form of repeated actions, words and phrases. They stated that each of these pieces of data can be labeled with codes developed by the researcher and these codes can be categorized to organize the data.

While explaining the construction of categories, Merriam (1998) stated that although the most common way is that investigator comes up with categories that reflect what she or he sees in the data, categories also can be borrowed from sources outside the study. “Applying someone else’s schema requires that the categories be compatible with the purpose and theoretical framework of the study” (Merriam, 1998; p.183). In this study, I used both categories from an outside source and those that I constructed from the interview data. Categories borrowed from an outside source were constructed and defined by Carlson (1998). These categories of mental actions (shown below) are completely consistent with the research purpose and a part of the theoretical framework of this study.

**Table 3.1. Mental actions.**

<table>
<thead>
<tr>
<th>Mental Action</th>
<th>Description of Mental Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mental Action 1 (MA1)</td>
<td>Coordinating the value of one variable with changes in the other</td>
</tr>
<tr>
<td>Mental Action 2 (MA2)</td>
<td>Coordinating the direction of change of one variable with changes in the other</td>
</tr>
<tr>
<td>Mental Action 3 (MA3)</td>
<td>Coordinating the amount of change of one variable with changes in the other</td>
</tr>
<tr>
<td>Mental Action 4 (MA4)</td>
<td>Coordinating the average rate of change of function with uniform increments of change in the input variable.</td>
</tr>
<tr>
<td>Mental Action 5 (MA5)</td>
<td>Coordinating the instantaneous rate of change of function with continues changes in the independent variable for the entire domain of the function.</td>
</tr>
</tbody>
</table>
**Cross-case analysis.** After completing a comprehensive analysis of each case (within-case analysis) by applying the coding techniques listed above, cross-case analyses was employed to discover common themes in the two cases and to build general explanations that fit each of the individual cases. According to Miles and Huberman (cited in Merriam, 1998), one purpose of employing cross-case analysis is to see process and outcomes that occur across many cases and to develop more sophisticated descriptions and more powerful explanations.

**Quality Criteria**

Trustworthiness of the results is an important issue in all research studies. Ensuring the quality criteria in a research study is essential for trustworthiness of research results. Although validity and reliability are said to be two important aspects of positivist or in other words experimental or quantitative research paradigm, Guba and Lincoln (cited in Merriam, 1998) stated that the idea of validity and reliability is not too different in qualitative case studies. However qualitative and quantitative research paradigms employ different methods to show the validity and reliability of research results. Firestone (1987) for example, stated “the quantitative study must convince the reader that procedures have been followed faithfully because very little concrete description of what anyone does is provided. The qualitative study provides the reader with a depiction in enough detail to show that the author’s conclusion makes sense.” (p. 19)

**Credibility:** Specifically, it deals with how research findings are congruent with the reality. Reality is defined by Guba and Lincoln (cited in Merriam, 1998) as “a multiple set of mental constructions … made by humans; their constructions are on their minds, and they are accessible to humans who make them” (p.295). Because the primary resource for the data collection in qualitative research is human, Merriam (1998) stated “interpretation of reality is accessed directly through their observation and interviews. Guba and Lincoln (1989) suggested a criterion called “credibility” which has the same idea with internal validity which is mostly used in quantitative research paradigm. They also suggested several techniques to ensure meeting this criterion. In this study peer debriefing technique was used by discussing the findings with members of mathematics education program in order to ensure that research findings are harmonizing with the reality. Techniques of triangulation of data were also used to ensure the credibility criterion.
**Transferability:** From the constructivist perspective, it is a process for checking the degree of similarity between researcher’s and reader’s conceptualization (Guba & Lincoln, 1989). Guba and Lincoln suggest generating “thick description” of data and of time, place, context, and culture in which researcher starts developing an understanding to help the readers assess transferability based on their experience in familiar settings. In this study, thick description of data, time and context were provided to ensure the transferability of research findings.

**Dependability:** From qualitative research perspective, replications of the research study may not yield the same results or findings, because several interpretations of the same data can be made by different researchers. Guba and Lincoln (1989) suggest “dependability” and they argue that dependability relies on the thorough recording of the research process (how data was actually collected and analyzed) which could be tracked back to its source. From positivist research viewpoint, changes in the construction decreases the reliability of research study. But Guba and Lincoln (1989) claimed that these changes in construction are expected for reaching more sophisticated constructions as long as these changes are tracked and reported.

**Confirmability:** Guba and Lincoln (1989) defined confirmability as “assuring that data, interpretations, and outcomes of inquiries are rooted in context and personas apart from the researcher and are not simply figments of the evaluator’s (researcher’s) imagination” (p.243). Samples of recorded raw data (parts from transcripts of recorded audio data), were provided in this study to enable reader to determine if the conclusion and interpretations can be traced to their sources.
CHAPTER IV
DATA ANALYSES AND FINDINGS

Overview

Considering the purpose of the investigation and the research questions, this chapter discloses the analysis of data from a detailed examination of students’ thinking and reasoning processes as they attempt to coordinate simultaneous changes of variables in continuously changing dynamic functional situations. Data obtained from students’ verbal expressions and graphical representations were analyzed in light of the theoretical lens. Carlson et al.’s (2002) covariation framework provided a skeletal structure for the description and interpretation of findings in each case. Specifically, five mental actions defined in the framework were used to describe each student’s covariational reasoning.

Subjects participated in this study were two students, who were enrolled in Calculus with Analytic Geometry – III course in the fall semester of 2005. Each student was interviewed five times. Pseudo-names were used in order to ensure anonymity identities. This chapter is arranged into three sections. The first two sections provide an analysis of individual cases. In each of these two sections, a) a brief introduction of student, b) analysis of student’s interviews, and c) a summary of the case was presented. After analysis of individual cases, the chapter also presents a cross-case analysis in the third section where the findings for each student was compared considering the five mental actions. In addition, several assertions are made and reported in the cross-case analysis section.

CASE OF JAY
Introduction

Jay was from one of the South American country. He is an industrial engineering student and in his second year in college when this study took place. He stated that he went to high school in his country. Upon graduation from high school he came to the US to study engineering. Like other youngsters in his age in his country, studying in the United States has always been a dream for him. When asked why, he smiled and said: “I believe that the idea of studying in the United States excites most of the young people all around the world not only in my country”. He said that finding a good job and building a good career is not easy in his country. He talked about how competitive employment is in even average companies and government agencies. He believes that a degree from a known US college may open many doors and may help him to be ahead of some other people.

He stated that in the first year in high school he was not a good student and “…it was because lack of interest to most of the subjects being taught in school. Nothing in school interested me in first year high school. For example, mathematics at first to be honest did not mean anything to me. I did not pay attention to it. I got away with it unfortunately by copying and by last minute studying.” Then in his junior year he said he started to realize that he was not learning much. He also mentioned that this is the milestone in his education. Then he realized his hunger for learning and he tried to get away from copying, memorizing and last minute studying. When I asked how things changed suddenly, he smiled like he has been waiting to answer this question. And he started to talk about how his second year physics teacher has played an important role by encouraging him to study. He said: “Physics teacher, who was very good at mathematics too, was gathering students who want to learn more stuff after school hours. He said he was going to teach us more advance mathematics. One of my friends and I decided to go. He was explaining everything very clear. After this activity I felt like most of the stuff is a lot easier than I thought of. He was the one who changed my vision.”

Jay has stated that he took algebra, geometry, trigonometry, pre-calculus, and the brief introduction to calculus in his high school education. He mentioned that he did not know about the placement test before coming to US so he just started allover again by taking college algebra, pre-calculus and trigonometry. He stated that he has taken the SAT before coming to the US and he was surprised with the content of the test. When I asked why he was surprised, he stated that he did not think the test was going to be very easy “I was like it is really what is required for
college and I said OK”. He claimed that the level of high school education is higher in his country than it is in the US. He claimed that the educational system in his country is stricter and more demanding. He said:

For example, here in calculus course we are doing first five practice examples in every chapter as homework but in my country you are required to do every single question at the end of the chapter. That is why grades are little lower in my country [Peru] than in the US.

He mentioned that there were hard college entrance examinations to get into universities and academies. He said that everyone in high school wants to go to college and it is very competitive.

When I asked why he had chosen industrial engineering, he said that he was not really informed about industrial engineering and he wanted to study mathematics or science. Then he said that he decided to study mechanical engineering. Upon this decision his uncle who is a manager in a big company invited him to the company to observe how things work. He said that after spending almost a month in the company he has observed electrical engineers, mechanical engineers and industrial engineers then decided to pursue a degree in industrial engineering.

The next question I asked him was if he liked mathematics. He said “Yes, I love it”. When asked why, he said that he was enjoying working on the problems and trying to solve challenging problems. And also he stated that mathematics gives a mental discipline and logic that is why he likes the mathematics. Then I asked him if he could define a mathematical function. He said: “A function is the way of expressing the relationship between two or more things.”

In terms of his grades, he has been a successful student in his college education. He has taken calculus I and passed with a B+ grade and also taken the calculus two and passed it with grade A. And he was also expecting an A in calculus III.
Analysis of the Case

The section provides an analysis of Jay’s covariational reasoning based on his cognitive activities and mental actions that were expressed by behaviors in face-to-face task-based interviews. Several tasks were posed in order to determine mental actions accompanied by collection of common behaviors that were exhibited while responding to these tasks. In these tasks, the student was given functional situations and was asked to construct graphs representing the situations. The student was also engaged in an interpretation task. In this task, he was given a graphical representation of a dynamic functional event and asked to describe the overall behavior of the graph by interpreting important features in the shape of the graph.

The task shown in figure 4.1 required him to draw a graph that represents the distance between two people as a function of time.

Two people start at opposite corners of a room and walk toward each other. As they walk, they both slow down as they get closer to each other, pass, and then they both speed up as they get farther apart. Draw a graph showing the distance between two people as a function of time. Describe your graph.

Figure 4.1 Task-1

After reading the question he started to think and then he read the question again slowly. After a second reading he stayed silent for a while. Then he started to draw the first graph as shown below (figure 4.2). He labeled the y-axis as “D” and x-axis as “S” and drew a straight line graph. When he was asked why he labeled like that. He said:

I am trying to draw Distance vs. Speed graph … to see the situation clearer. After that I will draw the Distance vs. Time graph…I guess it is going to be easier to see the situation over here [pointing the Distance vs. Speed graph] to… graph the other one [Distance vs. Time graph]
Although the above graph he provided was not an accurate representation of simultaneous changes of distance and speed, his attempt in this point was important to investigate. This attempt first indicated that the situation did not automatically evoke an image of how the distance between two people changes over fixed intervals of time. That might be because as Leinhardt et al. (1990) stated, the time variable is taken into account implicitly. In fact, the variable speed involved in the task in explicit considering the situation, although the task involves coordinating the simultaneous changes of distance and time. In order to figure out his reasoning, I asked him to explain how the Distance vs. Speed graph helps him to construct the Distance vs. Time graph; he stayed silent again for a while and he answered:

I am not quite sure …but usually drawing a different graph helps you to understand the situation…from my experiences we do that very often in class. But…here…actually without having specific function it is very hard to graph it because we need to have a function to find exact values…[looking at the question again and graph he drew] I am not quite sure that I am doing right.

His remarks shown in the above excerpt indicated that he was trying to remember a process or action to use in order to sketch the distance vs. time graph. Then I asked him how he was planning to use the first graph (distance vs. speed). He replied again he was not sure but he also said “in calculus I remember we were doing same kind of jobs”. I assumed that he was
trying to differentiate the speed to produce accumulated distance. Then I gave him a hint by asking if the speed vs. time graph works or not instead of distance vs. speed. He seemed very confused and not sure about what he was doing and seemed to be struggling to find a starting point. He did not use Distance vs. Speed graph at all to draw the Distance vs. Time graph. His verbal expression “…actually without having specific function it is very hard to graph it because we need to have a function to find exact values…” shown in the above excerpt revealed that he was looking for a algebraic representation of function to sketch the graph of relationship.

Then I asked him if he could explain why he graphed a straight line and he answered “when speed is increasing, distance should be increasing”. Afterwards he started to draw a second graph (Distance vs. Time) as shown below (figure 4.3).

![Figure 4.3 Jay’s graph of distance vs. time](image)

When he finished sketching the graph shown above, I wanted him to explain how his graph represented the distance between two people when they are speeding up and slowing down. He said that the first part [pointing the first straight line] represents that they are slowing down but after a little silence he stated that his graph was wrong and he started to draw another one shown in figure 4.4. When asked why he changed the graph, he said:

The distance is between them decreasing obviously and then they meet and it [distance] is increasing again…ummm the distance… is changing as slower rate as they approach each other. As they get apart…distance changes faster…in relation to time…ummm… I mean more distance faster speed and less distance slower speed…as the amount of distance…ummm…how much you can move per amount of time… So when speed is going down very small distance covered… in the second part [pointing the steeper straight line] it is opposite. This should be steeper to show the faster speed. That is why
I changed the graph. Because in this one, [pointing the previous graph he produced that is shown in figure 4.3] slopes of two lines are same that is why it was wrong.

Figure 4.4 Jay’s second graph of distance vs. time

Although he seemed very confused at the beginning when he was drawing first 2 graphs, he was very confident with his final graph. In all his attempts to graph the situation he exhibited the behaviors indicating an awareness of initial coordination of variables. According to Carlson et al. (2002), this coordination can be recognized by observing the student in labeling the axes and student’s verbal statements supporting his/her awareness that as one variable changes, the other variable changes. Jay labeled all the graphs he drew and he seemed to treat independent and dependent variables adequately. In the first task (figure 4.1) he was asked to draw a graph showing the distance between two people as a function of time. He labeled the x-axis (independent variable) as time and y-axis (dependent variable) as distance. As mentioned in the excerpt above, his verbal statements such as “distance changes [faster]… in relation to time” was also suggestive of his mental actions related to initial coordination of variables.

Moreover, his behaviors also appeared to support mental action of coordinating the direction of change. When we zoom in to look at these behaviors in detail, we can see that he obviously provided a graph consisting of decreasing and increasing parts claming that “The distance is between them decreasing obviously and then they meet and it [distance] is increasing again”. Consequently his above statement and his graph seemed to be important behaviors signifying his awareness of the direction of change of dependent variable (distance) while considering the change in independent variable (time).

Even though his remarks such as “as the amount of distance… how much you can move per amount of time” provided evidence of his intention to coordinate the amount of change of
distance between two people over fixed intervals of time, the graph he provided demonstrated the uniform changes in the amount of distance. Obviously his verbal statements suggested that he was trying to imagine the magnitude of the changes in distance between two people “how much you can move” while imagining the uniform increments or small intervals of time “…per amount of time…” However, he did not attempt to plot any points in order to represent the relative changes in magnitudes of distance and time in the graph. Therefore his graph shown in figure 4.4 was inconsistent with his verbal expressions.

When we look at his explanations stated in the above excerpt again, we can identify statements such as “…the distance… is changing as slower rate as they approach each other. As they get apart…distance changes faster…in relation to time…” These statements suggested that his reasoning involves mental actions of coordinating the rate of change of the distance while imagining the fixed increments of time. But again he did not provide a graph that represents various rate of change of distance. He did not even attempt to demonstrate contiguous line segments on the graph with different slopes representing the rate of change of the distance with respect to the time. Again there were inconsistencies observed between his graph and his verbal statements.

At this point, when we turn back and analyze his verbal remarks, the graphs he provided and the explanations that he provided for his graphs together, we see that his reasoning did not support mental actions of coordinating the continuously changing rate of change of the distance between two people. Although he stated that the distance between two people is changing “as slower rate” when they approach each other, he constructed a straight line to show the decreasing distance. Similarly he drew a steeper straight line for representing the change in distance when two people get apart. When asked how these lines demonstrated a “slower rate” or a “faster rate” he said that the slope of first line was less than the slope of second line and this was the difference between “slower rate” and “faster rate”. Apparently, he used the terms “slower rate” and “faster rate” for the purpose of comparing the two phases of this dynamic situation. He distinguished these two phases by drawing the distance vs. time graph which consisted of two straight lines differs from each other only in being more or less steeper. He did not seem to examine the rate of change of the distance over smaller and smaller intervals of time. Therefore his behaviors did not appear to support mental actions of imagining the successive
transformations of the distance between two people in order to form images of continuous change over small intervals of time.

Upon completion of the first task, a second task (figure 4.1.5) was given in the same interview session. Basically, he was required to draw a graph that represents the distance between two people as a function of time again.

These same two people decide again to start at opposite corners of the room and walk toward each other. But this time they both decide to maintain same steady pace the whole way. Draw a graph showing the distance between two people as a function of time. Describe your graph.

Figure 4.5 Task-2

After reading the problem, he repeated the words “same steady pace the whole way” several times. After a while, he constructed a graph shown below (figure 4.6). When I asked him to explain why he constructed the graph like that, he said:

“They are moving same amount of distance for …for each unit of time. So graph will be increasing straight line. It is a relation of one to one or unit to unit.”

Figure 4.6 Jay’s graph of distance vs. time

Then I asked him to explain what he meant from “one to one” relation or “unit to unit” relation. He stayed silent for a little while and responded: “for every unit of time the distance is same this is one to one. I mean it is not changing it is a direct relationship.” He appeared to be struggling with explaining the linear relationship in the situation by using the inappropriate expressions, such as “one to one” or “unit to unit” and “direct relationship”. These expressions can also be true for a non-linear functional relationship. When we look at the words he used, we
can see that he made an effort to express that there is no change in distance for same intervals of time. After his explanation, I asked him to explain how his graph represented that two people were approaching each other and getting apart from each other. He read the problem again and looked at what he drew. Then he smiled and said: “ohh this is not the correct graph. I did not pay attention to situation I just drew the regular time vs. distance graph.” He constructed another graph shown below (figure 4.7) and explained:

Here the situation is different from the first one. These two straight lines have same slope instead of different slopes like in the first one. Because they don’t change their speed …I mean they are moving with same speed whole time. That is why these lines have to have same slope.

![Figure 4.7 Jay’s second graph of distance vs. time](image)

Even though he provided an appropriate graph representing the situation, his verbal statements such as “I mean they are moving with same speed whole time…That is why these lines have to have same slope” suggested that he retained his tendency towards using the magnitudes of the slopes of these two lines, when he tried to demonstrate changing rate of change and constant rate of change in distance. His remarks in this task provided supplemental evidences that suggest that his reasoning did not appear to support mental actions of coordinating the rate of change of the distance with changes in time.
Similarly, on the third task, shown in figure 4.8 below, he was asked to draw a graph representing the distance between two people as a function of time again.

These same two people decide once more again to start at opposite corners of the room and walk toward each other. But this time as they walk they both speed up as they get closer to each other; pass, and then they both slow down as they get farther apart. Draw a graph showing the distance between two people as a function of time. Describe your graph.

Figure 4.8 Task-3

He consistently attempted to draw two straight lines with different slopes again as shown in figure 4.9 below. He seemed very confident while responding this task and drawing the graph. He did not read the question several times as he did in the first two tasks. While drawing the graph, he claimed that graph was going to be the “opposite” of the “first one” (the final graph he provided for the first task, shown in figure 4.4). When asked why, he replied: “When we read the question we can see that it is just opposite situation…I mean first they are speeding up and then slowing down. In the first one it was opposite”. Again he was asked to explain why he drew graph like that. He again used the same reasoning and said:

Since they are speeding up in the first part [when two people are approaching each other] it is going to be more distance covered and there is going to be less distance covered in more time in the second part [when two people getting apart from each other].

Figure 4.9 Jay’s graph of Distance vs. Time
Then I asked him to explain what he meant by “there is going to be less distance covered in more time”. He said: “It is going to take more time to get the other end of the room in second part because they are slowing down.”

Considering his verbal expression, he did not appear to coordinate the amount of change of the distance with respect to uniform increments of time. In his explanation of the statement he only mentioned the total amount of time. His explanation did not appear to support mental activities of coordinating the rate of change in the dependent variable (distance) while imagining the uniform increments in the independent variable (time). Also the statement “less distance covered in more time” did not suggest sufficiently that his reasoning resulted from mental actions of coordinating the rate of change in small intervals of time and of coordinating the continuously changing rate of change over smaller and smaller refinements of the average rate of change. In addition, it is also observed that his difficulties in imagining the transformation of the distance between two people are consistent. Carlson et al (2002) and Saldanha & Thompson (1998) stressed the role of mental enactment of the dynamic situation to be able to imagine transformation of the object. The word object here does not refer only tangible things but also implies mathematical objects such as variables. The term mathematical object have been defined and discussed by many researchers in mathematics education. According to Chiappini & Bottino (2000): “Mathematical objects are abstract objects; indeed mathematical objects are not amenable to any concrete imagination or manipulation; they are immaterial and accessible only to our thinking” (p.1)

After the first interview session, Jay was engaged in several other tasks in the second interview three days later. Before starting the interview session, he said that he was wondering if he did well on the first interview tasks. I again stressed that the main goal of interviews were trying to understand his reasoning rather than assessing the work as right or wrong.

In the second interview, again he was presented several tasks involving dynamic situations. Then he was asked to demonstrate these given situations in graphical representations and explain his reasoning afterwards. The task shown in figure 4.10 was given as the first one in this session.
After the task was presented, he read the problem two times slowly. Then he started to sketch a graph by drawing the both axis, while he was thinking. He drew the axis and stopped. I asked about what he was thinking and he responded:

I am confused about the starting point of the graph. But I guess I will start at 100. Yes, I would say that the amount of chemical X is 100 at the beginning….ummm I think in 20 second whole X will change into Y.

Then he labeled the starting amount of “chemical X” as “100” on the Y axis and drew a decreasing straight line as shown in figure 4.11. When asked why he thought that whole process in the situation would take 20 seconds, he replied: “in the problem it is stated that every second 5% of X changes into Y, so it will take 20 seconds”. By this observation he was exhibiting behaviors that indicate an initial coordination of variables. He labeled the y-axis as “X” to represent the “amount of chemical X” and x-axis as “t” to represent “time”. In addition to initial coordination of two variables, his behaviors also provided evidence that his reasoning built upon the mental actions of coordinating the direction of change of amount of chemical X (dependent variable) while imagining the changes in time (independent variable). He demonstrated these behaviors by drawing a decreasing straight line in his graph.
However, his behaviors did not appear to provide evidence of forming accurate images of the amount of change in the dependent variable (amount of chemical X) while considering the fixed changes in the independent variable (time). His statements revealed that he basically treated the “5% of change in every second” as a fixed amount of change in every second. It appeared that he did not consider how 5% of change in every second would be reflected with respect to fixed amounts of time intervals. To understand whether this reasoning is rooted in a conceptual obstacle or lack of attention, I gave him another situation by changing the percent amount into a fixed amount. I basically asked him to explain if there would be any difference, if it is stated that 10 gr of chemical X are changing into chemical Y every second. He said: “Yes. The situation would be different. For example if I say 5%, I know that there is a limitation of amount and time because 5% out of 100%.” Upon this answer I rephrased my question and told him that we have total 100 gr of Chemical X and 10 gr of that are changing into chemical Y. He said:

It is also different because it makes the situation two times faster … ummm … instead of 5% now 10 grams … yes two times faster…..[long silence-thinking] … the only thing I am trying to figure out is 5% of 95% is not the same 5%. …[thinking]… I am thinking if you have 100 gr and you are taking out 10 gr each time it is same 10 gr each time. If you think in terms of percentage it is not same.

His answer presented in the excerpt above demonstrated the evidence that his reasoning concerning the way he treated 5% change every second was not because of a conceptual misunderstanding. It seemed that he did not pay enough attention to form the images of changes in the amount. He realized that 5% change in every second reflects different amount of change in every second. Upon this answer, I asked him if there was any change needed to be done in the graph. He said:

Yes, it is going to be different. 5% out of 100%, then 5% out of 95 percent and so on …. ummm [Thinking]… I don’t know …. I am not sure…. I think I am moving faster… yeah of course I am moving faster. So X starts to fall down faster. It [straight line that he drew in the first graph] is going to be steeper. But it still takes 20 second.

Then he sketched the second graph shown in figure 4.12. Although he realized the difference between percent change in amount and fixed change in amount with time, it was observed in his reasoning that his mental actions were limited to the coordination of the direction.
of change. He was able to construct images to coordinate the direction of change but his explanations and his graph did not provide any evidence of coordinating the amount of change of the chemical $X$ with considering changes in the time. In fact, he retained his view that graph he looked for was still shaped as a decreasing straight line even though his later realization of “5% of 95% is not the same 5%” requires changes in the graph.

Moreover, he neither attempted to plot any point nor constructed any contiguous line segments with different slopes on the graph to represent the relative changes in the amount with time. Instead, he just sketched a steeper straight line by claiming that “I think I am moving faster” (chemical $X$ changes faster with time) and “it [straight line] is going to be steeper”. These behaviors signified that his reasoning did not give any appearance of mental actions of coordinating the average rate of change of the amount of chemical $X$ for equal amounts of time and of coordinating the instantaneous rate of change of the amount of chemical $X$ with changes in the time. Although he stated that 5% of 100% is not the same as 5% of 95%, he did not attempt to construct contiguous lines with different slopes or a smooth curve to represent this change. In addition, his statement “I think I am moving faster…yeah of course I am moving faster. So $X$ starts to fall down faster” also suggests that his reasoning does not involve mental actions of coordinating the amount of change of dependent variable (amount of chemical $X$) with respect to fixed amount of change in independent variable (time).
Similarly, in the next task shown below in figure 4.13, he exhibited the similar reasoning which did not give any appearance of coordinating the amount of change in chemical X with change in time and of coordinating the average rate of change and instantaneous rate of change of amount of chemical X while imagining the changes in time.

<table>
<thead>
<tr>
<th>5% of chemical X changes into chemical Y every second. 5% of chemical Y changes back to chemical X every second. Starting with all chemical X, sketch a graph that represents</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. the amount of chemical X as a function of time.</td>
</tr>
<tr>
<td>b. the amount of chemical Y as a function of time.</td>
</tr>
</tbody>
</table>

Figure 4.13 Task-5

After reading the problem stated, again he seemed very confused. He sketched two graphs shown in figure 4.14 by claiming:

It is very complicated…. After first second, I mean in the 2nd second Y will be giving back to X 5%. After first second in the 2nd second,…ummm… 5% changes back to X so after this point [pointing the lower end of the decreasing straight line shown in the graph] it is going to be constant. Because Y is always giving back what it get from X. The second graph [the amount of chemical Y as a function of time] is obviously the opposite of the first one [the amount of chemical X as a function of time].

Figure 4.14 Jay’s two graphs of amount of chemical X and Y vs. time

Obviously his reasoning did not provide any appearance of constructing accurate images in terms of coordinating changes in the independent variable (amount of X or amount of Y) with respect to dependent variable (time). When we look at his remarks stated in the excerpt above, we can see that he consistently treated the percent change in the amount as fixed change in the amount by claiming: “Because Y is always giving back what it get from X”.

54
When I asked him to explain why chemical Y starts changing back to chemical X in the 2nd second, he claimed that chemical Y has nothing to give back until the 2nd second. This explanation indicates that his reasoning does not involve mental actions of coordinating the simultaneous changes of two variables. He apparently constructed an image which consisted of one (1) second intervals of time and this image did not allow him to imagine the whole situation as a continuously changing dynamic situation. In other words, the student built a static image rather than imagining “a continuum of states” in a dynamic situation. Carlson et al (2002), Monk (1994) and Thopmson (1994b) have expressed that simultaneous coordination of two variables in a dynamic situation requires imagining “running through” a continuum of states very rapidly. Although these researchers used different terminology to express the difference of static and dynamic conceptualizing of functions, they share the perspective discussed above.

Jay was engaged in the last task shown in figure 4.15 below in the second interview session to search for further evidences of his reasoning about coordinating the simultaneous changes in two variables in a graphical context.

Imagine a pebble is thrown into a lake, creating a circular ripple that travels outward at a constant speed. Sketch a graph that represents the area, A, of the circle as a function of time that have passed since the ball hits the lake.

Figure 4.15 Task-6

After reading the problem, he stated that this problem is much easier than the previous problems. When asked why, he said that he could imagine the situation better. And he also stated that he has never been good at chemistry, although previous problems do not require any knowledge of chemical computation. Upon reading the question he provided the following two graphs shown in figure 4.16 by claiming: “The area of a circle is \( A=\pi r^2 \) since \( \pi \) is constant area depends on \( r^2 \). Since \( \pi \) is constant…it is like \( y=x^2 \)…but when I think of area vs. time…ummm…the Area increases as time goes by….[thinking] …it [ripple] seems to go faster when I imagine but the problem says ripple travels outward at a constant speed. So if it travels at a constant speed it should be straight line”.

55
Based upon his remarks shown above excerpt, I asked him to clarify what is increasing or traveling with constant speed. He claimed that the area is increasing a constant speed. Then I asked him to explain his reasoning about why he thought that area is increasing as constant. He again pointed out the problem expressing ripple traveling outward at a constant speed. Obviously he appeared to assume a linear relationship between speed of “ripple” and change in area. It is also significant to note that he attempted to utilize the area formula of a circle to come to a conclusion at first and sketched the area vs. radius graph. But he constructed inappropriate mental image of the situation and appeared to assume inappropriate connections between speed of ripple and change in area.

The evidence from the student’s behaviors such as drawing an increasing straight line and assuming the linear relationship between speed of the ripple and change in the area of the circle suggests weaknesses in coordinating the amount of change of the area with respect to time while communicating with the information given in the situation. Moreover, his reasoning did not provide any appearance of mental actions of coordinating the average rate of change of the area with respect to time. He did not attempt to construct contiguous line segments on the graph, with different slopes to reflect relative changes in the area for fixed amount of time segments. His behaviors also did not support mental actions of coordinating the instantaneous rate of change of the area with changes in time. Although he provided a smooth concave up graph or parabola for area vs. radius as seen in figure 4.16(a), his verbal expression “The area of a circle is $A=\pi r^2$ since pi ($\pi$) is constant area depends on $r^2$. Since pi ($\pi$) is constant… it is like $y=x^2…$” revealed that this utterance is an application of a memorized rule. According to Carlson et al (2002) mental actions of coordinating the instantaneous rate of change can be observed through the student’s behaviors of constructing a smooth curve and the student’s remarks that suggest an
understanding that the smooth curve resulted from considering the changing nature of the rate while imagining the time changing continuously. Although he provided a smooth curve, that was concave up, to represent the relationship between area and radius, this behavior is considered as a memorized action.

The third interview took place one week after the second interview. Again Jay was engaged in several tasks in order to gather further information about his reasoning. In these tasks, he was again given dynamic functional situations and asked to construct graphs representing the situations. The task shown in figure 4.17 was presented as the first task of the interview session.

Tom sees a ladder against a wall (in an almost vertical position). He pulls the base of the ladder away from the wall by a certain amount, and so forth. Each time he does this he records the distances by which the top of the ladder drops down. Do the amounts by which the top of the ladder drops down remain constant as Tom repeats this step; or do they get bigger, or do they get smaller? Explain

Figure 4.17 Task-7

Jay read the question two times out loud while he was thinking. And he said:

So the amount of which he pulls the ladder from the bottom is constant. That is going to resemble constant rate of change between the height it’s changing and distance it is moving from the base. So it [amounts by which the top of the ladder drops down] does not get smaller. If each time he pulls the ladder by the same amount, the top is going to drop the same amount each time.

In his remarks shown in above excerpt, again we can see similar patterns with his previous behaviors in terms of reliance on linear relationships between variables. Here, his expression “That is going to resemble constant rate of change between the height it’s changing and distance it is moving from the base” provides evidence that his reasoning does not support the mental actions of coordinating the amount of change of the height with considering the equal increments in the amount of change in the horizontal distance. Then I wanted him to provide more explanation about his reasoning. He then stayed silent for a little while and said “I don’t know what to say more but when I imagine the situation… since ladder is a one solid thing when you pull from one end by certain amount the other end will
move by same amount”. While he was providing this explanation, he sketched two figures shown below in figure 4.18 to represent the situation.

![Figure 4.18 Jay’s representation of the position of ladder](image)

The figure he provided revealed that he performed mental enactments of the ladder falling down the wall. He presented a succession of pictures of the ladder in different positions. Even though he appeared to engage in a behavior that suggested that he was attempting to coordinate the amount of change of the height with respect to equal increments of horizontal distance, he constructed an inappropriate image which prevented him from coordinating the independent and dependent variables accurately.

After this task, he was given the next task in figure 4.19. Basically, this task is very similar to the previous task (figure 4.17) with some changes. Instead of amount of vertical change, student was asked to analyze the amount of change in the speed of the top of the ladder as the top of the ladder drops down.

Newt, the science nerd, then comes along and puts wheels on the bottom of the ladder. He connects them to a motor so that the bottom rolls away at a constant, but very slow, speed. Does the top of the ladder move down at a constant speed; or does it speed up, or does it slow down? Explain

![Figure 4.19 Task-8](image)

In this second task, he consistently retained his reasoning by claiming that if the bottom of the ladder moves at a constant speed the top of the ladder drops down at a constant speed. He also expressed that this situation is not very different from the previous one and he said:

I am thinking if certain things just come out of my head are true, this should be the same thing just there is no interval time between each pull. It is just dropping at a constant rate. I mean it is dropping at a same speed as motorist pulling the bottom of the
ladder. If I don’t allow the weight to push my hand back faster than I am pulling, it will remain constant. So if I am pulling the ladder with certain speed, the ladder is going to be dropping same amount and same speed.

When asked to explain his reasoning, he relied on his reasoning on the first task. He stated that since the ladder is a single solid object, two ends of the ladder cannot move at different speeds. He expressed that it is a physical rule. When I asked him to explain this physical rule he stated: “now I do not remember exact principle but different points of same solid object cannot move at different speed.” His remarks exhibited a tendency toward the use of a general principle for describing the overall behavior of variables even though the principle does not apply to the given situation. He also seemed to perform inaccurate mental enactments of the situation to visualize the transformation of the object. Therefore his behaviors did not give any appearance of appropriate coordination of two variables.

In the last part of the “ladder problem”, Jay was asked to sketch a graph that represents the relationship between the horizontal and vertical position of the ladder as it slides down a wall, starting at a vertical position and finally resting on the ground. He provided the graph shown in figure 4.20. The graph he provided appeared to be accurate. When Jay was asked to explain his reasoning, he said:

I wanted to do it [graph] in terms of the angle which the ladder makes the floor 90 being vertical position starting position 0 being end position I mean horizontal position. It would look like this [pointing the graph] …actually… each portion of these represents change of angle.

Figure 4.20 Jay’s presentation of change in the angle

Figure 4.21 Jay’s presentation of horizontal disposition
As a matter of fact, his graph did seem to represent an accurate relationship between the horizontal and the vertical positions of the ladder. However, the way he sketched appeared that he was presenting a “unit circle”. The sketch in figure 4.21 indicates that he was trying to represent the relationship between the ladder’s horizontal disposition and change in the angle it makes with the ground. Even though his initial reasoning about the situation and his previous behaviors such as his remarks and previous graphs verified that his mental enactments of the situation did not allow him to visualize the transformation of the object and to coordinate the changing rate of change, the above graph he provided, which is an acceptable graph for representation of horizontal vs. vertical positions of the ladder, invoked a suspicion about whether he changed his mind and it was a conscious act or it is just an unconscious behavior. In order to clarify this matter, I asked him if he could draw a graph that represents the relationship between the horizontal and vertical position of the ladder as it slides down a wall. Then he provided the graph shown in figure 4.22 claiming:

In terms of vertical distance in relation to horizontal distance which is change in vertical and change in horizontal, it would be a straight line since it [the change] is constant.

![Figure 4.22 Jay’s graph of vertical vs. horizontal position of the ladder](image)

Even though his second graph appeared to be representing the relationship between vertical vs. horizontal displacement rather than representing the relationship between vertical vs. horizontal position of the ladder, his above expressions and the graph he provided clarified that he was still retaining the same view (constant rate of change) in terms of coordinating the dependent and independent variables.

Although he provided the unit circle to represent the angular change, he did not attempt to utilize this graph (unit circle) in his reasoning to coordinate the changing rate of change in the
sitting. Considering all evidences discussed above, his reasoning did not seem to support mental actions of coordinating the changing rate of change.

After the ladder problems, Jay was engaged in another task which was the last task in the third interview. The task shown in figure 4.23 required him again to represent the given situation in a graphical context.

Imagine this bottle filling with water at a constant rate. Sketch a graph that represents height of the water as a function of amount of water that is in the bottle.

Jay again read the problem two times. And he started to think about the situation. I reminded him to speak aloud while he was thinking. Then he provided an increasing straight-line graph again by claiming: “when the amount of water in the bottle increases height should go up too.” But just after finishing the sketch, he stated that he might be wrong and provided the second graph shown in figure 4.24 by claiming:

The shape of the bottle has to do something with it….ummm…[thinking]… taking into consideration of the shape of the bottle, the bottom part is spherical so volume equals to four third pi r cube [writing the equation $V= \frac{4}{3} \pi r^3$] and the top part is cylindrical … ummm…the volume is [writing down the formula $V= \pi r^2 h$] there are three depths I mean there are three parts… the bottom part or half circle… if the water is filling with a constant rate, the height has to increase slowly, because the bottle is getting wider up until this point [pointing the midpoint of spherical part of the bottle]. After this point the
height has to increase faster than it increases in the bottom part. I mean once the container starts inward; the water starts to fill up faster until it gets to last part of the container. Then in the last part…ummm… it is going to increase fastest, because container is getting narrower.

Figure 4.24 Jay’s first graph of height vs. amount of water

Then I asked him to explain how his graph represented the height as “increasing slowly” or “increasing faster”. He pointed to the graph he provided and said: “The slope of first line [ he refers the line connecting the origin and the point A] is less than the slope of second straight line [ he refers the line connecting the point A and the point B], so this shows that in the first part it is increasing with slower rate”.

The graph he provided and his remarks indicated that he had difficulties with coordinating the changing rate of change of the height of the water while imagining the equal increments of the amount of water in the bottle. Obviously his reasoning did not involve visualizing the transformation of the height of the water over smaller and smaller intervals of the amount of water in the bottle. Although his remarks “height has to increase slowly”, “water starts to fill up faster”, “increasing with slower rate” may be taken as indicators of his awareness of changing rate of change but when we look at the his explanations and his graph we can see that he was obviously using these phrases to make comparisons of general trend of change in the height in three parts of the bottle.

After he provided the graph (figure 4.24), and his explanation regarding the shape of the graph, he seemed to be less confident with his answer and started to mark lines on the bottle figure. After a little silence, I asked him if he could speak aloud what he was thinking. He said
that the graph might be wrong and started to sketch another graph as shown in figure 4.25 by claiming:

Every time same amount of water fills less distance because bottle is getting bigger….ummm…so this graph [the first graph he provided] is wrong it should be something like that [pointing the second graph]. Then as soon as water hits here [midpoint]… it [water] is going to cover more distance every time.

As seen in the above excerpt, Jay altered his graph and constructed smaller contiguous line segments on the graph by claiming: “…every time same amount of water fills less distance because bottle is getting bigger”. He was also observed placing marks on the shape, as seen figure 4.26, to represent how much the height of the water would go up for equal amounts of water. His reasoning, which is revealed by his observed behaviors, supported the mental actions of coordinating the amount of change of the height while imagining the changes in the amount of water. Also, behaviors such as constructing contiguous line segments on the graph and expressing “…the height has to increase slowly…” in the above excerpt showed that his reasoning supported the mental actions of coordinating the changing rate of change of the height while considering the changes in the amount of water. However, he did not make any attempt to move from contiguous line segments to smooth curve to represent the continuously changing rate of change. He was very much focused on the marks that he placed on the bottle and the process of adding equal amounts of water each time. These behaviors suggested that he altered the original dynamic situation (water is continuously filling the bottle) to a static situation in which the same amount water is added to the bottle every time.

After his explanations regarding why he changed the initial graph, I asked him to explain how the shape of the bottle made him to change his graph. He said:
When I looked at the top part of the bottle I realized that the circular part should be different. Because top is cylindrical shape so that means water is rising with same rate everywhere in this cylinder. But the these parts[pointing the lower and upper parts of the spherical shape] are in different shape I mean they are curved and that made me think that water would go up at different rates.

Then I asked him if he could explain his reasoning about the statement “water would go up at different rates”. He answered: “I imagined that I am putting same amount of water every time. And I know that bottle is getting wider so for every same amount of water there is less volume to fill up. The top part is the opposite obviously.” Even though his initial reasoning about the situation and his previous behaviors such as his remarks and previous graph verified that his mental enactments of the situation did not allow him to visualize the transformation of the object and to coordinate the changing rate of change, his later explanations suggested that the visual presentation played an important role in his reasoning. His responses and the graph he provided suggested that he imagined the successive pictures of the situation and performed accurate mental transformation of the height of the water. Drawing contiguous line segments in order to represent relative changes in height considering fixed amount of water was suggestive that his reasoning was built upon the mental activities of coordinating the average rate of change of height over small intervals of volume (amount of water in the bottle). However, he did not appear to form an image in which the situation is evaluated as “running through in a continuum of states” (Thompson, 1994b). Therefore, his reasoning did not seem to be supported by the mental actions of coordinating the continuously changing rate of change of height by imagining smaller and smaller refinements of the average rate of change.

The next interview session took place 3 days after the third interview. Interview session was similar to previous one when Jay engaged in several tasks. For the purpose of gathering more information about his covariational reasoning in graphical context, he was again asked to construct graphs of given situations which are involved in simultaneous changes of two variables. The first task shown in figure 4.27 was given after reminding him of the interview protocol.
Imagine this water tank is full of water and there is a hole in the bottom of the tank. Water leaves the tank at a rate proportional to the height of the water remaining in the tank. The tank is full when the hole is opened. Sketch a graph that represents amount of the water in the tank as a function of time.

Figure 4.27 Task-11

Upon reading the question, Jay said: “The main thing here is the water leaves the tank at a rate proportional to the height of the water”. And he stayed silent for a while. He read this part of the problem over and over several times. And he said:

This is a height of water [pointing a point in the y-axis]. So amount leaves proportional to the amount of left…ummm…that is a straight line but starts up full. So the amount of water is at its maximum. Let say 100%…ummm… 100% water in the tank. With the time goes by this is going to be less water in the tank. So the amount of water is going to leave at a certain time then it is going to be zero [0] water left….It is a straight line [drawing a straight line].

Figure 4.28 Jay’s graph of amount of water vs. time

Jay provided a graph (figure 4.28) representing decreasing straight line for the relationship. Then I asked him to explain why he thought that it was a straight line. He did not answer my question at first. He seemed to be confused and he started to read the question again. Then he stated:
The amount of water leaves the tank proportional to the height...ummm...the amount of water is dropping with a same rate with the height is dropping so the height ... I think it is how it looks [pointing the graph again]. Each second amount of water drops.

Instead of explaining his reasoning he again stated that the graph he provided represented the situation. Then I again asked him to explain why he constructed straight line. Then he stated: I am thinking that if it is just the graph representing the amount of water in the tank and the time. I am just thinking that as time goes by amount of water drops so...if you start at full call this point [pointing the starting point of the straight line on the y-axis] zero...that is where I am starting...

In all his attempts he failed to explain his reasoning about why he decided to draw a straight line. He appeared to have trouble imagining the situation to coordinate the simultaneous changes of variables. In addition, in his last attempt he was communicating with irrelevant ideas such as trying to set the starting point as zero [0]. When I asked why he was starting at zero, he said:

I am just thinking right now if I set this beginning as zero [0] instead of 100% (he called the starting amount of water as 100% in his previous remarks.). I call that zero [0] and it starts here...ummm...but I have to towards negative because it is loosing amount. So it doesn’t make sense. I am going to leave it as 100%.

Then I asked him again if he could explain why he constructed a straight line. Then he replied: Amount of water coming out is going to remain constant because hole is not changing so same water is coming out...I mean that the height of water is going to be dropping proportional to each second goes by for example. So if the hole is same size then height is dropping at constant rate same as time at a constant rate. So I can say that since the diameter of the hole is same the amount of water coming out is constant.

His remarks as seen in the excerpts above and graph that he provided suggested that he had difficulties to perform mental enactments of the situation considering the given information. In other words, he had difficulties with imagining the transformation of the object (amount of water in the tank) and with imagining the successive pictures of the transformation of the object. Although, in the beginning of the task, he claimed: “The main thing here is the water leaves the tank at a rate proportional to the height of the water”, he did not seem to pay enough attention to coordinate the variables considering this given circumstance which is stated in the task. Instead,
he focused on the size of the hole in the bottom of the tank. His statements revealed that he built his reasoning about the coordination of simultaneous changes in two variables on the condition of the hole in the bottom of the tank. He stated that amount of water in the tank is proportional to the height of the water in the tank and the amount it leaves is proportional to the height. Then he concluded that since the size of the hole is not changing the amount of water leaving the tank is constant so the height of the water is decreasing at a constant rate since it is proportional to the amount of leaves and then finally since the amount of water in the tank is proportional to the height of the water, amount of water is decreasing at a constant rate. As we can see, he set the whole situation under one condition which is the size of the hole. Thinking of the size of the hole all the time apparently hindered his reasoning about the situation. His behaviors revealed by his verbal statements and graph he constructed provided evidences that his reasoning supported his mental actions of coordinating the direction of change of the amount of water in the tank with respect to time. However, his reasoning did not support the mental actions of coordinating the amount of change, average rate of change and instantaneous rate of change of the dependent variable (the amount of water in the tank) with respect to dependent variable (time).

The next task shown in figure 4.29 below was presented to Jay upon completion of first task. The context of the problem was similar with previous one. He was again engaged in a same kind of task which required him to sketch a graph that represents the situation given in the problem.

Again there is a hole in the bottom of the tank. The tank starts off empty and water is filling in at a constant rate where the level increases. Water leaves tank at a rate which is proportional to the height of the water in the tank. Sketch a graph that represents relationship between amount of water in the tank and time.

Figure 4.29 Task-12

After he read the problem two times he stated that the situation is very similar with the previous problem and he said: “so the more water in the tank the more water comes out”. When I asked him to explain his statement, he pointed the problem and said: “Because in the problem, it
says water leaves the tank at a rate which is proportional to the height of the water at the tank.” Then I asked if he could provide more detailed explanation what this statement meant to him. He replied:

Basically if the amount coming out increases as height goes up, probably it wont ever increase…I mean height won’t be able to fill…. [reading the question again]…okay it starts filling up. Water starts leaving. Change in height is proportional to amount of water coming down. It won’t be able to fill up. If it starts at zero [0] in zero [0] second, as time goes by there won’t be no change in amount of water.

After his remarks, he provided the graph shown in figure 4.30 below. Then I asked him to explain his reasoning why the amount of water stays the same. Then he said:

Because, water comes in and comes out as same rate. Yes we start out empty and the water poured in a constant rate which the level increases…so amount of water comes in with constant rate which is the same rate as height increases and as the height increases water comes out with same rate. So it is going to be in and out situation here. That would be zero [0] change in the amount of water.

In this task, Jay appeared to misinterpret the phrase “proportional” which is stated in the question by claiming “…water comes out with same rate” as seen in the above excerpt. Although he tried to explain in his previous remarks, I wanted him to explain why water comes out with the same rate as the height increases to verify his reasoning. He replied: “because it [problem] says water leaves tank at a rate which is proportional to the height of the water.” Then I asked him if he could explain what the word “proportional” meant. He attempted to explain the meaning of “proportional” several times but he appeared to be confused. Then I asked him if the
word “proportional” meant “same”, he replied “not necessarily” but he again failed to explain the difference between these two words. Then he said: “I know that proportional does not mean that two things necessarily are same… I am trying to give an example but I can’t think of a good example to explain it”. Then I gave a hint by making up a situation and his answer to the situation satisfied that he was aware of the meaning of “proportional”. Then I wanted him to explain why he thought that “water comes out with same rate”. He stayed silent for while and claimed:

In here [pointing the problem] since there is no value for amount of water comes in and comes out I cannot say anything about the proportion between them. Since I do not know the values….I mean I am imagining the situation in my head and I am assuming that they need to be same.

Jay again seemed to be distracted by his inaccurate imaging of the situation. Therefore his reasoning did not allow him to coordinate the amount of water in the tank with respect to time. In other words, the image he constructed in his head prevented him to perform mental enactments of the situation to mentally picture the transformations of the object (the amount of water in the tank) with respect to time. His behaviors rooted in his reasoning, which is also rooted in the image that he created in his head, did not support even mental actions of coordinating the direction of change of the amount of water while considering the equal increments in the time. His reasoning pattern provided evidences that he had difficulties in visualizing the transformation of this dynamic situation as Carlson et al (2002) stated that this results from operation of coordinating.

Task 13 shown in figure 4.31 was presented to gather further information about his reasoning pattern supported by his mental actions.

Figure 4.31 Task-13
Jay read the question once and had a brief look at the figure shown above. He confidently claimed that this was the almost same situation with previous one. He also stated that there are two things different from the previous task. The first difference he mentioned was obviously the position of the hole and the other difference he claimed was: “it [the problem] does not say that the amount of water coming in proportional to height of the water”. He claimed that the previous problem stated that the amount of water coming in was proportional to height of the water. Then he went back to look at the previous problem and he read the problem two times and said: “it did not say so either” he seemed very confused. I reminded him that he put the relationship between amount of water coming in and the height of the water in previous problem. After a short silence, he smiled and said: “of course container is totally symmetric. That’s why height is increasing as same rate with the amount of water coming in.” Afterwards, he started to sketch the graph and provided a graph shown in figure 4.32 by claiming:

It starts up empty. And it is increasing like that [pointing the increasing straight line in the graph] until it hits the hole. What I want to say like 40% and after this point there will be no change. The height of the water is going to have proportional relationship to amount of time. So I am thinking that amount of water poured is proportional to the height of the water. Since container’s shape does not change. When it hits the hole…in this point [hole] same amount of water coming in and out so there is no change in height and amount of water.

![Figure 4.32 Jay’s graph of amount of water and time](image)
Apparently he used the same reasoning as he did in the previous task. Then he was asked again to explain what made him think that the amount of water coming and amount of water going out are same. He said:

Since problem says water leaves the tank at a rate proportional to the height of the water above the hole…. [thinking]… everything seems proportional to each other so amount of water coming out is same with the amount coming in.

He appeared to communicate with inaccurate images of this dynamic situation again. And again the expression “everything is proportional to each other” led him to think that amount of water coming in to tank is same with the amount of water leaving the tank. These behaviors signified that his reasoning did not support the mental actions of coordinating the changing rate of change of amount of water in the tank. The image he constructed in his head did not enable him to visualize the successive pictures of the transformations.

The task shown in figure 4.33 below was given as the last construction task to gather further evidences regarding Jay’s covariational reasoning.

Imagine air is pumped into spherical balloon at a constant rate. Sketch a graph that represents radius of the balloon as function of amount of air in the balloon.

Figure 4.33 Task-14

Jay read the problem once and started to sketch the graph. He seemed very confident while he was drawing the graph. He provided again a straight line graph by claiming:

So it is a regular graph one axis is radius the other one amount of air. Radius is going to increase proportional to the amount of air. So it is straight line. As time goes by amount of air pumped into balloon is increasing constantly … so radius increases constantly and balloon gets bigger and bigger. It is obviously a straight line.

Then I asked him if he could explain his reasoning. He stated “it is already given in the problem. I mean air is being pumped at a constant rate so there is a constant increase… I mean radius should be increasing constantly too.” His explanations and the graph he provided indicated that his behaviors did not support the mental actions of coordinating the amount of change of the radius while imagining the equal increments of the amount of air in the balloon. Since his mental enactments of the given dynamic situation did not support imagining the
appropriate transformations of the radius, he did not exhibit the behaviors supporting the mental actions of coordinating changing rate of change.

After he completed his explanations, he appeared to be thinking about the situation. When I asked what he was thinking, he said: “I am just thinking about the shape of the balloon”. I asked him if the shape of the balloon had an impact on the situation or not and wanted him to explain if there is a relationship. He said:

For example, if inside the balloon …. [thinking]… here radius is constant because shape is sphere…I do not know for example if it [the shape of the balloon] is ellipse. There are two different radii in the ellipse…But I thing it is going to be straight line again but different slope I guess.

At the first moment it appeared to me that he was going to make inferences about changing rate of change of radius considering the spherical shape of the balloon, when he mentioned that he was thinking about the shape of the balloon. Instead, he seemed to be communicating relatively insignificant information by making configuration in the shape of the balloon. His statements in the excerpt above indicated that he was not imagining the transformations or successive pictures of change in the radius by mentally enacting that air is being pumped in to the balloon to coordinate the continuous changes of rate of change in the radius. Rather he constructed again a static image in which he was comparing the radius of two shapes. As a matter of fact that his starting point, which revealed the image in his head, “here radius is constant because shape is sphere” has already restricted him to reason about dynamic situation by performing relative mental actions.

Jay was engaged in the interpretation task in the fifth interview which took place one week after the prior interview session. In the interpretation task the student was given a graphical representation of a dynamic functional event and asked to describe the overall behavior of the graph by interpreting important features in the shape of the graph. Basically Jay was asked to interpret global features of a given graph such as the general shape of the graph, pattern change (interpreting direction of change or changes from concave up to concave down or vise versa), intervals of increase or decrease and intervals of extreme increase and decrease (e.g. determining the part in which dependent variable increasing or decreasing most).
I presented the task shown in figure 4.34. Then I asked him to describe the overall graph. He said:

The graph shows car is going through city A to city E…as you arrive to B you start slowing down until you get to the middle between B and C…you…as time goes by no distance…kind of start slowing down. When you get to city D…you slow down a lot more as you go down and then as you go into E you start speed up again.

When I asked him “how do you know that the car is slowing down?” he said that the car was gaining little distance as time went by. And similarly he explained the other parts in terms of “gaining distance”. In his statement, he somehow appeared to be aware of rate of change in distance by expressing “…as you arrive to B you start slowing down until you get to the middle between B and C”. However, he did not mention about decreasing rate of change starting from beginning of the journey. Although his remarks indicated that he was communicating with an image of the situation he constructed in his mind, his reasoning did not seem to be built upon imaging the small intervals of time then coordinating the rate of change over these small intervals of time. For instance, he did not make any attempt to mark fixed intervals of time in the x-axis in order to demonstrate corresponding average rate of changes in these intervals by constructing sequence of second lines.

In order to gather further evidences regarding his reasoning, I then posed another question: “Between which consecutive cities is the average speed of the car maximum?” He said:
“I would say between A and B is the fastest in the whole section...I am not sure but E could be fastest too. But I say between A and B.” Then I asked him to explain his reasoning. At this point he stated that he wanted to draw speed vs. time graph to see the situation clearer. Then he first sketched following pair of graph. As shown in figure 4.35, he seemed to be using a mapping method. At this point, I asked him if he could explain what he was doing. He stated that the graph shown in the top is model of distance vs. time graph and the graph in the bottom is a correspondent velocity vs. time graph. He stated “when I was in the high school physics teacher thought this model to transfer the distance vs. time graphs to velocity vs. time graphs. We used this a lot in high school for solving that kind of problem.”

![Figure 4.35 Model of conversion from distance vs. time graph to velocity vs. time graph](image)

Then he constructed speed vs. time graph shown in figure 4.36 by looking at the parts at model he drew. Then I asked him again to describe the graph that he constructed. He said:

The car is slowing down until it arrives to BC [he called the top point as BC as seen in the graph below]. It stops at this point that I called BC. Then in this part [pointing the section that velocity is negative], car is speeding up but going to other direction until it hits to D. In here [pointing D] it starts slowing down. After that it is speeding up again. And it stops at point DE also because the speed is zero.
I was really surprised to see this method then I posed my second question again: “Between which consecutive cities is the average speed of the car maximum?” Then he came to a stopping point for a while. When I asked him what he was thinking, he said: “I am just thinking how to find average speed.” After he stayed silent for a while I observed him drawing horizontal lines to the y-axis from the points. Then he said: “I guess I have to look at the midpoints in the y-axis. But I do not have any value...I am just thinking...it looks like average speeds between points A and B and points C and D are same. I mean I can’t tell exactly but they look like same.

His behaviors suggested that he was not able to build his reasoning upon directly constructing an image of speed changing over intervals of time. Rather his reasoning involved in activation of memorized rules and procedures. Carlson et al. (2002) classified this kind of behavior as pseudo-analytical behavior in which necessary underlying understanding for performing the specific behavior is absent.
Summary of the Case

In the present section of this chapter, first case of this study was introduced, described and analyzed in the light of the data collected from clinical interviews with Jay. Jay was engaged in several tasks which consisted of construction of a graph for a given dynamic situation and interpretation of given graphical information. Mainly, in these tasks Jay’s behaviors such as graphs he presented, explanations he provided were described in detail and analyzed in order to understand his covariational reasoning. In other words, his mental actions or mental activities engaged in coordinating two variables given in a dynamic situation were investigated to understand his reasoning. His behaviors and corresponding mental actions were analyzed.

(a) Initial Coordination of Change. In all his attempts in given tasks Jay’s behaviors suggested that his mental actions were involved in initial coordination of the two varying quantities. It was observed that he labeled graphs he provided. His verbal expressions also supported that his mental actions were involved in initial coordination of change. For example in response to task 1 shown in figure 4.1, his verbal statements such as “distance changes [faster]…in relation to time” was suggestive of his mental actions related to initial coordination of variables. In other words, this statement revealed his awareness of one variable changes when other variable changes. For another example, when he was presented task 6 shown in figure 4.15, his verbal statement “The area increases as time goes by…” revealed that his mental actions were involved in initial coordination of area of circular ripples change with respect to time. These kind of verbal expressions have been heard while he was working on other tasks too.

(b) Coordinating Direction of Change. Jay’s behaviors also suggested that his reasoning involved in mental actions of coordinating the direction of change by forming an image of the dependent variables increasing or decreasing with respect to changes in the independent variables. In most of the tasks he was observed sketching decreasing and increasing straight lines showing the appropriate direction of change in the output variables with respect to changes in input variables. And his verbal expressions in these tasks also revealed that he constructed appropriate mental images of the direction of change of dependent variables. For example while he was working on task 1 shown in figure 4.1 he stated “The distance is between them decreasing obviously and then they meet and it [distance] is increasing again”. This statement clearly reveals his awareness of direction of change in distance between two people with respect to time. He was observed making this kind of statements in response to other tasks too.
However, he was observed having difficulties in forming an image of the direction of change in some cases. In the analysis of these cases, it was noted that these difficulties of coordinating the direction of change were based on communicating with the irrelevant information by ignoring or violating the stated situation in task: For example, in task 11 (figure 4.27), task 12 (figure 4.29) and task 13 (figure 4.31) his verbal statements such as “…since the container’s shape does not change…same amount of water coming in and coming out…” and “Amount of water coming out is going to remain constant because hole is not changing so same water is coming out…” revealed that he built his reasoning on the size of the hole or shape of the container which were considered as irrelevant information in these tasks. Because neither size of the hole nor the shape of the container were main variables to be considered in these tasks.

(c) Coordination of Amount of Change. Jay was observed having difficulties when reasoning about amount of change in dependent variables in most of the tasks in this study. His behaviors did not seem to support mental actions of coordinating the amount of change in dependent variables while imagining the uniform or fixed changes in the independent variables. His graphs and remarks indicated that the student had difficulties to imagine patterns in the magnitude of the output variables. In other words, he did not seem to form images of the transformation of output variable or images of successive pictures of change in the amount of output variable. Although it is observed that he performed mental enactments of the situation in some tasks, mostly these enactments did not seem to assist him to form accurate images of successive pictures. For example, in task 7 (figure 4.17), the figure he provided revealed that he performed mental enactments of the ladder falling down the wall. He presented succession of pictures of the ladder in different positions. Even though he appeared to engage in a behavior that suggested that he was attempting to coordinate the amount of change of the vertical displacement with respect to equal increments of horizontal displacement, he constructed an inappropriate image. His statement “…if each time he pulls the ladder by the same amount, the top is going to drop the same amount each time…” revealed that he did not appear to construct images representing amount of drop of top of the ladder while imagining the uniform or fixed amounts of increments in the displacement of the bottom of the ladder. It is also observed that sometimes these mental enactments to imagine the transformations of output variable distracted by communication with irrelevant information. For example, in task 11 (figure 4.27) he attempted to relate the magnitude
of change of amount of water in the tank to the size of the hole in the bottom of the tank (“Amount of water coming out is going to remain constant because hole is not changing so same water is coming out”) as discussed before.

In only task 10 shown in figure 4.23, he constructed contiguous line segments to represent the relative amount of changes in the height considering uniform changes in the amount of water (figure 4.26). He was observed placing marks on the figure of the bottle to construct an image of change in height. This behavior supported the mental actions of coordinating the amount of change in the height. He also supported his reasoning by his verbal expressions. It is also observed that this behavior did not occur immediately rather it came after student’s realization that bottles shape had to do something with the graph.

**d) Coordination of Average Rate of Change.** Collection of his behaviors did not seem to be supporting the mental actions of coordinating the rate of change of dependent variable with respect to uniform changes of independent variable. In his all attempts, excluding task 10 (figure 4.23), he was not observed evaluating the change in dependent variable over fixed intervals of independent variable. Although in task 1 (figure 4.1) he provided statements such as “…the distance… is changing as slower rate as they approach each other. As they get apart…distance changes faster…in relation to time…” he did not provide a graph that represents various rate of change of distance. He did not even attempted to demonstrate contiguous line segments on the graph with different slopes representing the rate of change of the distance with respect to the time. Considering his later explanations, it was noted that he used the terms “slower rate” and “faster rate” for the purpose of comparing the two phases of this dynamic situation. He did not seem to examine the rate of change of the distance over fixed intervals of time. In interpretation task shown in figure 4.34, it was observed that his reasoning about cars average speed between consecutive cities was not built upon evaluating the change in distance over uniform changes in time rather he applied some memorized procedures to construct velocity vs. time graph to find out average speed of car between consecutive cities. This behavior did not support mental actions of coordinating the average rate of change in dependent variable considering the uniform increments in the independent variable.

**e) Coordinating the Instantaneous Rate of Change.** Jay did not exhibit behaviors supporting the mental actions of coordinating the continuously changing rate of change of dependent variable over smaller and smaller refinements of average rate of change. He was not observed
providing a graph consists of smooth curves with the awareness of instantaneous rate of change resulted from coordinating the change over smaller and smaller intervals of input variable.

**CASE OF KARL**

**Introduction**

Karl is a mechanical engineering student and it was his second year in college when this study took place. He stated that he was in the honor program in high school. He also mentioned that he is in the honors program at college as well. Then he said that he was also in the band “on top of bunch of honor classes”. And he did think about joining the band in college in his first semester but then he thought that focusing on his studies rather than joining the band was much better idea. He said that he did not know what he was going to face in the classes and he thought that it was not going to be as easy as high school classes. Therefore he decided not to join the band. Also he said that he decided to stay in campus and try to be involved in campus life rather than renting a place out of campus.

When asked how he decided to major in mechanical engineering, he said that his physics teacher in high school have played great role on his decision. He said: “My physics teacher, I had him for two years, he encouraged the more advance students to look at engineering colleges. He showed us bunch of different schools and programs. And he was bringing the videotapes about engineering colleges.” then he said that he was really interested in physics and mathematics. He attended several engineering labs in the first year to get an idea and he said that the mechanical engineering picked his interest.

He took Calculus-I and Calculus-II classes before taking the Calculus-III. He said that since he got good scores on both Advance Placement Test and SAT, he did not have to take College Algebra and Pre-Calculus courses. He said that he got As in both Calculus-I and Calculus-II classes. When I asked him if he liked math or not, he said that he liked the math very much and added “Honestly, I have to say my favorite subject is functions. I like everything about functions such as derivatives, antiderivatives. If you give me just a function and you want me to graph it out, I just totally understand how it works. I do it in seconds”. As seen in his words, he is claming that he is very confident with and very good at graphing the functions. Then I asked him how he found Calculus classes. He said: “instructors of all three classes were very good but they
could have used more drawings, shapes and figures. It helps us a lot to visualize everything. I mean some things were too abstract”.

He was planning to take Engineering Mathematics I and II and Ordinary Differential Equations. He said “that is all as far as I know required for my major”. After graduation from college he said that he wanted to work in a good company rather than staying in college for further graduate studies. I asked him if he could give me some examples of how he could use the knowledge of functions or calculus in his area. He said he wasn’t sure how he could use this knowledge in mechanical engineering and he said that he would figure it out later on his college education.

The last question I asked him was if he could define a mathematical function. He said: “A function is set of points in a coordinate system. It can be in different size and shape and it is not limited to certain numbers of points. I mean there can be infinite number of points.” Then I asked him if he could give some examples of functions. Upon this question he listed several algebraic expressions of functions. All of the examples he stated were linear functions. When I asked him if he could give some more examples involving other ways of representing functions. Instead of giving graphical or tabular representations of functions, he listed exponential functions and polynomials. He was unclear about different representational systems to present a function. Then I rephrase my question and asked him if he could give some examples of graphical representations of functions. He said, “Do you want me to write down a function and graph it?” His question was an indicator to some extent that his understanding of functions is concentrated around formula dependency. It was noted that Karl showed his tendency to utilize algebraic expressions and declared his need for an algebraic expression to be able to construct a graph.
Analysis of the Case

The section provides analysis of Karl’s covariational reasoning based on his cognitive activities and mental actions that have been expressed by behaviors in face-to-face task based interviews. Several tasks have been posed in order to find out mental actions accompanied by collection of common behaviors that were exhibited while responding to these tasks. In these tasks, student was given dynamic functional situations and was asked to construct graphs representing the situations. The student was also engaged in an interpretation task. In this task, he was given a graphical representation of a dynamic functional event and asked to describe the overall behavior of the graph by interpreting important features in the shape of the graph.

The task shown in figure 5.1 required him to draw a graph that represents the distance between two people as a function of time.

Two people start at opposite corners of a room and walk toward each other. As they walk, they both slow down as they get closer to each other, pass, and then they both speed up as they get farther apart. Draw a graph showing the distance between two people as a function of time. Describe your graph.

Figure 5.1 Task-1

After reading the question he started to think out loud: “They start at opposite corners and walk towards each other…the distance between them is maximum…” and then he started drawing the graph shown in figure 5.2 below by claiming:

This [pointing the x-axis] is the time and this [pointing the y-axis] is the distance. Then at the point if I call “t = 0” when they are opposite corners…then the distance is up here [showing a point in the y-axis] and then they get closer to each other [drawing a straight line graph decreasing] then eventually they pass each other…distance will be zero at some point and then they both speed up as they get farther apart…so that means graph is increasing [drawing an increasing straight line] … But as they walk towards each other they both slow down…so that means that the distance that they are…the distance doesn’t decrease as fast so …let me think …
As seen in the above excerpt, first he attempted to label the graph while he was thinking upon the situation. He accurately labeled the graph and then started to sketch straight lines to demonstrate the decreasing and increasing distance over time. But in some point he stopped drawing and said “…but as they walk towards each other they both slow down…so that means that the distance that they are…the distance doesn’t decrease as fast so …let me think …”. He seemed that he was not satisfied with his graph. After a while, I asked him to talk about what he was thinking. He said that he was trying to imagine the situation and then said: “I would like to draw velocity-time graph”.

Karl’s attempt first indicated that the situation did not automatically evoke an image of how the distance between two people change over fixed intervals of time. In fact, the variable “speed” involves in the task explicitly considering the situation, although task involves in coordinating the simultaneous changes of distance and time. In order to figure out his reasoning, I asked him to explain how velocity vs. time graph helps him. Then he said: “I would like to draw velocity vs. time graph to…to relate them…since the derivative of the velocity is the distance”. As seen in his words, his behaviors signified that Karl was having difficulty in forming appropriate images of situation in order to coordinate simultaneous changes in two variables. Therefore he looked for an alternative way or in other words, more concrete and analytic way to draw the distance vs. time graph. However, part of his explanation (“since the derivative of the velocity is the distance”) also indicated that he was communicating with inaccurate information. Since distance is differentiation of the velocity (or speed) over time
rather than derivative of velocity, he was considered to be communicating with wrong information.

Afterwards, he provided a “velocity vs. time” graph shown in the figure 5.3 by claiming: “They will both slow down as they get closer but they will keep walking ... I mean they won’t stop at any point”. Then after this explanation, he tried to translate velocity vs. time graph to produce distance vs. time graph. He drew the axis of distance vs. time graph and labeled it and then looked at the velocity vs. time graph for a quite long time. I asked him to tell what he was thinking and he said: “well...I know this is not going to be like straight line...every time the velocity is decreasing... so distance that they cover every time will be less and less ...”.

![Figure 5.3 Karl’s graph of velocity vs. time](image)

It is observed that although he drew the velocity vs. time graph, he was struggling to make connections and to imagine the situation. Then he started to mark equal time intervals on the X axis. Afterwards, he plotted points by matching the points in the X axis which represents equal time intervals with points in the Y axis which represents relative distances. He again expressed that the distance covered in each time interval will be less than previous interval. Then he provided the following distance vs. time graph shown in figure 5.4 below as his final graph. I asked him if he could explain his reasoning. He said that the velocity vs. time graph guided him to draw this graph. He said:

you see this decreasing line here [pointing the decreasing line he drew in the velocity vs. time graph]...when the velocity is decreasing like that it means the distance between
them decreasing…ummm… but every time they move less and less because speed is decreasing…so the distance between them is decreasing slowly not so fast.

Figure 5.4 Karl’s second graph of distance vs. time

After his explanation, I want him to explain what he meant by “…the distance between them is decreasing slowly not so fast”. Then he said: “I mean they are approaching each other less and less every time so the distance is not changing fast as at the beginning” Considering his verbal explanations and the graph he provided, his behaviors suggested that he was able to form an image of successive pictures of the movement. These images supported the mental actions of coordinating the amount of change of the distance between two people. Although he drew the velocity vs. time graph for the purpose of utilizing the derivative rule, he only used the graph as a mediator to see a clearer picture of the situation to figure out the changes in distance over intervals of time.

Before going into next task, I asked several interpretation questions to get an idea about his awareness of the rate of change presented in his distance vs. time graph shown in figure 5.4 above. First I asked him “considering the graph you provided, can you tell in which point or points do you think the distance is changing most?” He wanted me to state the question again and I repeated the question. After a short period of silence, he said:

The distance is changing most when they [two people] are in here and here [pointing the first and last intervals on the graph]. As they walk towards each other they slow down
so the change in distance... [he was showing the successive lines on the graph] ...the change should be maximum here and here [pointing the end intervals of the graph]

I asked him to explain his reasoning and he said that he determined these parts by looking at slopes of consecutive lines. Then I asked him if he could open up his explanation in terms of the way he used the slope of line in determination of the parts or points where the distance changes most. He said: “I know that if the slope of a line is steeper that means the change is greater...I mean the change in whatever the variable in y-axis” Then I asked him to explain why that is so. Then he said: “Because if the line is steeper that means you are moving more in vertical then you do in horizontal...so every time the distance is changing less and less”. His above verbal explanations (“I know that if the slope of a line is steeper that means the change is greater”) indicated that he might be applying some memorized rules and processes, his later verbal statement (“...if the line is steeper that means you are moving more in vertical then you do in horizontal”) was suggestive of an awareness of the rate of change in graphical context.

Overall his behaviors when constructing the distance vs. time graph shown in figure 5.4 appeared to support the mental actions of coordinating the change over intervals of time. He constructed contiguous line segments to represent the relative changes in distance for the specified amount of time in the graph, his verbal statements (“every time the distance is changing less and less”) indicated that he formed an image of successive transformation of the distance in his mind. His behaviors both verbal statements and the graph he provided supported the mental actions of coordinating the amount of change of one variable with changes in the other variable. In addition, constructing adjacent line segments with different slopes to represent the relative changes in distance with uniform increments of time was also suggestive behaviors of mental actions of coordinating the average rate of change of the function. However, the graph he provided, did not consist of smooth curves, it indicated that he was not able to imagine the continuous changes of distance over smaller and smaller intervals of time. Therefore, his behaviors, both verbal statements and the graph he provided did not support the mental actions of coordinating the instantaneous rate of change of the distance between two people over time.

Upon finishing the first task the second task shown in figure 5.5 below was presented. Again, he was required to draw a graph that represents the distance between two people as a function of time again.
These same two people decide again to start at opposite corners of the room and walk toward each other. But this time they both decide to maintain same steady pace the whole way. Draw a graph showing the distance between two people as a function of time. Describe your graph.

Figure 5.5 Task-2

After reading the second task, he started to draw the distance vs. time graph shown in figure 5.6 by claiming: “They started maximum distance. At exactly halfway point here [showing a point where graph touches the x-axis (where distance between two people is zero)] they meet…the distance between them constantly decrease and as they walk away distance will constantly increase”.

Figure 5.6 Karl’s graph of distance vs. time

The important point is that he did not need to draw velocity vs. time graph. When asked why he did not sketch the velocity vs. time graph as he did in the previous task, he said that it is not necessary to draw it because the situation was not as much complicated as in the previous task. His behaviors suggested that since there is a linear relationship between distance and time, he did not need to imagine the change in distance over equal small intervals of time.

He was observed labeling the both axis accurately. And his behaviors were also suggestive of mental actions of coordinating the direction of change of distance over time.

Similarly, on the third task, shown in figure 4.2.7 below, he was asked to draw a graph representing the distance between two people as a function of time again.
These same two people decide once more again to start at opposite corners of the room and walk toward each other. But this time as they walk they both speed up as they get closer to each other; pass, and then they both slow down as they get farther apart. Draw a graph showing the distance between two people as a function of time. Describe your graph.

Figure 5.7 Task-3

After reading, he said: “another distance vs. time graph”. Then he drew the axis and labeled them. Afterwards, he read the question again and stayed silent for a while then he said: “They again start at maximum distance point. As they walk towards each other they speed up...the distance between them is decreasing ...then they slow down...and the distance is increasing again.” When I asked him to explain how “the distance [between them] is decreasing” or “the distance is increasing [again]”, he said: “they are consistently speeding up so every time speed goes up at a certain rate so the distance is decreasing faster and faster I think…and when they slow down the distance increasing slowly because they are moving less and less every time.”. Then he again wanted to draw a “velocity vs. time” graph. He sketched the velocity vs. time graph shown in figure 5.8 below.

Figure 5.8 Karl’s graph of velocity vs. time

After looking at the graph for a while he started to draw distance vs. time graph. He provided the graph shown in figure 5.9 below. First he drew and named the axis and then he plotted several points on the graph then attempted to construct contiguous line segments on the
graph as he did in the first task with different slopes to reflect relative changes in the distance for fixed amount of time segments. When I asked him to explain his reasoning, he stated that in each consecutive intervals of time they are moving faster and faster therefore in each intervals of time the distance should be decreasing more than the previous interval of time. He again did not make any attempt to use integration rule to figure out the distance vs. time graph. Rather he seemed to use the velocity vs. time graph again as a visual guide to see the picture and to imagine the situation. This behavior again indicated that the situation did not automatically evoke an image of how the distance between two people change over fixed intervals of time. Therefore he looked for a more visual presentation of the situation to construct the images of changes in distance over small intervals of time.

![Figure 5.9 Karl’s graph of distance vs. time](image)

Considering his verbal explanations as shown in the above excerpt (“…distance is decreasing faster and faster…” and “…they are moving less and less every time…”) and the graph he provided, his behaviors suggested that he was able to form an image of successive pictures of the movement. These images supported the mental actions of coordinating the amount of change of the distance between two people over a period of time. His behaviors also supported the mental actions of coordinating the average rate of change of the distance with respect to equal amounts of time or over equally fixed intervals of time. However his behaviors, like in the first task, did not support the mental actions of coordinating the continuously changing rate of
change of distance over time. He was not observed constructing smooth curves by making smaller and smaller refinements in contiguous line segments to represent the continuously changing rate of change.

After the first interview session, Karl was engaged in several other tasks in the second interview three days later. Before presenting the second set of tasks I asked him if he had any questions and comments about the previous interview. He stated that he had no question. Then he said that he really enjoyed the previous interview found the questions very interesting. I again reminded that the purpose of these interviews is to understand his reasoning rather than assessing the work as right or wrong.

In the second interview, I presented several other tasks involving the dynamic situations. Then he was asked to demonstrate the relationship of two variables in graphical representations and explain his reasoning afterwards. Following task shown in figure 5.10 was given as the first task in this session.

5% of chemical X changes into chemical Y every second. Chemical Y never changes back. Starting with all chemical X, sketch a graph that represents the amount of chemical X as a function of time.

Figure 5.10 Task-4

After reading the task he drew the axis first by placing the amount of chemical X in the Y-axis and time in the X-axis. Then he read the task again slowly word by word. After a short period of silence I asked him if he could tell me what he was thinking and he responded: “Well…we have chemical X versus time. And it starts at its highest point and then eventually in certain point it will all become chemical Y”. Then I observed him placing the markings in the X-axis (time axis) showing equal time intervals. Then he stated:

%5 percent change every time… ummm… if this is one second right here [pointing the t=1 point in the X-axis], then %5 will be gone and it will be %95. And then it will continue to decrease like that… every time amount of decrease should be less and less, because if you have specific amount of decrease every time then it would be a straight line. But here I mean in this situation it is not a straight line since it is %5 changes…I mean amount of change is different at each moment…it depends on the amount that there was in the previous situation.
He provided the graph shown in figure 5.11 above by plotting the several points on it and drawing the contiguous line segments to represent the relative changes in the amount of chemical X over equal intervals of time. His verbal statements such as “…every time amount of decrease should be less and less” and “…I mean amount of change is different at each moment…it depends on the amount that there was in the previous situation” have provided evidences of mental actions of coordinating the amount of change of the chemical X with the changes in time. However, since he presented contiguous line segments by connecting the points he plotted; this behavior was not considered as sufficient evidence of his awareness of the rate of change of the amount of chemical X while considering the uniform increments in time. Then I wanted to find out whether his overall image supports the mental actions of coordinating the average rate of change of the amount of chemical X with respect to time or this image is limited to support only coordinating the amount of change over equal time intervals. Then I asked him if he could describe his graph in terms of how it reflected relative changes in the amount of chemical X. Upon this question he confidently stated that the every line segment has different slope by expressing “…as you see each of these lines [pointing the adjacent lines] has different slopes…that means amount of decrease is getting less and less”. These verbal remarks expressed his awareness of the rate of change of the amount of chemical X with respect to time.
However, his overall images of covariation did not seem to support mental actions that involve coordinating the images of continuously changing rate of the amount of chemical X with respect to continuous changes in time. His did not make any attempt to form smooth curve that shows instantaneous rate of change and resulted from smaller and smaller refinements of the average rate of change. It was also observed that there were some inconsistencies between his verbal statements and the way he demonstrate the situation in graphical representation. In the above excerpt he stated: “every time amount of decrease should be less and less, because if you have specific amount of decrease every time then it would be a straight line. But here I mean in this situation it is not a straight line since it is %5 changes…I mean amount of change is different at each moment”. Although this explanation reflected that he might be imagining the continuous changes of amount of chemical X over time, he only presented fixed rate of change in fixed intervals of time and did not attempt to transform contiguous line segments into a smooth curve.

In the same interview session Karl was engaged in second task shown in the figure 5.12 below. In a given dynamic situation Karl was asked to coordinate two variables in graphical representation.

Imagine a pebble is thrown into a lake, creating a circular ripple that travels outward at a constant speed. Sketch a graph that represents the area, A, of the circle as a function of time that have passed since the ball hits the lake.

Figure 5.12 Task-5

When the task is presented, Karl read the question and started to think out loud:
I am just thinking a lake just calm. I am throwing the pebble in to the lake and circle is getting larger and larger. It is increasing every single direction and whole circle is expanding…ummm…the circular ripple travels outward at a constant speed. So I can say that change in r (\(\Delta r\)) over change in time (\(\Delta t\)) would be constant. Because in every moment it travels at a constant speed.

After verbal expressions seen in the above excerpt, he basically provided an increasing straight-line graph shown in figure 5.13 below. Then I asked him to explain why the area is increasing constantly. He said:
As I said before the \( r \) [radius] is expanding constantly so the circle is growing at a constant rate…or it is not…let me think about it….the area is \( A = \pi r^2 \)…when we plug different \( r \) every time the increase in the area would not be constant.

![Figure 5.13 Karl’s graph of area vs. time](image)

Then he wanted to sketch a radius vs. area graph. Afterwards he plugged several integers in to \( r \) and plotted the corresponding area in the graph. After couple plotting he just connected the plotted points and provided following graph shown in figure 5.14. Then he stated;

You see the slopes of the lines [contiguous line segments]…they are different it is getting steeper and steeper…so the area increases more and more every time. But as a matter of fact the area depends on \( r^2 \)…I mean it should be a parabola like in \( y = x^2 \) [giving an example]

![Figure 5.14 Karl’s graph of area vs. radius](image)
Then he changed his graph to a smooth curve as seen in the figure 5.15 below by claiming: “…but as a matter of fact the area depends on \( r^2 \)...I mean it should be a parabola like in \( y = x^2 \)”. Then he said that since the relationship between time and radius is linear, graphs of radius vs. area and time vs. area should be same.

![Figure 5.15 Karl’s second graph of area vs. radius](image)

His behaviors such as drawing a straight line and assuming the linear relationship between speed of the ripple and change in the area suggested a weakness in constructing the appropriate images while communicating with the information given in the situation at first. He automatically presumed an inappropriate linear relationship between change in radius of the circle and change in the area of the circle. After this immediate decision, he wanted to use the formula of the area of a circle to verify the relationship between radius and the area. Although his later behaviors such as constructing adjacent line segments with different slopes may appear to be evidence of coordinating the amount of change in area with respect to change in radius, it is observed that his reasoning was not resulted from constructing successive pictures in mind by imagining the mental enactments of the situation. Although his earlier statements such as “I am just thinking a lake just calm. I am throwing the pebble in to the lake and circle is getting larger and larger. It is increasing every single direction and whole circle is expanding” provided evidence that he was imagining the successive pictures of expanding circles, it is obvious that these successive pictures did not help him to coordinate amount of change in the area with respect to time. Rather he used analytical processes to find out appropriate relationship between radius and area and similarly between time and area. In the same way, although he finally
provided a smooth curve, this behavior did not demonstrate that his reasoning was based on understanding the continuous changes in the rate of change of the area for the entire domain of the function resulted from smaller and smaller refinements of the average rate of change. His verbal statements such as “…but as a matter of fact the area depends on \( r^2 \)...I mean it should be a parabola like in \( y= x^2 \)” indicated that he applied some memorized rules.

The third interview took place one week after the second interview. Again Karl was engaged in several tasks in order to gather further information about his reasoning. In these tasks, the student was given dynamic functional situations again and was asked to construct graphs representing the situations. The task shown in figure 5.16 below was presented as first task of the interview session.

Tom sees a ladder against a wall (in an almost vertical position). He pulls the base of the ladder away from the wall by a certain amount, and so forth. Each time he does this he records the distances by which the top of the ladder drops down. Do the amounts by which the top of the ladder drops down remain constant as Tom repeats this step; or do they get bigger, or do they get smaller? Explain

Figure 5.16 Task-6

After reading the question, Karl immediately drew a picture (figure 5.17) representing the situation then he started talking while he was working on this picture. He said:

Okay…So if it [the ladder] is almost in vertical position. I am guessing he is right here. If he pulls the base of the ladder by certain amount…since the length of the ladder stays same…so it is just going to be similar triangles by how much he moves it until it is basically horizontal and….but the amount of drops as he repeats pulling the base depends on how much he moves the base of the ladder every time…Without knowing the amount that he moves the base of the ladder every time I cannot say whether top of the ladder drops down constantly or bigger or smaller.
Then I told him that he [Tom] pulls the base of the ladder same amount every time. Then he quickly replied “If the base of the ladder moves same amount every time, the top drops down constantly.” After his response, in order to understand the pattern of his reasoning more clearly, I asked Karl if he could explain how the situation would be different if the base of the ladder was moved bigger and bigger or smaller and smaller amounts every time. He basically drew pictures of different situations shown in figure 5.18 while he was explaining:

As you see here if the base is moved by bigger and bigger amounts every time the amounts of drops would be bigger and bigger too. If the base is moved by smaller and smaller amounts, the ladder drops down by smaller and smaller amounts.
I reminded him his expression “since the length of the ladder stays same…so it is just going to be similar triangles by how much he moves it until it is basically horizontal” shown in the excerpt above then I asked him to explain how the idea of similar triangles works in this situation. After a short silence he provided following picture shown in figure 5.19 and stated that the amount of horizontal displacement “a” is proportional to the vertical displacement “b”. then he said that if the base of the ladder moves in the amount of “a” every time, top of the ladder drops down in the amount of “b” every time.

Figure 5.19 Karl’s representation of constant changes in vertical and horizontal displacement of the ladder

Considering all his explanations shown in the above excerpts and the graph he provided, his images of covariation is limited to direction of change. In other words, his behaviors and corresponding reasoning suggested that his mental actions did not go beyond the coordinating the direction of change of amount of displacement of the top of the ladder in the given situation. He consistently relied on a linear relationship between the amount of change in horizontal position and of vertical position of the ladder. These behaviors suggest that Karl’s strict reliance on linearity, keeps him away from constructing appropriate images of covariation supporting the mental actions of coordinating the amount of changes in horizontal and vertical displacements of the ladder.

After this task, he was given the next task shown below in figure 5.20. Basically, this task is very similar to the previous task (figure 5.16) with some changes. Instead of amount of vertical change, student was asked to analyze the amount of change in the speed of top of the ladder as top of the ladder drops down.
Newt, the science nerd, then comes along and puts wheels on the bottom of the ladder. He connects them to a motor so that the bottom rolls away at a constant, but very slow, speed. Does the top of the ladder move down at a constant speed; or does it speed up, or does it slow down? Explain

Figure 5.20 Task-7

In the second task, Karl appeared to be confused. In one hand he expressed that the situation is “almost same” with the previous task and he maintained the same reasoning by claiming that if the bottom of the ladder moves at a constant rate top of the ladder would drops down at a constant speed. But then he said that he was trying to visualize the situation. I encouraged him to state verbally what he was visualizing. He said:

Okay. I am just trying to visualize the situation in my head… actually I think…initially it would move at a quicker speed so then as time went on it slows down. Because…then there would be a point where the distance from the wall …I call it X …and the distance from the ground…I call it Y… will be equal. After this point the top of the ladder slows down I think.

After these expressions, I asked Karl if he could explain why the top of the ladder moves down faster at the beginning and slows down later on. He said that he was uncertain and he just told me how he visualized the situation. The important point is although his visualization of the situation provided evidence that his mental actions did not support constructing appropriate images of covariation, it was not consistent with his reasoning based on assuming a linear relationship between two variables. When asked to explain this inconsistency, he said:

Well…When I think theoretically speed of top of the ladder should be constant same with the speed of the bottom of the ladder. Because when you move an object, speed should be same in every part of that object. But when I try to visualize it, it seems it is going down faster at the beginning and then slowing down.

Considering all the evidences provided by his remarks shown in above excerpt, “…when you move an object, speed should be same in every part of that object…” his behaviors suggested signs of using a general principle for describing the relationship between two variables (speed of the bottom of the ladder and speed of the top of the ladder) even though the principle does not apply the given situation.
Karl’s behaviors in this task did not demonstrate any evidence of constructing appropriate images of the situation and again his behaviors and reasoning did not support mental actions that go beyond the coordinating direction of change.

In the last part of the “ladder problem”, Karl was asked to sketch a graph that represents the relationship between the horizontal and vertical position of the ladder as it slides down a wall, starting at a vertical position and finally resting on the ground. He provided two graphs shown in figure 5.21. He said that the first graph he provided was representing the situation if top of the ladder drops down at a constant rate of speed and second one was representing that top of the ladder drops down at a smaller and smaller rate of speed.

![Figure 5.21 Karl’s two graph of vertical vs. horizontal position of the ladder](image)

Although both graphs he provided were consistent with the images he produced in his mind, his reasoning revealed that his mental enactments of the situation did not allow him to coordinate an accurate relationship between to variables. It is also important to note that besides the problems in constructing accurate images of relationship between two variables, strict reliance on some general principles of science played significant role in his reasoning.

After the ladder problems, Karl was engaged in another task in the third interview. The task shown in figure 5.22 required him again to represent the relationship between two variables in a graphical context.
Imagine this bottle filling with water at a constant rate. Sketch a graph that represents height of the water as a function of amount of water that is in the bottle.

Upon reading the problem shown above, he started talking about the situation while he was thinking. First of all, he wrote down two formulas one for the spherical part of the bottle \( V_s = \frac{4}{3} \pi r^3 \) and the other for the cylindrical part \( V_c = \pi r^2 h \) as seen in the figure 5.23 by claiming that volume of the spherical part of the bottle does not depend upon the height \( h \). Then he said that he could not utilize the formula to find out the relationship between amount of water and height and stated:

Figure 5.22 Task-8

Figure 5.23 Karl’s work on the figure
The bottle actually is like 3-dimensional so we have specific volume I’ll do amount of water and height graph. Instead of amount of water I’ll call it volume. Initially the volume will be at “0” for the height “0”… I will call it H not or Ho and …here is the H final [placing marks on the figure for presenting the Ho and H final]. Eventually we will have maximum volume. Since the radius of… ummm… at each different time there is a certain radius “r” and it is increasing. When the height goes up volume slows down because the distance between two walls is increasing as height goes up and eventually going back down because it is sphere. So until it gets to the maximum distance between two sides of the bottle, the rate of change in volume is going to be slowing down and it will be speeding back up. And then up here [showing the point where cylindrical part starts] the rate of change would be constant. I mean initially the volume will increase at slower and slower rate and then right here [showing the mid point of the sphere] if I call this Hd… ummm… that would be the point where the graph changes from greater rate of increase to… I mean it slows down and it speeds back up.

While he was expressing what he was thinking he provided a graph as seen in the figure 5.24. Considering his verbal expressions such as “…as height goes up volume slows down” and the graph he provided (volume as a function of height), it is observed that Karl constructed images of height changing constantly and volume changes at a varying rate. This image indicated that Karl has switched the roles of independent and dependent variables. After assuming the height is changing constantly he put the variable “Height” aside and introduced another variable and tried to explain the rate of change in volume by changes in the distance between two walls of the bottle. However, his reasoning appeared to result in constructing inappropriate graph. In his verbal statements he said: “When the height goes up volume slows down because the distance between two walls is increasing as height goes up”. Although he presented accurate relationship between height and the distance between two walls “…the distance between two walls is increasing as height goes up”, he presented inconsistencies in his reasoning by saying volume slows down because of this relationship. Because if the variable “height” is taken as constant, amount of water or volume should be increasing at an increasing rate of change over every constant increments of height since the container is getting wider and wider.
Figure 5.24 Karl’s graph of volume vs. height

Afterwards I wanted him to provide more explanation about why the volume is slowing down. He basically repeated his thoughts and said that as the distance between walls is increasing the rate of increase in volume is getting smaller and smaller until the midpoint of the bottle. Then I asked him “how do you know?” After a short silence he said “since every time distance is getting bigger the bottle is getting wider therefore volume is going up less and less.” Considering this statement “volume is going up” he appeared to be distracted by the image he constructed in his mind and treating volume as height. In other words, his reasoning revealed that he was measuring the change of volume as changes in the level of water, which is in fact the measure of height.

Although his behaviors such as constructing fixed intervals of height in the X axis and representing the relative changes in the amount of water or volume over these fixed intervals of height seemed to be supporting the mental actions of coordinating the amount of change of independent variable over dependent variable, switching the roles of variables volume and the height and communicating with the inaccurate information prevented him to provide appropriate coordination of covariation between volume and the height.

Overall his behaviors such as his verbal expressions and the graph he provided suggested that inconsistencies in his reasoning prevented him to construct accurate images or successive
pictures of height changing at a varying rate as the amount of water in the bottle increases constantly. Therefore his mental actions were only limited to the coordination of the direction of change.

After finishing the task I wanted to get more information about his reasoning on same kind of task and I have created another task shown in figure 5.25 below.

Imagine this water tank filling with water at a constant rate. Sketch a graph that represents height of the water as a function of amount of water that is in the tank.

After reading the task, Karl asked several questions to make sure what he was required to do. First he asked if this was a cross sectional area of the tank. I told him that the tank consisted of several cylindrical parts with different radiiuses and this is the cross sectional view of the tank. Then he asked me if I wanted him to sketch a graph representing height as a function of amount of water in the tank. Then he started thinking upon the situation:

Let me call these Ho, H1, H2, H3 and H final [by placing marks on the figure to represent heights of different cylindrical parts]…between these different distances the volume increases at a constant rate… the volume is always increasing at a constant rate but between Ho and H1 it is certain constant rate and then there is a different one and different one and so on and here [pointing the first cylindrical part or height between Ho and H1] the volume is actually increasing the slowest and then little bit faster here [pointing the second cylindrical part] and faster and faster here [pointing the third cylindrical part] and finally the greatest rate of increase up here [pointing the cylindrical part at the top]. Now let me draw the graph.
As obviously seen in the above excerpt he maintained the same reasoning as he had in the previous task. Instead of thinking “height” as a function of amount of water in the tank or “volume”, as he called it, he constructed an image in his mind by considering the “volume” as a function of “height”. He was also observed violating the conditions of the task. First he said “between these different distances the volume increases at a constant rate… the volume is always increasing at a constant rate” which is consistent with the statement used in the task but his following statement provided some evidence that he was dealing with inaccurate information by violating the conditions in the task such as saying “the volume is always increasing at a constant rate but between Ho and H1 it is certain constant rate and then there is a different one and different one and so on”. This violation resulted in dealing with inaccurate information and forming inappropriate images of relationship between amount of water in the tank and the height. Then with parallel to his thoughts he provided a graph shown in figure 5.26 below.

![Figure 5.26 Karl’s graph of volume vs. height](image)

I then asked him if he could explain why amount of water in the tank is increasing at increasing rate across each cylindrical part of the tank. He confidently replied:

Because in the first part [bottom cylindrical part of the tank] the distance between the walls I mean radius of the cylinder is wider than one in the second part and the radius in the second part is wider than one in the third part and so on so forth…so volume is increasing slowly then it increases faster and faster rate …I mean since the first part is
wider water increases at a slower rate… when I visualize the situation I see water goes up slowly in the first part and little faster in the second part and so on.

His statements revealed that he was actually treating the volume as height. In other words, he imagined the change in volume as change in the water level. After a while I asked him if he could sketch a graph representing volume as a function of time. Then he provided the graph shown in figure 5.27 claiming:

So instead of height being like spaced out evenly, we should have equal time intervals…umm… I would say that here it is at the slowest rate of increase but it is still constant because radius does not change. Then here the rate is faster so I drew as a greater slope. They are all straight lines. So it should be the same as this one [pointing the previous graph he provided].

![Figure 5.27 Karl’s graph of volume vs. time](image)

When we look at the verbal expressions seen in the above excerpt, it is obvious that his mental actions and his reasoning prevented him from constructing accurate images of the relationship between two variables. In this case he believes that the volume or amount of water in the tank is increasing at different constant rates (with increasing rate of change) in each different intervals of time. Although in the statement of the task shown in figure 5.25 it is
expressed that the water comes in at a constant rate, Karl was observed violating this situation by constructing an image of volume changes in varying rate over intervals of time.

The next interview session took place 3 days after the third interview. In this interview session Karl was engaged in several tasks as he has been previous interviews. For the purpose of gathering more information about his covariational reasoning in graphical context, he was again asked to construct graphs of given situations which are involved in simultaneous changes of two variables. The first task shown in figure 5.28 was given after reminding the interview protocol again.

Imagine this water tank is full of water and there is a hole in the bottom of the tank. Water leaves the tank at a rate proportional to the height of the water remaining in the tank. The tank is full when the hole is opened. Sketch a graph that represents amount of the water in the tank as a function of time.

After reading the question, he seemed little confused and he read the question again out loud. Then he started talking about he was thinking about the situation. He said:

The amount of water in the tank should be decreasing at a constant rate because there is a certain hole in the bottom and the hole is not changing. But… ummm …here [pointing the statement of the task] it says “Water leaves the tank at a rate proportional to the height of the water remaining in the tank.” So….it can not be a constant rate because amount of water that leaves the tank depends on the height which is varying. I think this means more height there is then faster it [water] is going to run out. So initially it is going to rush out faster because there is more height. Over here [placing a mark on the figure to show the lower height] when the water is very low there is going to be this much volume here then until it gets to final point it is going to be slower and slower.
After these explanations he basically drew the axis and placed equal time intervals on the X axis then he said that for every interval of time water would leave the tank at a rate smaller than in previous interval. Finally he provided the graph shown in figure 5.29 below.

![Figure 5.29 Karl’s graph of volume vs. time](image)

His verbal remarks such as “...this means more height there is then faster it [water] is going to run out.” and graph he provided suggested that his reasoning is resulted from the mental actions of coordinating the amount of change in the amount of water in the tank over intervals of time. His behaviors also revealed that his reasoning seemed to support the mental actions of coordinating the average rate of change of the amount of water in the tank for equal intervals of time. While he was drawing the contiguous line segments with slope of each segment represent the relative rate for specified time interval, he said: “...until it [volume of the water] gets to final point it [water leave] is going to be slower and slower.”

However he did not provide any smooth curve to represent the continuously changing rate of change of the amount of water in the tank while over time. He did not make any attempt to refine contiguous line segment over smaller and smaller intervals of time and construct a curve which is resulted from considering the changing nature of the rate while imagining the amount of water changing continuously.

The next task shown in figure 5.30 below was presented to Karl upon completion of first task. The context of the problem was similar with previous one. He was again engaged in a same
kind of task which required him to sketch a graph that represents the situation given in the problem.

Again there is a hole in the bottom of the tank. The tank starts off empty and water is filling in at a constant rate where the level increases. Water leaves tank at a rate which is proportional to the height of the water in the tank. Sketch a graph that represents relationship between amount of water in the tank and time.

![Figure 5.30 Task-11](image_url)

After reading the question two times, Karl said that the problem was little more complicated than the previous one. And he stayed silent for slightly long time. Then I asked him if he can state his thoughts verbally while he was thinking upon the situation. He said:

Okay. So…it is going to start up “0” the initial time. I think it is going to take really really long time for it to eventually slow down…ummm…Since we do not have any numbers here you can’t really tell if it is going to level off where the rate of water coming in is equal to rate of water going out. I actually think that there would probably be sometime where it would where the water level off. I will call \( V_l \) where the volume levels off. I think it would be increasing faster at first until the point where height is big enough so that the rate of water coming in and rate of water going out will be equal.

After the verbal expressions as seen in the excerpt above, he provided a graph shown in figure 5.31 below. Although he mentioned that the water increases at greater rate and it slows down until it gets to certain point, he was observed having difficulties to demonstrate his accurate reasoning pattern in the graphical representation. Karl was observed having initial images of amount of water in the tank changing and the rate of change is getting smaller and smaller but when we look at the graph he just provided line segments to represent the relationship between variables. His verbal expressions such as “…I think it would be increasing faster at first…” and “…I think it is going to take really really long time for it to eventually slow down…” are some evidences suggesting that his reasoning supported the mental actions of coordinating the rate of change of the amount of water in the tank over time.
Figure 5.31 Amount of water in the tank vs. time

Following task shown in figure 5.32 was presented as the last task in the forth interview session to obtain further information about his reasoning pattern supported by his mental actions. The task again is very similar to previous “water tank” tasks. Once again Karl was required to sketch a graph representing the relationship between two variables.

There is a small hole part way up to the side of a water tank and no hole in the bottom. The tank stats off empty and water is poured in at a constant rate where the level increases. Water leaves the tank at a rate which is proportional to the height of the water above the hole. Sketch a graph that represents amount of water in the tank as function of time.

Figure 5..32 Task-12

Karl basically used the same reasoning with the previous task. After reading the task he said that the task is very similar with the previous one and the logic is the same with minor differences. Then he started talking upon the task:

Now then. There is going to be some point I just call it (h) where the water starts to rush out here [placing a mark on the figure to represent the hole on the water tank]. So up
until that point...ummm...let me call it \( t \) for the time it [water level] takes to get there.

At that point it is going to be look like exactly the other graph. Up until that point water is increasing faster than it increases after that point.

He then provided the graph shown in figure 5.33 below. Considering his verbal expressions and his graph, again it is observed that he exhibited difficulties demonstrating his reasoning in the graphical representation. Although his reasoning pattern revealed that his mental actions involved in coordinating the changing rate of change of the amount of water in the tank over time, he failed to demonstrate the changing rate of change accurately in graphical representation.

Afterwards I asked him if he could sketch another graph representing amount of water in the tank as a function of height. I reminded him that all the conditions in the task were same. He read the statement of the task again and stayed silent for a while and then he said: “it seems more complicated”. Then he continued:

Again I would say that volume increases as constant rate of increase then it would level off before the tank filled up. If the tank is big enough, it [volume] is going to increase at a slower rate again. I am thinking that the graph would still be same. Because volume increases same...I mean when the height increases volume will increase in the same
way…first it [volume] increases faster then it increases but at relatively slower rate. Then finally it will level off.

After these explanations he provided the same graph as you see in figure 5.34 below by claiming that the situation is same and volume increases in the same way. His behaviors revealed that he had difficulties in constructing the accurate images of two variables.

![Figure 5.34 Karl’s graph of amount of water in the tank vs. height](image)

The task shown in figure 5.35 below was given as the last construction task to gather further evidences regarding to Jay’s covariational reasoning.

![Figure 5.35 Task-13](image)

Imagine air is pumped into spherical balloon at a constant rate. Sketch a graph that represents radius of the balloon as function of amount of air in the balloon.

After reading the task Karl starts to think out loud. At the beginning he seemed little confused. He said:
I have radius and... as air is being pumped in at a constant rate.... ummm... since it is spherical balloon, the radius is going to be changing at a constant rate as well... and I am just trying to think how it goes... but... I think the radius is going to be increasing faster at the beginning and it slows down when it gets to final volume that is how I visualize. Because initially when you blow up the balloon there is more space so radius increasing faster then when it gets to the end it [radius] is still increasing but slowly since there is not much room.... ummm... But I am not hundred percent sure... let me think.... ummm... I think I confuse myself [smiling]... I am going to write down the volume function [he wrote $V=\frac{4}{3} \pi r^3$]. This is much clear... since $4/3$ and pi [$\pi$] are constants, volume depends on $r^3$.

After his statements he basically plugged several $r$ values into the volume formula and came up with corresponding values of volume and sketched a graph of adjacent line segment by plotting the points and connecting these points. As seen in the above excerpt first he made an effort to visualize situation in his head. He constructed an image of radius increasing faster at the beginning and increasing slower at the end. First construction of this image suggested that he was mentally estimating the rate of change in radius over equal intervals of amount of air pumped into balloon. But his following statements such as “...there is more space so radius increasing faster...” and “...it [radius] is still increasing but slowly since there is not much room...” suggested that his reasoning was not built upon mental actions of coordinating the rate of change of radius over small intervals of amount of air pumped in. He was not observed conceptually coordinating the rate of change in radius rather he made deductions by communicating with irrelevant information such as “...there is more space” and “...there is not much room....”.

Afterwards he wanted to apply more analytical method to figure out how radius changes with respect to amount of air in the balloon by applying the volume formula. He basically stated that the volume of the spherical balloon depends on $r^3$ then plugged several $r$ values to obtain corresponding volumes. These behaviors suggested that he was applying memorized procedures rather than conceptually coordinating the two variables. He was also observed switching the roles of dependent and independent variables. This is an indication of student’s tendency towards using algebraic formulas or equation or memorized procedures to demonstrate in a graphical representation when run into a familiar situation.
Karl was engaged in interpretation task in the fifth interview. In the interpretation task the student was given a graphical representation of a dynamic functional event and asked to describe the overall behavior of the graph by interpreting important features in the shape of the graph. Basically Karl was asked to interpret global features of a given graph such as general shape of the graph, pattern change (interpreting direction of change or changes from concave up to concave down or vise versa), intervals of increase or decrease and intervals of extreme increase and decrease (e.g. determining the part in which dependent variable increasing or decreasing most).

Below is the distance vs. time graph of a car traveling through city A to city E.

![Graph of distance vs. time](image)

Figure 5.36 Task 14

I presented the task shown in figure 5.36 above. Then I asked him to describe the overall graph. He said:

We have distance vs. time graph as car traveling...so it is going through city A to city E...ummm... we have polynomial function here...I also see that between A and B looks different than between B and C...it [function] looks like piecewise because it is increasing here [pointing the interval of A and B] and levels out here [showing the maximum point just after point B]. I think that is all.

In his descriptions as seen in the above excerpt he did not provide detailed description including how distance changes over time or information about positions of the cities or. He only tried to classify this function by assuming “…we have polynomial function here…” or “…it
[function] looks like piecewise…” Then I posed the second question: “Between which consecutive cities is the average speed of the car maximum?” He said: “speed is... ummm... speed is represented by the integral of the distance... trying to think of that... speed is first derivative of acceleration. The question is between which consecutive cities the integral is biggest... ummm... I cannot compute the exact area under these curves but I will guess and say that... since velocity is distance over time... ummm ...I am confused. I think it is wrong. I am having trouble to remember the correct relationship. I am going to say between A and B. It is complete guess. I am not sure but it seems A and B.”

His verbal expressions indicated that he was communicating inaccurate information such as “…speed is represented by the integral of the distance…” and “…speed is first derivative of acceleration…”. Although information he tended to use is incorrect, this behavior is also suggestive of Karl’s tendency towards using memorized rules or procedures dominantly used in classes. It is also observed that because of strict tendency to use memorized derivative or integration rules he did not even attempt to evaluate distance over small intervals of time even though he was heard saying “…since velocity is distance over time…” (see excerpt above). He did not attempt to sketch velocity vs. time graph. I asked him if he could sketch it. But he responded “It seems complicated... ummm... actually without a specific function I cannot calculate the areas under the curves.” His expressions revealed his reliance on the need for algebraic representation of a function to graph it. In addition, his strict reliance on inaccurate information such as “…speed is represented by the integral of the distance…” and “…speed is first derivative of acceleration…” prevented him from interpreting the velocity.

Summary of the Case
In this section, second case in the study was introduced. Data were described and analyzed under the lens of theoretical underpinnings of the study. Karl was engaged in several tasks which consisted of construction of a graph for a given dynamic situation and interpretation of given graphical information. Mainly, in these tasks Karl’s behaviors such as graphs he presented, explanations he provided were described in detail and analyzed in order to understand his covariational reasoning. In other words, his mental actions or mental activities engaged in coordinating two variables given in a dynamic situation were investigated to understand his reasoning. His behaviors and corresponding mental actions were analyzed.

(a) Initial Coordination. In all his attempts in given tasks Karl’s behaviors suggested that his mental actions in all tasks supported initial coordination of the two varying quantities which is described by Carlson et al (2002) as constructing images of two variables that are in covarying relationship. According to Carlson et al. (2002), this level is called “Mental Action 1” in the proposed framework of “Mental Actions of the Covariation”. Karl was observed labeling all the graphs he provided and mostly treated the independent and dependent variables in appropriate ways. It is also noted that he also made verbal indications such as “This [pointing the x-axis] is the time and this [pointing the y-axis] is the distance” and “Well…we have chemical X versus time…”.

(b) Coordinating the Direction of change. Karl’s behaviors suggested that in his all attempts while working on the tasks his reasoning appeared to be involved in mental actions of coordinating the direction of change by forming images of the dependent variables increasing or decreasing with respect to changes in the independent variables. This is what Carlson et al (2002) indicates as “second level” or “direction level” or “Mental Action 2” in the proposed framework of Mental Actions of Covariation”. Collection of Karl’s behaviors (graphs he provided and his verbal statements) indicated that his images of covariation supported the mental actions of coordinating the direction of change in the dependent variable while considering the changes in the independent variable. For example, in task 9 he indicated: “…the volume is always increasing…” while he was working on the task. In addition to verbal statements, all the graphs he constructed revealed that his mental images supported the coordination of the direction of change in independent variables.
(c) Coordination of Amount of Change. Karl’s behaviors appeared to support mental actions of coordinating the amount of change in dependent variable while imagining the changes in the independent variable. In the framework of “Mental Actions of the Covariation” proposed by Carlson et al. (2002), this level is called “third level” or “quantitative coordination” or “Mental Action 3”. In most of the tasks, he was observed plotting points on the graphs by placing equal intervals in the X-axis and placing corresponding intervals appropriately representing the relative amounts in the Y-axis.

While he was representing the amount of change in the dependent variables on the graphs he also verbally expressed his awareness of how the amount of dependent variables changes over fixed intervals of independent variables. For example, while he was representing the relationship between two peoples distance from each other and time in task-1 shown in figure 5.1, he was observed placing eight equal time intervals on the X-axis and placing relatively changing intervals on the Y-axis to demonstrate change in the distance. His verbal expressions such as “…they are approaching each other less and less every time so the distance is not changing fast as at the beginning” and “… so distance that they cover every time will be less and less …” are significant utterances revealing his mental actions of coordinating the amount of change in the distance over fixed intervals of time.

Similarly, while working on the task 10 shown in figure 5.28, he constructed a graph again representing the relative amounts of water remaining in the tank over fixed intervals of time by plotting points and constructing contiguous line segments or secant lines. He was again observed placing equal time intervals on the X-axis and placing varying intervals on the Y-axis to show the relatively changing amounts of water in the tank. His reasoning appeared to support mental actions of coordinating the amount of change in the volume over fixed intervals of time. In addition to his graph, his verbal statements revealed that his reasoning involved in coordination of amount of change in the dependent variable. For example, his verbal remark such as “…this means more height there is then faster it [water] is going to run out.” suggested that his reasoning is resulted from the mental actions of coordinating the amount of change in the amount of water in the tank over intervals of time. In task 4 shown in figure 5.10, he again constructed adjacent line segments on the graph (figure 5.11) with different slopes by plotting points. And his verbal expressions such as “… every time amount of decrease should be less and less…” and “…I mean amount of change is different at each moment…” were indicated that his
reasoning reflected by his mental images supported mental actions of coordinating amount of change of chemical X over fixed intervals of time.

However in some cases he was observed having difficulties in constructing appropriate mental images. It is also noted that these inappropriate images he constructed in his mind and strict reliance on these images prevented him to coordinate amount of changes in the dependent variables. In these cases, as it is revealed in his reasoning, his mental actions were appeared to be limited to only coordination of direction of change. In other words, his behaviors did not support the mental actions beyond the coordination of direction of change. For example, in the task 8, his verbal expressions such as “…as height goes up volume slows down” and the graph he constructed (volume as a function of height) suggested that Karl constructed images of height changing at a constant rate and volume changes at a varying rate although it is stated in the task that he was asked to sketch a graph representing height as a function of amount of water in the tank as water fills in the tank at a constant rate. He was observed first switching the roles of dependent and independent variables then violating the condition of the task by thinking the volume changing at varying rate. In the following task shown in figure 5.25, he constructed similar mental images and maintained same reasoning. Although in the task it is stated that water tank is being filled with water at a constant rate, his verbal expressions such as “the volume is always increasing at a constant rate but between Ho and H1 it is certain constant rate and then there is a different one and different one and so on” revealed that he has constructed an image of amount of water changing at varying rate across the different levels of the tank. He was observed again switching the role of dependent and independent variables. Instead of thinking “height” as a function of amount of water in the tank or “volume”, as he called it, he constructed an image in his mind by considering the “volume” as a function of “height”.

Although it is observed that he performed mental enactments of the situation in some tasks, in some cases these mental enactments did not seem to assist him to form accurate images of successive pictures. For example, in task 6 (figure 5.16), the figure 5.17 revealed that he performed mental enactments of the ladder falling down the wall. He presented succession of pictures of the ladder in different positions. Even though he appeared to engage in a behavior suggesting that he was attempting to coordinate the amount of change of the height with respect to equal increments of horizontal distance, he constructed an inappropriate image. It is also observed that sometimes these mental enactments distracted by communication with irrelevant
concepts or a misuse of a concept or physical principle. For example in task 7 shown in figure 5.20, he was asked to explain the speed of the top of the ladder when the bottom rolls away at a constant speed. In response to this task, his statement “When I think theoretically, speed of top of the ladder should be constant same with the speed of the bottom of the ladder. Because when you move an object, speed should be same in every part of that object,” revealed that he was communicating with an irrelevant concept or principle by misusing the principle. This reasoning prevented him from constructing appropriate mental images to be able to coordinate amount of change in the speed of the top of the ladder while considering the fixed increments in the speed of the bottom of the ladder.

(d) Coordinating the Average Rate of Change. Karl’s behaviors in response to tasks seemed to be supporting the mental actions of coordinating the average rate of change of dependent variables with respect to independent variable in most cases. He was observed evaluating the change in dependent variables over fixed intervals of independent variables by constructing contiguous line segments or secant lines on the graph, with the slope of each segment adjusted to reflect the relative rate for the specified amount of dependent variables. Carlson et al (2002) specifically pointed out that mental actions of coordinating the amount of change and mental actions of coordinating the rate of change might both result in constructing contiguous line segments on the graphs.

While analyzing his behaviors carefully it was noted that his verbal expressions clearly indicated his awareness of the rate of change of the dependent variable or output while uniform increments of the independent variable or input. For example, in task 1 shown in figure 5.1 after constructing the graph he stated “…if the slope of a line is steeper that means the change is greater…I mean the change in whatever the variable in y-axis” Then I asked him to explain why that is so. Then he said: “Because if the line is steeper that means you are moving more in vertical then you do in horizontal…so every time the distance is changing less and less”. For another example, in task 4 shown in figure 5.10 again he constructed adjacent lines or secant lines and he stated “…as you see each of these lines [pointing the adjacent lines] has different slopes…that means amount of decrease is getting less and less”. These verbal remarks expressed his awareness of the rate of change of the amount of chemical X with respect to time. Generally he constructed an image of dependent variable changes with respect to an imagined fixed amount of independent variable.
(e) Coordinating the Instantaneous Rate of Change. Karl’s behaviors in response to all tasks in this study did not seem to be supporting the mental actions of coordinating the continuously changing rate of change of dependent variable with respect to changes in independent variables. He was not observed constructing smooth curves to represent continuously changing rate of change in these tasks. As an exception in task 5 shown in figure 5.12 he constructed a smooth curve (figure 5.15) that was concave up to represent the changes in the area of a circular ripple. But when he was asked to provide rationale for this construction he stated “…as a matter of fact the area depends on $r^2$…I mean it should be a parabola like in $y = x^2$”. This statement indicated that he was applying a memorized rule. Although his construction of a smooth curve gave an appearance of engaging in coordinating the continuous changes in the dependent variable, his verbal expressions revealed that his behavior was not more than applying a memorized procedure.

According to Carlson et al (2002) such behaviors (constructing smooth curves) would be considered to be supporting the mental actions of coordinating the continuously changing rate of change only if these behaviors demonstrate an understanding that the instantaneous rate resulted from smaller and smaller refinements of average rate of change. In a few words, he was not observed providing a graph consists of smooth curves with the awareness of instantaneous rate of change resulted from coordinating the change over smaller and smaller intervals of input variable.
CROSSCASE ANALYSIS

In the previous section of this qualitative study, Jay’s and Karl’s behaviors when representing and interpreting dynamic function events have been analyzed and reported in details in order to understand patterns of their reasoning, cognitive activities or mental actions in light of covariational framework developed and introduced by Marilyn Carlson. Summaries have also been provided at the end of each case taking into consideration of Carlson et al (2002)’s “Mental Actions of Covariation” framework. In this section findings for each student were compared considering five mental actions defined comprehensively in Carlson et al (2002)’s framework mentioned above. In addition, several assertions were made and reported at the end of the section with regards to findings in two cases.

(a) Initial Coordination. Observed behaviors of both students in this study supported the mental actions of coordinating two variables changing with respect to each other. Both Jay and Karl have been observed drawing X and Y-axis and labeling the axis. They also verbally expressed their awareness of one variable changes with respect to changes in other variable. Verbal expressions such as “distance changes [faster]…in relation to time”, “The area increases as time goes by…” “Well…we have chemical X versus time…” are some examples indicated their mental actions of initial coordination of two variables changing with respect to each other.

(b) Coordination of Direction of Change. With regard to coordinating the direction of change, although both students’ behaviors supported their awareness of direction of change, Jay was observed having difficulties in forming an image of the direction of change in some cases. In the analysis of these cases, it was noted that these difficulties of coordinating the direction of change were strongly bounded with the strict reliance on communicating the irrelevant information. For example while working on tasks 11, 12 and 13 he said: “Amount of water coming out is going to remain constant because hole is not changing so same water is coming out…”. His reason “…because hole is not changing…” was considered as irrelevant information here since the size of the hole has nothing to do with situations stated in these tasks. On the other hand Karl’s behavior appeared to be supporting the mental actions of coordinating the direction of change in all tasks presented in this study. Verbal expressions of both students also revealed that they constructed appropriate mental images of the direction of change. For example “The distance is between them decreasing obviously and then they meet and it [distance] is increasing again” and
“…the volume is always increasing…” are some examples clearly indicating their awareness of direction of change in one variable considering the changes in the other.

**(c) Coordination of Amount of Change.** It was observed that both students in this study varied in their ability to coordinate amount of change in dependent variables with respect to uniform changes in the amount of independent variables. Jay’s behaviors such as his graphs and verbal remarks indicated that the student had difficulties to coordinate amount of change in the output variables. In other words, he did not seem to form images of the transformation of output variable or images of successive pictures of change in output variable. Neither was he observed plotting points on the graph by mentally evaluating the values of dependent variables considering the fixed amounts of independent variables nor did he construct contiguous line segments in order to show relative changes in the amount of dependent variables. In addition, his verbal expressions such as “…if each time he pulls the ladder by the same amount, the top is going to drop the same amount each time…” indicated his difficulties in coordinating the amount of change in the output. Although he was able to coordinate the amount of change in height considering the equal increments in amount of water in one task (see task 10, figure 5.23) accurately, he did not exhibited same behaviors on the other tasks.

His mental actions seemed to be limited to coordinating the direction of change. On the other hand, Karl’s behaviors supported the mental actions of coordinating the amount of change in the output in most of the presented tasks in this study. He was observed plotting points on the graphs by placing equal intervals in the X-axis and placing corresponding intervals appropriately representing the relative amounts in the Y-axis. While he was representing the amount of change in the dependent variables on his graphs, he also verbally expressed his awareness of how the amount of dependent variables changes over fixed intervals of independent variables.

Generally his behaviors appeared to support mental actions of coordinating the amount of change in the dependent variables. However while responding to some tasks; he was observed having difficulties to construct accurate images in order to coordinate amount of change in the output. In some cases, these difficulties were associated with using irrelevant or inaccurate information in switching the roles of dependent and independent variables. In these tasks his behaviors only supported mental actions of coordinating the direction of change of output.
(d) Coordinating Average Rate of Change. Analysis of Jay’s and Karl’s behaviors demonstrated dissimilarities in their mental actions of coordinating the average rate of change of the dependent variables in given tasks. Collection of Jay’s behaviors suggested that his mental activities did not support coordinating the average rate of change of output while Karl’s behaviors suggested that his mental actions were mostly involved in coordinating the average rate of change of output with respect to uniform changes of input in given situations. It is observed that Jay did not attempt to construct contiguous line segment with relative slopes in his graphs in order to represent relative rate of changes in the output variables. On the other hand, Karl’s graphs and his verbal expressions revealed that he mostly constructed accurate images of successive rate of change of dependent variables while imagining the fixed changes of dependent variables. Karl demonstrated these changing rates on his graphs by constructing adjacent line segments with different slopes. However in some tasks he was observed switching the role of dependent and independent variables and violating the condition of the task or communicating with inaccurate or irrelevant information. Therefore he was unable to construct accurate images of relative rate of changes in the output variables in these tasks. Although in some tasks, Jay’s verbal expressions gave an appearance of engaging in mental actions of coordinating the relative rate of changes in the output variables, his graphs were found to be inconsistent with these verbal expressions. For example, in task 1(figure 5.1) he provided statements such as “…the distance…is changing as slower rate as they approach each other. As they get apart…distance changes faster…in relation to time…” he did not provide a graph that represents various rate of change of distance. He did not even attempted to demonstrate contiguous line segments on the graph with different slopes representing the rate of change of the distance with respect to the time.

(e) Coordinating the Instantaneous Rate of Change. Analysis of both Jay’s and Karl’s behaviors revealed that both students had difficulties in constructing images of continuously changing rate of change. Although both students were observed constructing graphs with smooth curves in some tasks, their behaviors mostly associated with applying memorized actions. In other words, their reasoning in these tasks did not give any appearance of awareness that constructing smooth curves in order to demonstrate the continuously changing rate of change resulted from smaller refinements of the average rate of change. Carlson et al (2002) emphasized in their study that mental actions of coordinating the continuously changing rate of change in the input variable should be supported by behaviors that demonstrates student’s understanding of
continuously changing rate of change resulted from smaller refinements of the average rate of change.

Although Karl was observed evaluating the change in dependent variables over fixed intervals of independent variables by constructing contiguous line segments or secant lines on the graph, with the slope of each segment adjusted to reflect the relative rate for the specified amount of dependent variables he did not attempt to make smaller and smaller refinement to demonstrate the continuously changing rate of change over entire domain. On the other hand, since Jay’s behaviors did not even support the mental actions of coordinating the amount of change and the average rate of change of dependent variables with respect to uniform changes of independent variables, he was not able to construct images of continuously changing rate of change in the output variables.

Table 4.1 demonstrates the summary of findings for each individual case according to five mental actions defined comprehensively in Carlson et al (2002)’s framework for covariational reasoning.
<table>
<thead>
<tr>
<th>Mental Action -1</th>
<th>Mental Action-2</th>
<th>Mental Action -3</th>
<th>Mental Action-4</th>
<th>Mental Action-5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Coordination</td>
<td>Coordination of direction of change</td>
<td>Coordination of amount of change</td>
<td>Coordination of average rate of change</td>
<td>Coordination of instantaneous rate of change</td>
</tr>
<tr>
<td><strong>JAY</strong></td>
<td>Initial coordination was consistently demonstrated. He verbally expressed his awareness of one variable changes with respect to changes in other variable. Jay has also been observed drawing X and Y axis and labeling the axis.</td>
<td>• Coordination of direction of change was mostly demonstrated. • Some difficulties were observed in constructing the accurate images of the direction of change due to strict reliance on irrelevant information (e.g. focusing on another variable whose variation has no effect on relationship between main variables.</td>
<td>His observed behaviors did not support mental actions of coordinating the amount of change in dependent variables while imagining the uniform or fixed changes in the independent variables. His graphs and remarks indicated that he did not seem to form images of successive pictures of change in the amount of output variable.</td>
<td>Although in some tasks, Jay’s verbal expressions gave an appearance of engaging in mental actions of coordinating the relative rate of changes in the output variables, his graphs were found to be inconsistent with these verbal expressions. He did not even attempt to demonstrate contiguous line segments on the</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>KARL</strong></td>
<td>Initial coordination was consistently demonstrated by verbal expressions and graphical representations.</td>
<td>Coordination of direction of change was consistently demonstrated with verbal expressions and the graphical representations.</td>
<td>Karl’s behaviors supported mental actions of coordinating the amount of change in dependent variable while imagining the changes in the independent variables. He was observed plotting points on the graphs by placing equal intervals in the X axis and placing corresponding intervals appropriately representing the relative amounts in the Y axis. He also verbally expressed his understanding.</td>
<td>He was observed evaluating the change in dependent variables over fixed intervals of independent variables by constructing contiguous line segments or secant lines on the graph, with the slope of each segment adjusted to reflect the relative rate for the specified amount of dependent variables.</td>
</tr>
</tbody>
</table>
CHAPTER V
CONCLUSIONS

Based on the research questions, aim of this study was to explore, describe and analyze college students’ covariational reasoning abilities. Data were collected through in-depth clinical interviews with the selected two high performing students, who are enrolled in a third semester Calculus course. Task based interviews focused on exploring these college students thinking and reasoning processes as they attempt to represent and interpret covariant aspects of the real world dynamic function situation. Both Jay and Karl were interviewed five times. Data obtained from interviews were analyzed in light of the theoretical lens to identify each student’s thinking and reasoning processes as they attempt to coordinate simultaneous changes of variables in continuously changing dynamic functional situations. Both students in this study have exhibited various behaviors while they were applying their reasoning in order to analyze, interpret and represent dynamic functional events. Table 5.1 demonstrates common patterns of students’ interpretations of simultaneous changes of two variables and of students’ graphical representations aligned with their interpretations.

Table 5.1 Common patterns of interpretation and representation of simultaneous changes.

<table>
<thead>
<tr>
<th>Students’ interpretations of simultaneous changes of two variables in functional situations</th>
<th>Students’ graphical representations of simultaneous changes of two variables aligned with their interpretations.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Conceiving of functional situations statically:</strong> functional situations are conceived as static rather than dynamic. Interpretations are based upon thinking about simultaneous changes between two variables one step at a time.</td>
<td><strong>Point-wise or interval–wise construction of graphs:</strong> Plotting points in order to obtain relative X and Y values and obtain adjacent linear lines for equal intervals in the domain.</td>
</tr>
<tr>
<td>Verbal expressions in some cases indicate the awareness of continuously changing rate of change in the interpretation of simultaneous change.</td>
<td>Difficulties in demonstrating the accurate interpretations in graphical representations. Graphs with no smooth curve indicating the continuously changing rate of change.</td>
</tr>
</tbody>
</table>
**Table 5.1 Continued.**

<table>
<thead>
<tr>
<th>Focusing on procedures such as memorized set of rules and formulas:</th>
<th>Constructing accurate graphs to represent continuous changes (e.g. showing smooth curves):</th>
</tr>
</thead>
<tbody>
<tr>
<td>Students demonstrate tendency to use procedures that hinder meaningful interpretations of simultaneous changes.</td>
<td>Appropriate graphs are presented without giving a rational for the construction.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Reliance on irrelevant arguments or inappropriate principles:</th>
<th>Constructing erroneous graphs:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interpretations of simultaneous changes of two variables are concentrated around and built upon misuse of some general principles of science or insignificant arguments which have no effect on simultaneous changes of two variables and lead to erroneous conclusions.</td>
<td>Interpretations strictly based upon irrelevant arguments or inappropriate principles lead to construction of erroneous graphs in which even direction of change is inaccurately demonstrated.</td>
</tr>
</tbody>
</table>

**Assertions**

In the previous section, students’ covariational reasoning were analyzed and described regarding their cognitive activities in five categories of mental actions that are defined in covariation framework (Carlson et al, 2002). In addition to these macro level categories, analysis of data also disclosed other common issues in their reasoning patterns. Following assertions were made as a result of analysis of two cases.

1. **Thinking about simultaneous changes between two variables one step at a time and conceiving of functional situations statically leads to difficulties in coordinating the continuously changing rate of change over entire domain.**

Analysis of both cases disclosed that functional situations are conceived as static rather than dynamic. Students have been observed plotting points and obtaining adjacent linear lines for equal intervals in the domain rather than reasoning dynamically and disregarding the step by step evaluation of change. This static approach prevents them from evaluating the whole process as it is happening at once. In other words, students are not able to reason about continues change in a functional situation and coordinate the simultaneous changes of two variables on entire domain.
(2) Difficulties in graphical representations produce inconsistencies between interpretations and representations of simultaneous changes of two variables.

Both Jay and Karl had difficulties in demonstrating their accurate interpretations in graphical representations in some tasks. For example, in task 1 shown in figure 4.1, Jay states, “the distance… is changing as slower rate as they approach each other. As they get apart…distance changes faster…in relation to time”. Although he thinks that distance between two people is changing “as slower rate” when they approach each other, he demonstrates this situation with a straight line to show the decreasing distance. Similarly he drew a steeper straight line for representing the change in distance when two people get apart. When asked how these lines demonstrated a “slower rate” or a “faster rate” he said that the slope of first line is less than the slope of second line and this is the difference between “slower rate” and “faster rate”.

In task 11 shown in figure 5.30, Karl believes that water increases at greater rate and it slows down until it gets to certain point. However, he had difficulties with demonstrating his accurate reasoning pattern in the graphical representation. Karl had initial image of amount of water in the tank changing and the rate of change is getting smaller and smaller but when we look at the graph shown in figure 5.31, he provided line segments to represent the relationship between variables. Although his verbal expressions such as “…I think it would be increasing faster at first…” and “…I think it is going to take really really long time for it to eventually slow down…” are some evidences suggesting that his reasoning supports the mental actions of coordinating the rate of change of the amount of water in the tank over time, he did not graph smooth curve to demonstrate the continuous change.

(3) Strong procedural tendency hinders reasoning and meaningful interpretations about change in functional situations.

Both Jay and Karl were observed applying memorized set of rules and procedures to construct a graph for a given situation. It was noted that they were not able to provide a rationale for their construction. Carlson et al (2002) identified this kind of behavior as pseudo-analytical which was defined by Vinner (1997). According to Carlson, et al (2002) when students do not provide evidence that they possess an understanding that supports their behavior, this behavior is classified as pseudo-analytical behavior.

In the present study, Jay and Karl were observed constructing smooth curves in their graphs without demonstrating a rational for their construction. For example Karl used a formula
of area of a circle to find out the respective amounts of area of a circle in task 5 shown in figure 5.12. He ended up an area vs. radius graph consisted of contiguous line segments (figure 5.14). Then he converted this graph to a concave up graph by claiming: “…but as a matter of fact the area depends on \( r^2 \)… I mean it should be a parabola like in \( y = x^2 \)”. Although his construction of a smooth curve gave an appearance of engaging in coordinating the continuous changes in the dependent variable, his verbal expressions revealed that his behavior was not more than applying a memorized rule. Jay was also observed demonstrating the similar behaviors. In the interpreting task, for example, shown in figure 4.1.36, Jay utilized a memorized method to produce velocity vs. time graph from a given distance vs. time graph. His verbal expressions did not provide any evidence that supported the behavior.

Both Jay and Karl were also observed expressing their tendency to use intermediary graphs which were claimed to be providing clearer picture of a situation. Then they both attempted to use these mediator graphs to be able to obtain main graph by applying some calculus concepts such as derivative and integration. However their verbal expressions did not provide any evidence that they possessed an understanding of these concepts.

In light of the analysis of cases, it was noted that both students showed their tendency to utilize algebraic expressions from time to time by either applying an algebraic expression, when available, or declared their need for an algebraic expression to be able to construct graph that represents the relationship between two variables. While working on the task 1 (figure 4.1), Jay stated: “…actually without having specific function it is very hard to graph it because we need to have a function to find exact values…”. Jay has also expressed his need for numerical information in several other tasks too, although it was not necessary. For instance, while he was working on the task 12 (figure 5.29), he claimed “…since there is no value for amount of water comes in and comes out I cannot say anything about the proportion between them. Since I do not know the values….”. Both Jay and Karl were observed utilizing the area of a circle formula and volume of a sphere formula in other tasks. Tall and Vinner (1981) stated that students’ restricted concept of function is rooted in the predominant use of functions given by algebraic formulas in traditional instructional methods. Leinhardt, Zaslavsky, and Stein (1990) have emphasized that construction of graphs from algebraic formulas were highly dominant in traditional instructional methods.
(4) Reasoning based on irrelevant arguments or inappropriate principles leads to erroneous conclusions about simultaneous changes of two variables.

Analysis of data also revealed several factors in students’ reasoning which resulted in constructing inappropriate images of relationship between two variables. Both Jay and Karl were observed communicating with information that was not related to situations presented in the tasks. For example, while working on the ladder problem Jay claimed: “… since ladder is a one solid thing when you pull from one end by certain amount the other end will move by same amount” and Karl claimed: “…When I think theoretically speed of top of the ladder should be constant same with the speed of the bottom of the ladder. Because when you move an object, speed should be same in every part of that object…”. As seen in their verbal expressions their reasoning strictly bounded with irrelevant information considering the situation in the specific task. Similarly in task 11 (figure 4.27), task 12 (figure 4.29) and task 13 (figure 4.32) Jay’s verbal statements such as “…since the container’s shape does not change…same amount of water coming in and coming out…” and “Amount of water coming out is going to remain constant because hole is not changing so same water is coming out…” revealed that he built his reasoning on the size of the hole or shape of the container which were considered as irrelevant information in these tasks, because neither size of the hole nor the shape of the container were main variables to be considered in these tasks. While coordinating the change in the radius of a spherical balloon over time in task 13, Karl communicated inappropriate information considering the situation stated in the task. He claimed: “… I think the radius is going to be increasing faster at the beginning and it slows down when it gets to final volume that is how I visualize. Because initially when you blow up the balloon there is more space so radius increasing faster then when it gets to the end it [radius] is still increasing but slowly since there is not much room….” He built his reasoning upon irrelevant information.

Patterns of Students’ Interpretations and Graphical Representations of simultaneous changes of two variables

Implications

The results of this qualitative inquiry demonstrated high performing third semester calculus students’ covariational reasoning when they attempt to coordinate simultaneous changes of two variables in continuously changing functional situations. Therefore this study has
implications for instructional methods in classrooms, design and development of curricular activities, alternative assessment methods.

Analysis of data and findings revealed that even high performing calculus students have exhibited difficulties in constructing accurate images of continuously changing rate of change of functions over domains. Taking into consideration of this finding, this study suggests:

First, since students’ underlying function conception is said to be very important in imagining the simultaneous covariation of variables, the concept of function could be introduced to students in alternative ways in order to develop dynamic conceptions rather static conceptions. More specifically, instead of introducing the concept of function as correspondence, which is concentrated around the application of certain rules and formulas to describe how to obtain the output value from a given input value, it will be more helpful for students if the concept of function is introduced as covariation which focuses on coordinating changes between input and output values and emphasizes the changing nature of functions. This introduction would provide an incredible foundation for students to develop better conceptual understanding of functions.

Second, utilizing computer technology such as dynamic software in the instruction may provide more visual representations in order to enhance students’ conceptualization of changing nature of functions. Third, study suggests design and development of alternative curricular activities in order to develop students’ covariational reasoning abilities. In order to decrease negative effects of overexposure of routine formula dependent problems and task, providing students with alternative qualitative problems and tasks would be more constructive.

Finally, study also suggests design and development of alternative assessment tools for classroom teachers in order to assess and understand students’ thinking and reasoning.

**Limitations**

From several perspectives this study has limitations. Starting from the selection of participant to the analysis of data there are several issues that will possibly limit the purpose of this study. Most of these limitations are rooted in method of inquiry.

Methodological limitations: Present study was conducted in a frame of case study techniques and procedures. Thus findings of this study extremely depended upon each student’s level of understanding and cognitive structure. Considering this fact, findings of this study may not be applicable to subjects who have different background and in different settings.
Researcher limitations: This study is also limited by researcher’s integrity and sensitivity. In analyzing students’ responses to the problems in interviews researcher’s background, beliefs and language difficulties might have an effect on the interpretations of findings of the study. In order to reduce this negative effect, peer debriefing was employed by discussing the findings with members of the mathematics education program.

**Issues for Future Research**

This study investigated students’ covariational reasoning based on their reflection of dynamic situations in graphical representations. Therefore, this study calls for further research to investigate students’ covariational reasoning in different representational systems.

This study also suggests investigations of the effectiveness of different curricular activities and different instructional methods in developing students’ covariational reasoning abilities. These studies would also be very informative for curriculum developers in terms of designing more effective methods for instruction.

In this study, while working on the tasks students’ constructed mental images of situations to be able to coordinate simultaneous changes of two variables. Although they reflected their mental images by verbal statements and graphical representations of situations, further investigations will be useful to better understand the role of mental imagery in students’ covariational reasoning.
Walking Problems

1. Two people start at opposite corners of a room and walk toward each other. As they walk, they both slow down as they get closer to each other, pass, and then they both speed up as they get farther apart. Draw a graph showing the distance between two people at each moment in time. Describe your graph.

2. These same two people decide again to start at opposite corners of the room and walk toward each other but this time they both decide to maintain the same steady pace the whole way. Draw a graph showing the distance between two people at each moment in time. Describe your graph.

3. These same two people decide once more again to start at opposite corners of the room and walk toward each other. But this time as they walk, they both speed up as they get closer to each other; pass, and then they both slow down as they get farther apart. Draw a graph showing the distance between two people at each moment in time. Describe your graph.

Adopted from Hauger, G.S. (1998)

Ladder problems

4. Tom sees a ladder against a wall (in an almost vertical position). He pulls the base of the ladder away from the wall by a certain amount, and so forth. Each time he does this he records the distances by which the top of the ladder drops down. Do the amounts by which the top of the ladder drops down remain constant as Tom repeats this step; or do they get bigger, or do they get smaller? EXPLAIN

5. Newt, the science nerd, then comes along and puts wheels on the bottom of the ladder. He connects them to a motor so that the bottom rolls away at a constant, but very slow, speed. Does the top of the ladder move down at a constant speed, or does it speed up, or does it slow down? EXPLAIN

6. Draw a graph which represents the relationship between the horizontal and vertical positions of a ladder as it slides down a wall, starting at a vertical position and finally resting on the ground. EXPLAIN.

Adopted from Monk, S. (1992)

Water filling problems
7. Imagine this bottle filling with water. Sketch a graph that represents the relationship between amount of water that is in the bottle and the height.

8. Imagine this water tank filling with water. Sketch a graph that represents the relationship between amount of water that is in the tank and the height.
9. Imagine this water tank filling with water. It starts out empty. Sketch a graph that represents relationship between amount of water in the tank and height.

10. Now there is a hole in the bottom of the tank. Water leaves the tank at a rate is proportional to the height of the water remaining in the tank. The tank starts off with full when the hole is opened. Sketch a graph that represents relationship between amount of water in the tank and time.

11. Again there is a hole in the bottom of the tank. The tank starts off empty and water is poured in at a constant rate at which the level increases. Water leaves tank at a rate which is proportional to the height of the water in the tank. Sketch a graph that represents relationship between amount of water in the tank and time.
12. There is a small hole part way up to the side of a water tank and no hole in the bottom. The tank starts off empty and water is poured in at a constant rate. Water leaves the tank at a rate which is proportional to the height of the water above the hole. Sketch a graph that represents the relationship between the amount of water in the tank and time.

Pebble problem

13. Imagine a pebble is thrown into a lake, creating a circular ripple that travels outward at a constant speed. Sketch a graph that represents the relationship between the area, $A$, of the circle and time, that have passed since the ball hits the lake.
Chemical changes

14. 5% of chemical X changes into chemical Y every second. Chemical Y never changes back. Starting with all chemical X, sketch a graph that represent the amount of chemical X vs. time.

15. 5% of chemical X changes into chemical Y every second. 5% of chemical Y changes back to chemical X every second. Starting with all chemical X, sketch a graph that represent
   a. The amount of chemical X vs. time
   b. The amount of chemical Y vs. time

Below is the distance vs. time graph of a car traveling through city A to city E.
APPENDIX B

CONSENT LETTER
Human Subject Committee
Letter of Consent for Adults

"An Investigation of College Students’ Covariational Reasoning Abilities."

I have been informed that:

1. Onder Koklu, a graduate student under the direction of Dr. Elizabeth Jakubowski in the Mathematics Education Program, Department of Middle and Secondary Education, at Florida State University, has requested my participation in a research study of college students’ covariational reasoning abilities.

2. The purpose of this research is to better understand the college students’ covariational reasoning abilities.

3. I understand that I will be asked to fill out paper and pencil Questionnaires. I may also be asked to participate in one-hour interviews with researcher. If I participate in the interview, I will receive $10.00 compensation for my time. I understand that my participation is voluntary during the 2005 fall semester, and if I choose not to participate or to withdraw from the study at any time, there will be no prejudice or penalty, and my information will not be used.

4. There are no foreseeable risks or discomforts if I agree to participate in this study.

5. The results of this research study may be published but my name or identity will not be revealed. The researcher will use pseudonyms to represent myself and my university, and will maintain confidentiality of my records, to the extent allowed by law. I understand that interviews will be audio taped by researcher. I also understand that all documents and audio tapes will be safely kept in a locked filing cabinet located in 219 MCH building-researcher’s office. I understand that only the research will have access to these documents and audiotapes. I understand that all documents and audiotapes will be destroyed by April 28, 2007.

6. Any questions I have concerning the research study or my participation in it, before or after my consent, will be answered by Onder Koklu at (850) 656-6569, 219 MCH, Florida State University, Tallahassee, FL, 32306-4490, or okk3078@fsu.edu; or by Dr. Elizabeth Jakubowski, his major professor, at (850) 644-8428 or ejakubow@coe.fsu.edu.

7. If I have questions about my rights as a subject/participant in this research, or if I feel I have been placed at risk, I can contact the Chair of the Human Subjects Committee, Institutional Review Board, through the Office of the Vice President for Research, at (850) 644-8633.

Human Subject Committee
Letter of Consent for Adults

I have read the above informed consent form. I understand that information obtained during the course of the study will remain confidential, to the extent allowed by law.

I understand that I may withdraw my consent and discontinue participation at any time without penalty or loss of benefits to which I may otherwise be entitled. In signing this consent form, I am not waiving any legal claims, rights or remedies. A copy of this consent form will be offered to me.

Sincerely,

Participant's Signature ________________________________ Date ________

REFERENCES


BIOGRAPHICAL SKETCH

Onder Koklu was born in Sivas, Turkey. He started his undergraduate study in 1991, Cukurova University, Adana where he studied Civil Engineering. He completed his undergraduate study in 1996 with a Bachelors of Science degree. Until March 1998, he worked for a private international engineering company in Ankara, Turkey, as a project engineer where he worked on highway projects.

In March 1998, he was awarded with a full scholarship by Turkish Ministry of National Education to pursue masters and PhD degrees in the field of Mathematics Education in the US. In fall 1999 he started his masters program in mathematics education at Florida State University.
After receiving the Master of Science degree in May 2001, he continued his doctoral program and earned Doctor of Philosophy degree in Summer 2007.